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On unified integral associated with the generalized Mittag-Leffler function

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Abstract

The main object of the present article is to provide an interesting double integral involving generalized Mittag-Leffler function defined by Khan and Ahmed [3], which is expressed in terms of generalized (Wright) hypergeometric function. A, further extension of our main result and their associated special cases are also considered.

AMS subject classification: 33C45, 33C60, 33E12.
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1. Introduction

The well known Mittag-Leffler function $E_\alpha(z)$ [5] (which is the generalization of exponential function), occurs as the solution of fractional order differential and integral equation is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function~~; $\alpha \geq 0$.~~

A generalization of $E_\alpha(z)$ was introduced by Wiman [13] as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$, which is also known as Mittag-Leffler function or Wiman's function.

Afterwards, Prabhakar [6] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form (see also Killbas et al. [4]):

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

In (2007), Shukla and Prajapati [9] introdued and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.4}$$

where

$$\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0, 1) \bigcup \mathbb{N}$$

and $(\gamma)_{qn} = \dfrac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$, denotes the generalized Pochhammer symbol.

A new generalized Mittag-Leffler function was defined by Salim [11] as:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n}, \tag{1.5}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0.$$

Further, Salim and Faraj [10] introduced the following extension of Mittag-Leffler function:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{1.6}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}, \min\{\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0\} > 0; \ p, q > 0$$

and $q < \Re(\alpha) + p$.

Very recently, Khan and Ahmed [3] introduced a further generalization of the Mittag-Leffler function defined as:

$$E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n + \beta)(\nu)_{\sigma n}(\delta)_{pn}} z^n, \tag{1.7}$$

where

$$\alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C},$$

$$\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0;$$

$p, q > 0$ and $q \leq \Re(\alpha) + p$.

Equation (1.7) is a generalization of equation (1.1)-(1.6).

- On setting $\mu = \nu$, $\rho = \sigma$, (1.7) reduces to the Mittag-Leffler function defined by (1.6).
- On setting $\mu = \nu$, $\rho = \sigma$, and $p = q = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.5).
- On setting $\mu = \nu$, $\rho = \sigma$, and $\delta = p = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.4), which further for $q = 1$, gives the known generalization of Mittag-Leffler function given by (1.3).
- On setting $\mu = \nu$, $\rho = \sigma$, and $\delta = p = q = 1$, (1.7) reduces to (1.2), which further for $\beta = 1$, reduces to the Mittag-Leffler function defined by (1.1).

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [2] and Wright ([14], [15], [16]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [12, p.21]; see also [8]):

$$ {}_p\Psi_q\left[\begin{matrix} (\alpha_1\ ,\ A_1), & \ldots.., & (\alpha_p\ ,\ A_p); \\ & & \\ (\beta_1\ ,\ B_1), & \ldots.., & (\beta_q\ ,\ B_q); \end{matrix}\ z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\ \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q}\ \Gamma(\beta_j + B_j k)}\ \frac{z^k}{k!}\ , \tag{1.8} $$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$ \text{(i) } 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \text{ and } 0 < |z| < \infty;\ z \neq 0. \tag{1.9} $$

$$ \text{(ii) } 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \text{ and } 0 < |z| < {A_1}^{-A_1} \ldots {A_p}^{-A_p} {B_1}^{B_1} \ldots {B_q}^{B_q}. \tag{1.10} $$

A special case of (1.8) is

$$ {}_p\Psi_q\left[\begin{matrix} (\alpha_1\ ,\ 1), & \ldots.., & (\alpha_p\ ,\ 1); \\ & & \\ (\beta_1\ ,\ 1), & \ldots.., & (\beta_q\ ,\ 1); \end{matrix}\ z\right] = \frac{\prod_{j=1}^{p}\ \Gamma(\alpha_j)}{\prod_{j=1}^{q}\ \Gamma(\beta_j)}\ {}_pF_q\left[\begin{matrix} \alpha_1, & \ldots.., & \alpha_p\ ; \\ & & \\ \beta_1, & \ldots.., & \beta_q\ ; \end{matrix}\ z\right], \tag{1.11} $$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [7])

$$ {}_pF_q\left[\begin{matrix} \alpha_1, & \ldots.., & \alpha_p\ ; \\ & & \\ \beta_1, & \ldots.., & \beta_q\ ; \end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\ \frac{z^n}{n!} $$

$$ = {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z), \tag{1.12} $$

where $(\lambda)_n$ is the Pochhammer's symbol [7].

Furthermore, we also recall here the following interesting and useful result due to Edward [1, p.445]:

$$\int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} \, dx \, dy = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.13}$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. On unified integral associated with the generalized Mittag-Leffler function

Theorem 2.1. If

$$\alpha, \beta, \gamma, \delta, \eta, \nu, \rho, \sigma, \lambda, \mu, p, q \in \mathbb{C},$$

$$a \neq 0, \Re(\delta) > 0, \Re(\nu) > 0, \Re(\eta) > 0, \Re(\gamma) > 0,$$

$p, q > 0$, and $q < \Re(\alpha) + p$, then

$$\int_0^1 \int_0^1 y^{\lambda}(1-x)^{\lambda-1}(1-y)^{\mu-1}(1-xy)^{1-\lambda-\mu} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dxdy$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)} {}_5\Psi_4 \left[\begin{array}{c} (\lambda, 1), (\mu, 1), (\eta, \rho), (\gamma, q), (1, 1); \\ (\beta, \alpha), (\nu, \sigma), (\delta, p), (\lambda+\mu, 2); \end{array} a \right], \tag{2.1}$$

where ${}_p\Psi_q$ is defined by (1.8).

Proof. In order to establish our main result (2.1), we denote the left-hand side of (2.1) by I and then using (1.7), we get:

$$I = \int_0^1 \int_0^1 y^{\lambda}(1-x)^{\lambda-1}(1-y)^{\mu-1}(1-xy)^{1-\lambda-\mu} \times \sum_{n=0}^{\infty} \frac{(\eta)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)\Gamma(\nu)_{\sigma n}(\delta)_{pn}} \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]^n. \tag{2.2}$$

Now changing the order of integration and summation (which is guaranteed under the given conditions), and then by applying the result (1.13), we get:

$$I = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\lambda+n)\Gamma(\mu+n)\Gamma(1+n)}{\Gamma(\beta+\alpha n)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\lambda+\mu+2n)} \frac{a^n}{n!}. \tag{2.3}$$

Finally, summing up the above series with the help of (1.8), we easily arrive at the right-hand side of (2.1). This completes the proof of our main result. ∎

Next, we consider other variation of our main result in which the left-hand side of (2.1) is expressed in terms of generalized hypergeometric function ${}_pF_q$.

Variation of (2.1): Let the conditions of our main result be satisfied, then the following integral formula holds true:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu)}$$

$$\times {}_{\rho+q+3}F_{\alpha+\sigma+p+2}\left[\begin{matrix} \Delta(\rho;\ \eta), & \Delta(q;\gamma), & \lambda, & \mu, & 1; \\ \Delta(\alpha;\beta), & \Delta(\sigma;\nu), & \Delta(p;\delta), & \Delta(2;\lambda+\mu); & \end{matrix}\ \frac{a\rho^{\rho}q^{q}}{4\alpha^{\alpha}p^{p}\sigma^{\sigma}}\right], \tag{2.4}$$

where $\Delta(m;\ l)$ abbreviates the array of m parameters

$$\frac{l}{m},\ \frac{l+1}{m},\ \cdots,\ \frac{l+m-1}{m},\ m \geq 1.$$

Proof. In order to prove the result (2.4), using the results

$$\Gamma(\alpha+n) = \Gamma(\alpha)(\alpha)_n$$

and

$$(l)_{kn} = k^{kn}\left(\frac{l}{k}\right)_n \left(\frac{l+1}{k}\right)_n \cdots \left(\frac{l+k-1}{k}\right)_n,$$

(Gauss multiplication theorem) in (2.3) and summing up the given series with the help of (1.12), we easily arrive at our required result (2.4). ■

3. Special Cases

(i) On setting $\eta = \nu$, $\rho = \sigma$ in (2.1) and then by using (1.6), we get the following interesting integral:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E^{\gamma,\delta,q}_{\alpha,\beta,p}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_4\Psi_3\left[\begin{matrix} (\gamma,\ q), & (\lambda,\ 1), & (\mu,\ 1), & (1,\ 1); \\ (\beta\ ,\ \alpha), & (\delta,\ p), & (\lambda+\mu, 2) & ; \end{matrix}\ a\right], \tag{3.1}$$

where $\Re(\delta) > 0, \Re(\gamma) > 0$.

(ii) Further, on setting $\eta = \nu$, $\rho = \sigma$ in (2.4) and then by using (1.6), we find:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta,p}^{\gamma,\delta,q}\left[\frac{ay(1-x)(1-y)}{1-xy^2}\right] dx\,dx$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu)}\,{}_{q+3}F_{\alpha+p+2}\left[\begin{matrix} \Delta(q;\gamma), & \lambda, & \mu, & 1 & ; \\ \Delta(\alpha;\beta), & \Delta(p;\delta), & \Delta(2;\lambda+\mu), & & ; \end{matrix}\ \frac{aq^q}{4\alpha^{\alpha}p^p}\right], \tag{3.2}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0, \Re(\beta) > 0$.

(iii) On setting $\eta = \nu$, $\rho = \sigma$ and $p = q = 1$ in (2.1) and then by using (1.5), we obtain:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta}^{\gamma,\delta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_4\Psi_3\left[\begin{matrix} (\gamma,\ 1), & (\lambda,\ 1), & (\mu,\ 1), & (1,1); \\ (\beta,\alpha), & (\delta,\ 1), & (\lambda+\mu,2) & ; \end{matrix}\ a\right], \tag{3.3}$$

where $\Re(\delta) > 0, \Re(\gamma) > 0$.

(iv) On setting $\eta = \nu$, $\rho = \sigma$ and $p = q = 1$ in (2.4) and then by using (1.5), we acquire:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta}^{\gamma,\delta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu)}\,{}_4F_{\alpha+3}\left[\begin{matrix} \gamma, & \lambda, & \mu, & 1; \\ \Delta(\alpha;\beta), & \delta, & \Delta(2;\lambda+\mu) & ; \end{matrix}\ \frac{a}{4\alpha^{\alpha}}\right], \tag{3.4}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0, \Re(\beta) > 0$.

(v) On setting $\eta = \nu$, $\rho = \sigma$, and $p = \delta = 1$ in (2.1) and then by using (1.4), we obtain:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta}^{\gamma,q}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{1}{\Gamma(\gamma)}\,{}_3\Psi_2\left[\begin{matrix} (\gamma,q), & (\lambda,\ 1), & (\mu,\ 1) & ; \\ (\beta,\ \alpha), & (\lambda+\mu,\ 2); & & \end{matrix}\ a\right], \tag{3.5}$$

where $\Re(\gamma) > 0$.

(vi) On setting $\eta = \nu$, $\rho = \sigma$, and $p = \delta = 1$ in (2.4) and then by using (1.4), we attain:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^{\gamma,q} \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\beta)\,\Gamma(\lambda+\mu)} {}_{q+2}F_{\alpha+2} \left[\begin{matrix} \Delta(q;\gamma), & \lambda, & \mu\ ; \\ \Delta(\alpha;\ \beta), & \Delta(2;\ \lambda+\mu); & \end{matrix} \ \frac{aq^q}{4\alpha^{\alpha}}\right], \quad (3.6)$$

where $\Re(\mu) > 0, \Re(\lambda) > 0, \Re(\beta) > 0$.

(vii) On setting $\eta = \nu$, $\rho = \sigma$, and $p = \delta = q = 1$ in (2.1) and then by using (1.3), we get:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^{\gamma} \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{1}{\Gamma(\gamma)}\, {}_3\Psi_2 \left[\begin{matrix} (\gamma,1), & (\lambda,1), & (\mu,1); \\ (\beta,\alpha), & (\lambda+\mu,2) & ; \end{matrix} \ a\right], \quad (3.7)$$

where $\Re(\gamma) > 0$.

(viii) On setting $\eta = \nu$, $\rho = \sigma$, and $p = \delta = q = 1$ in (2.4) and then by using (1.3), we find:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}^{\gamma} \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu)}\, {}_3F_{\alpha+2} \left[\begin{matrix} \gamma, & \lambda, & \mu; \\ \Delta(\alpha;\beta), & \Delta(2;\lambda+\mu) & ; \end{matrix} \ \frac{a}{4\alpha^{\alpha}}\right], \quad (3.8)$$

where $\Re(\mu) > 0, \Re(\lambda) > 0, \Re(\beta) > 0$.

(ix) On setting $\eta = \nu$, $\rho = \sigma$, and $p = q = \delta = \gamma = 1$ in (2.1) and then by using (1.2), we attain:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta} \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= {}_3\Psi_2 \left[\begin{matrix} (1,1), & (\lambda,1), & (\mu,1); \\ (\beta,\alpha), & (\lambda+\mu,2) & ; \end{matrix} \ a\right]. \quad (3.9)$$

(x) On setting $\eta = \nu, \rho = \sigma$, and $p = q = \delta = \gamma = 1$ in (2.4) and then by using (1.2), we acquire:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu)} {}_3F_{\alpha+2}\left[\begin{matrix} 1, & \lambda, & \mu; & \\ & & & \frac{a}{4\alpha^{\alpha}} \\ \Delta(\alpha;\beta), & \Delta(2;\lambda+\mu) & ; & \end{matrix}\right], \tag{3.10}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0, \Re(\beta) > 0$.

(xi) On setting $\eta = \nu, \rho = \sigma$, and $p = q = \delta = \gamma = \beta = 1$ in (2.1) and then by using (1.1), we find:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= {}_3\Psi_2\left[\begin{matrix} (1,1), & (\lambda,1), & (\mu,1); & \\ & & & a \\ (1,\alpha), & (\lambda+\mu,2) & ; & \end{matrix}\right]. \tag{3.11}$$

(xii) On setting $\eta = \nu, \rho = \sigma$, and $p = q = \delta = \gamma = \beta = 1$ in (2.4) and then by using (1.1), we get:

$$\int_0^1 \int_0^1 y^{\lambda} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} E_{\alpha}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} {}_3F_{\alpha+2}\left[\begin{matrix} 1, & \lambda, & \mu; & \\ & & & \frac{a}{4\alpha^{\alpha}} \\ \Delta(\alpha;1), & \Delta(2;\lambda+\mu) & ; & \end{matrix}\right], \tag{3.12}$$

where $\Re(\lambda) > 0, \Re(\mu) > 0$.

(xiii) On setting $\eta = \nu, \rho = \sigma$ and $p = q = \delta = \beta = \alpha = \gamma = 1$ in (2.1), we obtain:

$$\int_0^1 \int_0^1 y^{\lambda-1} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} \exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= {}_2\Psi_1\left[\begin{matrix} (\lambda,1), & (\mu,1) & ; \\ & & a \\ (\lambda+\mu,2); & & \end{matrix}\right]. \tag{3.13}$$

(xiv) On setting $\eta = \nu, \rho = \sigma$ and $p = q = \delta = \beta = \alpha = \gamma = 1$ in (2.4), we get:

$$\int_0^1 \int_0^1 y^{\lambda-1} (1-x)^{\lambda-1} (1-y)^{\mu-1} (1-xy)^{1-\lambda-\mu} \exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)}\,{}_2F_2\left[\begin{matrix} \lambda, & \mu\ ; \\ \Delta(2;\ \lambda+\mu); & \end{matrix}\ \frac{a}{4}\right], \tag{3.14}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0$.

References

[1] Edward, J, A treatise on the Integral calculas, Vol. **II**, Chelsea Publishing Company, New York, 1922.

[2] Fox, C, The asymptotic expansion of generalized hypergeometric functions, Proc. Lond. Math. Soc., **27**(1928), 389–400.

[3] Khan, M.A and Ahmed, S, On some properties of the generalized Mittag-Leffler function, Springer Plus 2013, 2:337 2013.

[4] Kilbas, A.A, Saigo, M, Saxena, R.K, Generalized Mittag-leffler function and generalized fractional calculus operators, Inte. Trans. Spec. funct., **15** (2004), 31–49.

[5] Mittag-Leffler, G.M, Sur la nouvelle fonction $E_\alpha(x)$, CR Acad. Sci., Paris, **137**(1903), 554–558.

[6] Prabhakar, T.R, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., **19**(1971), 7–15.

[7] Rainville, E.D, Special functions, The Macmillan Company, New York, 1960.

[8] Rathie, A.K, A new generalization of generalized hypergeometric function, Le Matematiche **LII**(II)(1997), 297–310.

[9] Shukla, A.K, Prajapati, J.C, On a generalized Mittag-Leffler function and its properties, J. Math. Anal. Appl., **336**(2007), 797–811.

[10] Salim, T.O and Faraj, A.W, A generalization of Mittag-Leffler function and Integral operator associated with the Fractional calculus, Journal of Fractional Calculus and Applications, **3**(5)(2012), 1–13.

[11] Salim, T.O, Some properties relating to generalized Mittag-Leffler function, Adv. Appl. Math. Anal., **4**(1)(2009), 21–30.

[12] Srivastava, H.M. and Karlsson, P.W, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester, U.K.), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[13] Wiman, A, Uber den fundamental satz in der theorie der funcktionen, $E_\alpha(x)$, Acta Math., **29**(1905), 191–201.

[14] Wright, E.M, The asymptotic expansion of the generalized hypergeometric functions, J. Lond. Math. Soc., **10**(1935), 286–293.

[15] Wright, E.M, The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. R. Soc. Lond., A **238**(1940), 423–451.

[16] Wright, E M, The asymptotic expansion of the generalized hypergeometric function II, Proc. Lond. Math. Soc., **46**(1940), 389–408.