Soft linear $pq$-functions and soft $\beta$ kernel in Vector Soft Topological Spaces

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Abstract

In this paper we prove some basic properties of the soft sets in a vector soft topological space (VSTS). Also we establish the soft linearity of $pq$-functions and soft $\beta$ kernel of a soft linear $pq$-function in a VSTS.

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1. Introduction

any two families of soft sets. And in 2015, Moumita Chiney and S. K. Samanta [9] introduced vector soft topology, connecting soft set theory and topological vector spaces. With that motivation, we like to study some more concepts of soft sets, soft linearity of $pq$-functions and soft $\beta$ kernel of a soft linear $pq$-function in a VSTS.

Section 2 deals with the preliminaries such as definition of soft sets, its basic operations, definition of soft topology and some properties. In section 3, we proved some theorems on soft sets in a VSTS. In Section 4 we present some properties of convex and balanced sets in a VSTS. Also we defined the concept of soft subspace topology and proved some results based on it. Section 5 contains a main results of the paper, the condition for linearity of a soft $pq$-function and the necessary and sufficient conditions for the continuity of a soft linear $pq$-function. In section 6 we defined soft $\beta$ kernel of a soft linear $pq$-function and proved that soft $\beta$ kernel of a soft linear map is a vector space. The definition of soft quotient topology and a necessary and sufficient condition for the continuity of a function on soft quotient topology is also proved in this section.

2. Preliminaries

**Definition 2.1.** [8] A pair $(F, A)$ is called a soft set over a universal set $X$, where $F$ is a mapping $F : A \rightarrow 2^X$, $A$ is a set of parameters.

Notation [2]: The family of all soft sets over $X$ is denoted by $SS(X, A)$.  

**Definition 2.2.** [8] The soft set $(F, A) \in SS(X, A)$ where $F(\alpha) = \phi, \forall \alpha \in A$ is called the null soft set of $SS(X, A)$ and is denoted by $\phi_A$.  
The soft set $(F, A) \in SS(X, A)$ where $F(\alpha) = X, \forall \alpha \in A$ is called the absolute soft set of $SS(X, A)$ and is denoted by $X_A$.  

**Definition 2.3.** [16] Let $\tau$ be a collection of soft sets over $X$. Then $\tau$ is said to be a soft topology if

1. $\phi_A, X_A$ belong to $\tau$
2. the soft union of any number of soft sets in $\tau$ belongs to $\tau$
3. the soft intersection of any two soft sets in $\tau$ belongs to $\tau$

The triplet $(X, \tau, A)$ is called a soft topological space.

**Definition 2.4.** [16] Let $(X, \tau, A)$ be a soft topological space over $X$, then the members of $\tau$ are said to be soft open sets in $X$.  
A soft set $(F, A)$ over $X$ is said to be soft closed set in $X$ if its soft complement $(F^c, A)$ belongs to $\tau$.  

**Proposition 2.5.** [3] Let $(X, \tau, A)$ be a soft topological space over $X$. Then for a fixed $\alpha \in A$, $\tau_\alpha = \{F(\alpha) : (F, A) \in \tau\}$ defines a topology on $X$.  

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**Definition 2.6.** [14] Let $SS(X, A)$ denote the set of all soft sets over $X$ under the parameter set $A$. A soft set $(F, A) \in SS(X, A)$ is said to be *pseudo constant* soft set if $F(\alpha) = X$ or $\phi$, $\forall \alpha \in A$.

Let $CS(X, A)$ denote the set of all pseudo constant soft sets over $X$ under the parameter set $A$.

**Definition 2.7.** [14] A soft topology $\tau$ on $X$ is said to be an *enriched soft topology* if (1) of the Definition 2.7 is replaced by (1') $(F, A) \in \tau, \forall (F, A) \in CS(X, A)$.

Then the triplet $(X, \tau, A)$ is called an *enriched soft topological space* over $X$.

**Proposition 2.8.** [9] Let $X$ be a non-empty set, $A$ be the set of parameters and for each $\alpha \in A, \tau_\alpha$ is a crisp topology on $X$. Then $\tau^* = \{(G, A) \in SS(X, A) : G(\alpha) \in \tau_\alpha, \forall \alpha \in A\}$ is an enriched soft topology on $X$.

**Proposition 2.9.** [13] Let $(X, \tau, A)$ be a soft topological space and if $\tau^* = \{(G, A) \in SS(X, A) : G(\alpha) \in \tau_\alpha, \forall \alpha \in A\}$, then $\tau^*$ is an enriched soft topology on $X$ such that $\tau \subseteq \tau^*$ and $[\tau^*]_\alpha = \tau_\alpha, \forall \alpha \in A$. And $\tau^*$ is called the *enriched topology derived from* $\tau$.

**Definition 2.10.** [13] Let $X$ and $Y$ be two non-empty sets and $f : X \rightarrow Y$ be a mapping.

1. the image of a soft set $(F, A) \in SS(X, A)$ under the mapping $f$ is denoted by $f[(F, A)]$ and is defined by $f[(F, A)] = (f(F), A)$ where $[f(F)](\alpha) = f[F(\alpha)], \forall \alpha \in A$.

2. the inverse image of a soft set $(G, A) \in SS(Y, A)$ under the mapping $f$ is denoted by $f^{-1}[(G, A)]$ and is defined by $f^{-1}[(G, A)] = (f^{-1}(G), A)$ where $[f^{-1}(G)](\alpha) = f^{-1}[G(\alpha)], \forall \alpha \in A$.

**Definition 2.11.** [13] Let $(X, \tau, A)$ and $(Y, \upsilon, A)$ be soft topological spaces. The mapping $f : (X, \tau, A) \rightarrow (Y, \upsilon, A)$ is said to be

1. *soft continuous* if $f^{-1}(F, A) \in \tau, \forall (F, A) \in \upsilon$.

2. *soft homeomorphism* if $f$ is bijective and $f$ and $f^{-1}$ are soft continuous.

3. *soft open* if $(F, A) \in \tau \Rightarrow f(F, A) \in \upsilon$.

4. *soft closed* if $(F, A)$ is soft closed in $(X, \tau, A) \Rightarrow f(F, A)$ is soft closed in $(Y, \upsilon, A)$.

**Definition 2.12.** [9] Let $(F, A)$ and $(G, A)$ be two soft sets over a vector space $V$, over $K$, the field of real or complex numbers. Then

1. $(F, A) + (G, A) = (F + G, A)$ where $(F + G)(\alpha) = F(\alpha) + G(\alpha), \forall \alpha \in A$.

2. $k(F, A) = (kF, A)$ where $(kF)(\alpha) = \{kx : x \in F(\alpha)\}, \forall \alpha \in A$ and $\forall k \in K$. 

3. \( x + (F, A) = (x + F, A) \) where \( (x + F)(\alpha) = \{ x + y : y \in F(\alpha) \} \), \( \forall \alpha \in A \) and \( \forall x \in V \).

4. If \( (E, A) \) is any soft set over \( K \), then \( (E, A) \cdot (F, A) = (E \cdot F, A) \) where \( (E \cdot F)(\alpha) = E(\alpha) \cdot F(\alpha) \), \( \forall \alpha \in A \).

**Definition 2.13.** [13] A soft set \( (E, A) \) over \( X \) is said to be a soft element if there exists \( \alpha \in A \) such that \( E(\alpha) \) is a singleton say \( \{ x \} \) and \( E(\beta) = \phi \), \( \forall \beta (\neq \alpha) \in A \). Such a soft element is denoted by \( E^x_\alpha \). A soft element \( E^x_\alpha \) is said to be in the soft set \( (G, A) \) denoted by \( E^x_\alpha \in (G, A) \) if \( x \in G(\alpha) \).

**Definition 2.14.** [13] Let \( (X, \tau, A) \) be a soft topological space over \( X \). A soft set \( (F, A) \) is said to be a soft neighbourhood of the soft set \( (H, A) \) if there exists a soft open set \( (G, A) \) such that \( (H, A) \subseteq (G, A) \subseteq (F, A) \).

If \( (H, A) = E^x_\alpha \), then \( (F, A) \) is said to be soft neighbourhood of the soft element \( E^x_\alpha \). The neighbourhood system of a soft element \( E^x_\alpha \) is denoted by \( N_\tau (E^x_\alpha) \), which is the family of all its soft neighbourhoods.

**Definition 2.15.** [5] Let \( SS(U, A) \) and \( SS(V, B) \) be two families of soft sets. Let \( q : U \to V \) and \( p : A \to B \) be mappings. Then a mapping \( f_{pq} : SS(U, A) \to SS(V, B) \) is defined as

1. Let \( (F, A) \) be a soft set in \( SS(U, A) \). The image of \( (F, A) \) under \( f_{pq} \) written as \( f_{pq}(F, A) = (f_{pq}(F), p(A)) \) is a soft set in \( SS(V, B) \) such that

\[
(f_{pq}(F))(y) = \begin{cases} \bigcup_{x \in p^{-1}(y)} q(F(x)) & \text{if } p^{-1}(y) \neq \phi \\ \phi & \text{otherwise} \end{cases}, \forall y \in B
\]

2. Let \( (G, B) \) be a soft set in \( SS(V, B) \). Then the inverse image of \( (G, B) \) under \( f_{pq} \) written as \( f_{pq}^{-1}(G, B) = (f_{pq}^{-1}(G), p^{-1}(B)) \) is a soft set in \( SS(U, A) \) such that

\[
(f_{pq}^{-1}(G))(x) = \begin{cases} q^{-1}(G(p(x))) & \text{if } p(x) \in B \\ \phi & \text{otherwise} \end{cases}, \forall x \in A
\]

The soft function \( f_{pq} \) is called surjective if \( p \) and \( q \) are surjective. The soft function \( f_{pq} \) is called injective if \( p \) and \( q \) are injective.

**Proposition 2.16.** [5] Let \( SS(U, A) \) and \( SS(V, B) \) be families of soft sets. For a function \( f_{pq} : SS(U, A) \to SS(V, B) \), the following statements are true:

1. \( f_{pq}(\phi_A) = \phi_B \)
2. \( f_{pq}(U_A) \supseteq U_B \)
3. \( f_{pq}((F, A) \sqcup (G, A)) = f_{pq}(F, A) \sqcup f_{pq}(G, A) \) where \( (F, A), (G, A) \in SS(U, A) \).

In general \( f_{pq}(\sqcup_i (F_i, A)) = \sqcup_i f_{pq}(F_i, A) \) where \( (F_i, A) \in SS(U, A) \).
4. If \((F, A) \subseteq (G, A)\) then \(f_{pq}(F, A) \subseteq f_{pq}(G, A)\) where \((F, A), (G, A) \in SS(U, A)\).

5. If \((G, B) \subseteq (H, B)\) then \(f^{-1}_{pq}(G, B) \subseteq f^{-1}_{pq}(H, B)\) where \((G, B), (H, B) \in SS(V, B)\).

**Proposition 2.17.** [18] Let \(SS(U, A)\) and \(SS(V, B)\) be families of soft sets. For a function \(f_{pq} : SS(U, A) \rightarrow SS(V, B)\), the following statements are true

1. \(f_{pq}^{-1}((G, B)^c) = (f_{pq}^{-1}(G, B))^c\)

2. \(f_{pq}(f_{pq}^{-1}(G, B)) \subseteq (G, B) \forall (G, B) \in SS(V, B)\). If \(f_{pq}\) is surjective, the equality holds.

3. \((F, A) \subseteq f_{pq}^{-1}(f_{pq}(F, A))\) for any soft set \((F, A)\) in \(SS(U, A)\). If \(f_{pq}\) is injective, the equality holds.

**Definition 2.18.** [18] Let \((U_1, \tau_1, A_1)\) and \((U_2, \tau_2, A_2)\) be soft topological spaces. Let \(q : U_1 \rightarrow U_2\) and \(p : A_1 \rightarrow A_2\) be mappings. Let \(f_{pq} : SS(U_1, A_1) \rightarrow SS(U_2, A_2)\) be a soft function and \(E^x_\alpha \in U_{1A_1}\)

1. \(f_{pq}\) is soft \(pq\)-continuous at \(E^x_\alpha \in U_{1A_1}\) if for each \((G, B) \in N_{\tau_2}(f_{pq}(E^x_\alpha))\), \(\exists a (H, A) \in N_{\tau_1}(E^x_\alpha)\) such that \(f_{pq}(H, A) \subseteq (G, B)\)

2. \(f_{pq}\) is soft \(pq\)-continuous on \(U_{1A_1}\) if \(f_{pq}\) is soft \(pq\) continuous at each soft points in \(U_{1A_1}\).

**Proposition 2.19.** [18] Let \((U, \tau, A)\) and \((V, \nu, B)\) be soft topological spaces. Let \(f_{pq} : SS(U, A) \rightarrow SS(V, B)\) be a function and \(E^x_\alpha \in U_A\). Then the following statements are equivalent.

1. \(f_{pq}\) is soft \(pq\)-continuous at \(E^x_\alpha\)

2. For each \((G, B) \in N_{\nu}(f_{pq}(E^x_\alpha))\), \(\exists a (H, A) \in N_{\tau}(E^x_\alpha)\) such that \((H, A) \subseteq f^{-1}_{pq}(G, B)\).

3. For each \((G, B) \in N_{\nu}(f_{pq}(E^x_\alpha))\), \(f^{-1}_{pq}(G, B) \in N_{\tau}(E^x_\alpha)\).

**3. Properties of soft sets in a Vector Soft Topological Space**

**Definition 3.1.** [9] Let \(K\) be the field of real or complex numbers, \(A\) be the set of parameters and \(v_\alpha\) be the usual topology on \(K\), \(\forall \alpha \in A\). Then the soft topology \(v\) derived from \(v_\alpha\) is called the soft usual topology on \(K\).

**Definition 3.2.** [9] Let \(V\) be a vector space over a scalar field \(K\), endowed with the soft usual topology, \(v\), \(A\) be the parameter set and \(\tau\) be a soft topology on \(V\). Then \(\tau\) is said to be a vector soft topology on \(V\) if the mapping:
1. \( f : (V \times V, A, \tau \times \tau) \to (V, A, \tau) \) defined by \( f(x, y) = x + y \) and

2. \( g : (K \times V, A, v \times \tau) \to (V, A, \tau) \) defined by \( g(k, x) = kx \)

are soft continuous, \( \forall x, y \in V \) and \( k \in K \).

**Proposition 3.3.** [9] Let \( \tau \) be a vector soft topology on a vector space \( V \) over the field \( K \), \( A \) be the parameter set and \( v \) be the soft usual topology on \( K \). Then \( \tau_\alpha \) is a vector topology on \( V \), \( \forall \alpha \in A \).

**Proposition 3.4.** [9] Let \( V \) be a vector space over a scalar field \( K \), endowed with the soft usual topology, \( v, A \) be the parameter set and \( \forall \alpha \in A \), \( \tau_\alpha \) is a vector topology on \( V \), then \( \tau^*_\alpha \) is a vector soft topology on \( V \), where \( \tau^*_\alpha \) is defined as in Proposition 2.2.

**Proposition 3.5.** Let \( \tau \) be a vector soft topology on a vector space \( V \) over the field \( K \), \( A \) be the parameter set. Then for any \( (F, A) \in SS(V, A) \) and \( x \in V \), \( [x + (F, A)]^- = x + (F, A)^- \) and \([\lambda(F, A)]^- = (\lambda F, A)^-, \forall \lambda \in K\).

**Proof.** \( (F, A)^- \) is the intersection of all soft closed sets containing \( (F, A) \).

Let \( (G, A) \) be a soft closed set containing \( (F, A) \).

Then \( x + (G, A) = (x + G, A) \), where \( (x + G)(\alpha) = \{ x + y : y \in G(\alpha) \} \).

\( x + (G, A) \) is also soft closed, since the addition map is continuous.

Also \( x + (F, A) \subseteq x + (G, A) \).

\[ [x + (F, A)]^- = \cap \{ (x + (G, A) : (G, A) \text{ is soft closed and } (G, A) \supseteq (F, A) \} \]

\[ = x + \cap \{ (G, A) : (G, A) \text{ is soft closed and } (G, A) \supseteq (F, A) \} \]

\[ = x + (F, A)^- \text{ Now } \lambda(G, A) = (\lambda G, A) \text{ where } (\lambda G)(\alpha) = \{ \lambda y : y \in G(\alpha) \}. \]

Since scalar multiplication is continuous, \( \lambda(G, A) \) is closed if \( (G, A) \) is closed.

And \( \lambda(F, A) \subseteq \lambda(G, A) \).

\[ [\lambda(F, A)]^- = \cap [\lambda(G, A) : (G, A) \text{ is soft closed and } (G, A) \supseteq (F, A)] \]

\[ = \lambda \cap \{ (G, A) : (G, A) \text{ is soft closed and } (G, A) \supseteq (F, A) \} \]

\[ = (\lambda(F, A))^- \text{ Thus the closure of } \lambda(F, A) \text{ is } (\lambda F, A)^-. \] ■

**Proposition 3.6.** Let \((V, \tau, A)\) be a vector soft topological space(VSTS). Then for any \((F, A), (G, A) \in SS(V, A), (F, A)^- + (G, A)^- \subseteq (F + G, A)^-\).

**Proof.** For a fixed \( \alpha \in A \), consider \( F^-(\alpha) + G^-(\alpha) \). Let \( x_\alpha \in F^-(\alpha) \) and \( y_\alpha \in G^-(\alpha) \).

Then \( (x_\alpha, y_\alpha) \in F^-(\alpha) \times G^-(\alpha) \), and \( F^-(\alpha) \times G^-(\alpha) \) is a closed set since it is the product of two closed sets. Then by the continuity of the addition map, \( x_\alpha + y_\alpha \in \) any closed set containing \( F(\alpha) + G(\alpha) \). Therefore \( x_\alpha + y_\alpha \in (F + G)^-(\alpha) \) Since this is true for all \( \alpha \in A \), \( (F, A)^- + (G, A)^- \subseteq (F + G, A)^-\). ■

**Proposition 3.7.** In the VSTS \((V, \tau^*, A)\), the sum of any soft set and a soft open set is soft open.

**Proof.** Let \((F, A)\) be any soft set in \((V, \tau^*, A)\) and \((U, A)\) be a soft open set. Fix \( \alpha \in A \)

\[ F(\alpha) + U(\alpha) = \cup_{\alpha} \{ x + U(\alpha) : x \in F(\alpha) \}. \]

Since \( U(\alpha) \) is open, \( x + U(\alpha) \) is open and
by the property of a topological space, \( \cup \{ x + U(\alpha) : x \in F(\alpha) \} \) is open in \( \tau_{\alpha} \).

i.e. \((F + U)(\alpha)\) is open in \(\tau_{\alpha}\).

Since this is true for all \(\alpha \in A\), \((F + U, A)\) is soft open in \(\tau^*\). ■

**Definition 3.8.** [18] A family \(\Psi\) of soft sets is a cover of a soft set \((F, A)\) if \((F, A) \subseteq \bigcup\{(F_i, A) : (F_i, A) \in \Psi, i \in I\}\). It is a soft open cover, if each member of \(\Psi\) is a soft open set. A subcover of \(\Psi\) is a subfamily of \(\Psi\) which is also a cover.

**Definition 3.9.** [18] A soft topological space \((U, \tau, A)\) is compact if each soft open cover of \(U_A\) has a finite subcover.

**Proposition 3.10.** Let \((V, \tau, A)\) be a VSTS. If \((C, A)\) and \((D, A)\) are two soft compact sets in \(V\), then \((C, A) + (D, A)\) is also a soft compact set.

*Proof.* If \((C, A)\) and \((D, A)\) are two soft compact sets, \(C(\alpha)\) and \(D(\alpha)\) are compact sets for all \(\alpha \in A\). This implies \(C(\alpha) \times D(\alpha)\) is compact, for all \(\alpha \in A\). Then \(C(\alpha) + D(\alpha)\) is compact, for all \(\alpha \in A\), by the continuity of addition in vector soft topology. Hence \((C, A) + (D, A)\) is compact. ■

**Proposition 3.11.** Let \((V, \tau, A)\) be a VSTS. If \((C, A)\) is a soft compact set in \(V\), then \((\lambda C, A)\) is also a soft compact set, \(\forall \lambda \in K\).

*Proof.* If \((C, A)\) is a soft compact set, \(C(\alpha)\) is compact set for all \(\alpha \in A\). This implies \(\lambda C(\alpha)\) is compact, for all \(\alpha \in A\). Then \((\lambda C)(\alpha)\) is compact, for all \(\alpha \in A\), by the continuity of scalar multiplication in vector soft topology. Hence \((\lambda C, A)\) is compact. ■

*Note:*
By the propositions 3.6 and 3.7, the set of all soft compact sets in a VSTS forms a vector space with addition of soft sets and scalar multiplication of soft sets.

### 4. Convex and balanced soft sets in a VSTS and soft subspace topology

**Definition 4.1.** [9] A soft set \((F, A)\) over a vector space \(V\) is said to be

1. convex if \(k(F, A) + (1 - k)(F, A) \subseteq (F, A), \forall k \in [0, 1]\).
2. balanced if \(k(F, A) \subseteq (F, A)\) for all scalar \(k\) with \(|k| \leq 1\).
3. absolutely convex if it is balanced and convex.

**Remark 4.2.** [9]

1. \((F, A)\) is convex (balanced) soft set if and only if for all \(\alpha \in A\), the ordinary set \(F(\alpha)\) is convex (balanced).
2. If \((F, A)\) and \((G, A)\) are two convex (balanced) soft sets in a vector space \(V\) over the scalar field \(K\), then \(k_1(F, A) + k_2(G, A)\) is convex (balanced) soft set in \(V\) for all scalars \(k_1, k_2 \in K\).

3. If \(\{(F_i, A)\}_{i \in I}\) is a family of convex (balanced) soft sets in a vector space \(V\), then \((F, A) = \cap_{i \in I}(F_i, A)\) is a convex (balanced) soft set in \(V\).

**Proposition 4.3.** The closure of a balanced soft set is balanced in any VSTS.

*Proof.* Let \((F, A)\) be a balanced soft set in \((V, \tau, A)\), a VSTS. Then by definition, \(k(F, A) \subseteq (F, A)\), \(\forall |k| \leq 1\)

\[k(F, A)^{-} = (kF, A)^{-} \subseteq (F, A)^{-}, \forall k \leq 1\]

Hence \((F, A)^{-}\) is a balanced soft set. ■

**Proposition 4.4.** The interior of a balanced soft set is balanced in any VSTS.

*Proof.* Let \((F, A)\) be a balanced set. Then by definition \((kF, A) \subseteq (F, A), \forall k \in K\). And for any soft set \((F, A)\), \(k(F, A) = (kF, A)\)

Hence \((F, A)^{o} = (k F, A)^{o} \subseteq (F, A)^{o}\)

So \((F, A)^{o}\) is a balanced soft set. ■

**Proposition 4.5.** The closure of a convex soft set is a convex soft set in any VSTS.

*Proof.* Let \((F, A)\) be a convex soft set. Then by definition \(k(F, A) + (1-k)(F, A) \subseteq (F, A), \forall k \in [0, 1]\)

Now \(k(F, A)^{-} + (1-k)(F, A)^{-} = (kF, A)^{-} + ((1-k)F, A)^{-}\)

\(\subseteq (kF + (1-k)F, A)^{-}\)

\(\subseteq (F, A)^{-}\).

Thus \((F, A)^{-}\) is a convex soft set. ■

**Definition 4.6.** Let \((V, \tau, A)\) be a vector soft topology and \(W\) be a subspace of \(V\). Then for \((F, A) \in \tau, \exists (F|_{W}, A) \in \tau|_{W}\) where \(F|_{W}(\alpha) = F(\alpha) \cap W, \forall \alpha \in A\).

Then clearly \(\tau|_{W}\) is a soft topology on \(W\).

If \(\tau|_{W}\) is a vector soft topology on \(W\), then \((W, \tau|_{W}, A)\) is called a soft subspace topology.

**Proposition 4.7.** Let \((V, \tau, A)\) be a VSTS, then the closure of a soft subspace in \(V\) is a soft subspace in \(V\).

*Proof.* Let \((FA)\) be a soft vector space in a vector space \(V\)

i.e. \(F(\alpha)\) is a vector space for all \(\alpha \in A\).

Let \(b\) and \(c\) be any two scalars.

\[b(F, A)^{-} + c(F, A)^{-} = (bF, A)^{-} + (cF, A)^{-}\]

\(\subseteq (bF + cF, A)^{-}\)

\(= (F, A)^{-}\), since \((bF + cF, A) = (F, A)\), by the definition of soft vector space.

Thus \((F, A)^{-}\) is a soft vector space in \(V\). ■
**Proposition 4.8.** Let \((L, \tau)\) be a topological vector space and \(A\) be any parameter set. Then the soft enriched topology \(\tau^*\) derived from \(\tau\), is a vector soft topology on \(L\), where for each \(\alpha \in A, \tau_\alpha = \tau\).

*Proof.* Since vector addition and scalar multiplication are both continuous in a topological vector space, proof follows directly from the definitions of vector soft topology.

**Proposition 4.9.** If \((L, \tau)\) be a topological vector space and \(M\) is a subspace of \(L\), then \(M\) is the closure of \(M\) in \((L, \tau)\). Then \((M, \tau^*|_M, A)\) is a soft subspace topology of \((L, \tau^*, A)\).

*Proof.* Since \((L, \tau)\) is a topological vector space and \(M\) is a subspace of \(L\), by the property of topological vector space we have \(M + M \subseteq M\) and \(kM \subseteq M\), \(\forall k \in K\). Thus \(M\) is again a subspace of \(L\). Then by the definition of soft subspace topology and the proposition 5.2, \((M, \tau^*|_M, A)\) is a soft subspace topology of \((L, \tau^*, A)\).

5. Soft \(pq\)–functions in a VSTS

**Definition 5.1.** A soft zero element \(E^0_\alpha\) is the soft element given by \(E(\alpha) = \{0\}\) and \(E(\beta) = \emptyset, \forall \beta (\neq \alpha) \in A\).

Result:
\[E^0_\alpha + E^x_\alpha = E^x_\alpha, \forall x \in X.\]

**Proposition 5.2.** Let \((X, \tau, A)\) be a VSTS and \(\tau^*\) be the enriched topology derived from \(\tau\). Let \((M, A) \in N_{\tau^*}(E^0_\alpha)\). Then \((M, A) + E^x_\alpha \in N_{\tau^*}(E^x_\alpha)\).

*Proof.* Since \((M, A) \in N_{\tau^*}(E^0_\alpha)\), there exists \((H, A) \in \tau^*\) such that \(E^0_\alpha \in (H, A) \subseteq (M, A)\). Then \(\{0\} \subseteq H(\alpha)\). So \(\{x\} \subseteq \{x\} + H(\alpha)\). Hence \(E^x_\alpha \in (H, A) + E^x_\alpha\). Since \((H, A)\) is soft open, \((H, A) + E^x_\alpha\) is soft open since \(H(\beta)\) is open for each \(\beta \in A\) and \(H(\alpha) + x\) is open by continuity of addition. Also \((H, A) \subseteq (M, A) \Rightarrow (H, A) + E^x_\alpha \subseteq (M, A) + E^x_\alpha\). Thus \((M, A) + E^x_\alpha \in N_{\tau^*}(E^x_\alpha)\).

**Theorem 5.3.** Let \((V_1, \tau^*_1, A_1)\) and \((V_2, \tau^*_2, A_2)\) be two enriched vector soft topological spaces. Let \(q : V_1 \to V_2\) and \(p : A_1 \to A_2\) be two mappings in which \(q\) is linear and \(T_{pq} : (V_1, \tau^*_1, A_1) \to (V_2, \tau^*_2, A_2)\). Then \(T_{pq}(E^0_\alpha) = E^0_{p(\alpha)}\).

*Proof.* Since \(q\) is a linear map \(q(0) = 0\).
By definition of \(pq\)–soft mapping,
\[ T_{pq}(E^0_\alpha)(y) = \begin{cases} \bigcup_{x \in p^{-1}(y)} q(E(x)) & \text{if } p^{-1}(y) \neq \phi \\ \phi & \text{otherwise} \end{cases} \]

\[ = \begin{cases} q(\{0\}) & \text{if } p(\alpha) = y \\ \phi & \text{otherwise} \end{cases} \]

\[ = \begin{cases} \{0\} & \text{if } p(\alpha) = y \\ \phi & \text{otherwise} \end{cases} \]

Thus \( T_{pq}(E^0_\alpha) = E^0_\alpha \).

**Corollary 5.4.** \( T_{pq} \) is soft \( pq \)-continuous at \( E^0_\alpha \) if for each neighbourhood \((M, A)\) of \( T_{pq}(E^0_\alpha) = E^0_\alpha \), \( \exists \) a neighbourhood \((L, A)\) of \( E^0_\alpha \) such that \( T_{pq}(L, A) \subseteq (M, A) \).

**Theorem 5.5.** Let \((V_1, \tau_1, A_1)\) and \((V_2, \tau_2, A_2)\) be two VSTS. Let \( q : V_1 \to V_2 \) and \( p : A_1 \to A_2 \) be two mappings in which \( q \) is linear and \( T_{pq} : (V_1, \tau^*_1, A_1) \to (V_2, \tau^*_2, A_2) \). Then \( T_{pq} \) is linear in the sense that \( T_{pq}[\alpha (L, A) + \beta (M, A)] = \alpha T_{pq}(L, A) + \beta T_{pq}(M, A) \).

**Proof.** Since \( p \) is one-one

\[ T_{pq}(L)(y) = \begin{cases} q(L(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{when } p^{-1}(y) = \phi \end{cases} \]

So

\[ T_{pq}[\alpha (L, A) + \beta (M, A)](y) = T_{pq}[\alpha (L, A) + \beta (M, A)](y) \]

\[ = \begin{cases} q(\alpha L(x) + \beta M(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{when } p^{-1}(y) = \phi \end{cases} \]

\[ = \begin{cases} \alpha q(L(x)) + \beta q(M(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{otherwise} \end{cases} \]

, since \( q \) is linear.

Now

\[ \alpha T_{pq}(L)(y) + \beta T_{pq}(M)(y) \]

\[ = \alpha \begin{cases} q(L(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{when } p^{-1} = \phi \end{cases} + \beta \begin{cases} q(M(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{when } p^{-1} = \phi \end{cases} \]

\[ = \begin{cases} \alpha q(L(x)) + \beta q(M(x)) & \text{when } x = p^{-1}(y) \\ \phi & \text{otherwise} \end{cases} \]
Thus $T_{pq}[\alpha(L, A) + \beta(M, A)] = \alpha T_{pq}(L, A) + \beta T_{pq}(M, A)$. ■

**Theorem 5.6.** Let $(V_1, \tau_1^*, A_1)$ and $(V_2, \tau_2^*, A_2)$ be two enriched vector soft topological spaces. Let $T_{pq} : (V_1, \tau_1^*, A_1) \to (V_2, \tau_2^*, A_2)$ be linear. Then $T_{pq}$ is soft $pq$-continuous at $E^x_{\alpha}$ if and only if for each $(M, A) \in N_{\tau_2^*}(E^0_{p(\alpha)})$, there exists $(L, A) \in N_{\tau_1^*}(E^0_{\alpha})$ such that $T_{pq}[(L, A) + E^x_{\alpha}] \subseteq (M, A) + T_{pq}(E^x_{\alpha})$.

**Proof.** Addition is a homeomorphism in a VSTS. Hence by Proposition 6.1 $(F, A)$ is a soft neighbourhood of a soft point $E^x_{\alpha}$ if and only if $-E^x_{\alpha} + (F, A)$ is a soft neighbourhood of $E^0_{\alpha}$. Thus any neighbourhood of $E^x_{\alpha}$ is obtained from a neighbourhood of $E^0_{\alpha}$ and vice versa. Hence $(L, A) + E^x_{\alpha}$ is a soft neighbourhood of the soft point $E^x_{\alpha}$ for any $(L, A) \in N_{\tau_1^*}(E^0_{\alpha})$. Now $T_{pq}[(L, A) + E^x_{\alpha}] = T_{pq}(L, A) + T_{pq}(E^x_{\alpha})$, by the linearity of $T_{pq}$. This shows that any neighbourhood of $T_{pq}(E^0_{\alpha})$ can be obtained from the image of the neighbourhood $(L, A)$ of $E^0_{\alpha}$. Also if $(L, A)$ is a neighbourhood of $E^0_{\alpha}$, $T_{pq}(L, A)$ is a neighbourhood of $T_{pq}(E^0_{\alpha}) = E^p_{\alpha}$. Let $(M, A) \in N_{\tau_2^*}(E^0_{p(\alpha)})$. Then $(M, A) + T_{pq}(E^x_{\alpha}) \in N_{\tau_2^*}(E^0_{p(\alpha)})$. $T_{pq}$ is soft $pq$-continuous at $E^x_{\alpha}$ if and only if there exists a neighbourhood of $E^x_{\alpha}$ say $(D, A)$ such that $T_{pq}(D, A) \subseteq (M, A) + T_{pq}(E^x_{\alpha})$.

And corresponding to $(D, A)$ we may find $(L, A) \in N_{\tau_1^*}(E^0_{\alpha})$ such that $(L, A) + E^x_{\alpha} = (D, A)$.

Thus $T_{pq}[(L, A) + E^x_{\alpha}] \subseteq (M, A) + T_{pq}(E^x_{\alpha})$. ■

6. **Soft $\beta$ kernel of a soft $pq$-linear map**

**Definition 6.1.** [14] Let $(X, \tau, A)$ be a soft topological space. If for the soft elements $E^x_{\alpha}, E^y_{\beta}$ with $E^x_{\alpha} \neq E^y_{\beta}$, there exists,

1. $(F, A) \in \tau$ such that $E^x_{\alpha} \in (F, A)$ and $E^y_{\beta} \not\in (F, A)$ or $E^x_{\alpha} \not\in (F, A)$ and $E^y_{\beta} \in (F, A)$, then $(X, \tau, A)$ is called a soft $T_0$-space.

2. $(F, A), (G, A) \in \tau$ such that $E^x_{\alpha} \in (F, A)$ and $E^y_{\beta} \not\in (F, A)$ and $E^x_{\alpha} \not\in (G, A)$ and $E^y_{\beta} \in (G, A)$, then $(X, \tau, A)$ is called a soft $T_1$-space.

3. $(F, A), (G, A) \in \tau$ such that $E^x_{\alpha} \in (F, A), E^y_{\beta} \in (G, A)$ and $(F, A) \cap (G, A) = \phi_A$, then $(X, \tau, A)$ is called a soft $T_2$-space.

**Proposition 6.2.** [14] A soft topological space $(X, \tau, A)$ is soft $T_1$ space if and only if all soft elements $E^x_{\alpha}$ is soft closed.
Definition 6.3. Let $T_{pq} : (V_1, \tau_1, A_1) \rightarrow (V_2, \tau_2, A_2)$ be a soft linear map. Then the soft-$\beta$ kernel of $T_{pq}$ denoted by $K_\beta(T_{pq})$ is the pre-image of the soft zero-element $E_\beta^0$ for some $\beta \in A_2$.

Note:

$K_\beta(T_{pq}) = \{(F, A_1) \in SS(V_1, A_1) | T_{pq}(F, A_1) = E_\beta^0\}$

Since $T_{pq}$ is soft linear,

$T_{pq}(F)(y) = \begin{cases} q(F(x)) & \text{if } x = p^{-1}(y) \\ \phi & \text{if } p^{-1}(y) = \phi \end{cases}$

and

$T_{pq}(F, A_1) = E_\beta^0 \Rightarrow K_\beta(T_{pq}) = \{(F, A_1) \in SS(V_1, A_1) | F(p^{-1}(\beta)) \subseteq \ker q \text{ and } F(\alpha) = \phi \forall \alpha (\neq p^{-1}(\beta)) \in A_1\}.$

Proposition 6.4. Let $T_{pq} : (V_1, \tau_1, A_1) \rightarrow (V_2, \tau_2, A_2)$ be a soft linear map between the VSTS $(V_1, \tau_1, A_1)$ and $(V_2, \tau_2, A_2)$ and $K_\beta(T_{pq})$ be the soft $\beta-$ kernel of the mapping. Then $K_\beta(T_{pq})$ is a vector space under addition of soft sets and scalar multiplication of a soft set.

Proof. Since $\ker q$ is a subspace of $V_1$, if $(F, A_1) \in K_\beta(T_{pq})$, $F(p^{-1}(\beta)) + G(p^{-1}(\beta)) \subseteq \ker q$ and $F(\alpha) + G(\alpha) = \phi \forall \alpha \neq p^{-1}(\beta)$. Thus $(F, A_1) + (G, A_1) \in K_\beta(T_{pq})$. Similarly $\lambda(F, A_1) \in K_\beta(T_{pq})$, $\forall (F, A_1) \in K_\beta(T_{pq})$.

$E_{p^{-1}(\beta)}^0$ acts as the zero vector for addition and for any $(F, A_1) \in K_\beta(T_{pq})$, $-(F, A_1) \in K_\beta(T_{pq})$, which is the additive inverse of $(F, A_1)$.

Remark 6.5. The soft union of all soft sets in $K_\beta(T_{pq})$ is the soft set $(K_\beta, A_1)$ given by

$K_\beta(x) = \begin{cases} \ker q \text{ if } x = p^{-1}(\beta) \\ \phi \text{ otherwise} \end{cases}$

Remark 6.6. If $q$ is one-one, $T_{pq}$ is one-one and then

$K_\beta(T_{pq}) = T_{pq}^{-1}(E_\beta^0) = E_{p^{-1}(\beta)}^0$.

Proposition 6.7. If $(V_2, \tau_2, A_2)$ is a soft Hausdorff space and $T_{pq}$ is soft-continuous, then each soft set in $K_\beta(T_{pq})$ is soft closed and if $A_1$ is finite $(K_\beta, A_1)$ is soft closed.

Proof. If $(V_2, \tau_2, A_2)$ is a soft Hausdorff space, $E_\beta^0$ is soft closed and if $T_{pq}$ is soft-continuous, then the inverse image of the closed set $E_\beta^0$ is soft closed set. That is each soft set in $K_\beta(T_{pq})$ is soft closed.

Then if $A_1$ is finite $(K_\beta, A_1)$ is soft closed being finite union of closed sets.

Definition 6.8. Let $(V, \tau, A)$ be a VSTS. Let $W$ be a subspace of $V$. Then $V/W$ is the quotient space and $Q : V \rightarrow V/W$ given by $Q(v) = v + W$ is the quotient map. The soft quotient topology, $\tau_Q$ on $V/W$ is defined such that a soft set, $(E, A)$ in $V/W$ is soft open if and only if the inverse of $(E, A)$ under the quotient map is soft open. $(V/W, \tau_Q, A)$ is called the vector soft topological quotient space.
Proposition 6.9. A soft $pq$—function $T_{pq}$ on $V/W$ is continuous (open) if and only if the composition $T_{pq} \circ Q$ is soft continuous (open).

Proof. By the definition of the soft quotient topology the map $Q : (V, \tau, A) \rightarrow (V/W, \tau_Q, A)$ is soft open and soft continuous. Now consider the map $T_{pq} : (V/W, \tau_Q, A) \rightarrow (X, \upsilon, B)$. If $T_{pq}$ is soft continuous (open), then clearly the composition $T_{pq} \circ Q$ is soft continuous (open). Assume that the composition $T_{pq} \circ Q$ is soft continuous (open). Let $(Y, B)$ be soft open in $(X, \upsilon, B)$. $T_{pq}^{-1}(Y, B)$ is soft open if and only if $Q^{-1}[T_{pq}^{-1}(Y, B)]$ is soft open, by the definition of $\tau_Q$. And $Q^{-1}[T_{pq}^{-1}(Y, B)] = [T_{pq} \circ Q]^{-1}(Y, B)$. But since the composition is continuous $[T_{pq} \circ Q]^{-1}(Y, B)$ is soft open and hence $T_{pq}^{-1}(Y, B)$ is soft open, showing that $T_{pq}$ is soft $pq$-continuous. Let $(F, A)$ be soft open in $(V/W, \tau_Q, A)$. Since $Q$ is soft continuous $Q^{-1}(F, A) = (G, A)$ is soft open in $(V, \tau, A)$. Since the composition $T_{pq} \circ Q$ is soft open $[T_{pq} \circ Q](G, A) = T_{pq}(Q(G, A)) = T_{pq}(F, A)$ is soft open, showing that $T_{pq}$ is soft $pq$-open. ■

7. Conclusion

The study of soft sets and soft topology has wide applications in classical and non-classical logic. The notion of soft mappings have been applied to medical diagnosis in medical expert systems [5]. We hope that our study connecting vector spaces, soft topology and soft mappings can be applied to many problems in several fields of uncertainty.

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References