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Fourier Type Integral Transform For Integrable Boehmians

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Abstract

In this paper we extend Xiao transform to class of integrable Boehmians. Further we prove that the extended Xiao transform have properties like linear, one-to-one, onto and continuous from one Boehmian space to another Boehmian space.

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1. Introduction

Boehmians was first introduced by T. K. Boehme in [1]. The construction of Boehmians is given by J. Mikusinski and P. Mikusinski. Boehmians are the new classes of generalized functions, has opened the door to new area of research in mathematics [2, 3, 5, 6]. P. Mikusinski has studied Fourier transform for integrable Boehmians in [4]. Fourier sine and cosine transforms on Boehmian spaces are studied by R. Roopkumar at.el. in [7]. Rcently, Xiao-Jun Yang introduced Fourier type new integral transform for solving the heat-diffusion problem in [9], we call it as Xiao transform. The Xiao transform for real function ψ on $\mathbb{R}$ is defined as

$$\Xi(\theta) = \wp[\psi(t)](\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-it}{\theta}} \psi(t)dt \tag{1.1}$$

provided the integral exist for $\theta \in \mathbb{R}/0$ and $\wp$ is Xiao transform operator. Let $\mathcal{L}^1(\mathbb{R})$ be space of all absolutely integrable functions on $\mathbb{R}$ with the norm of $f \in \mathcal{L}^1(\mathbb{R})$ is

$||f|| = \int_{\mathbb{R}} |f(t)|dt$. Let $f, g \in \mathcal{L}^1(\mathbb{R})$, then the convolution product $(f * g)$, that is $(f * g)(t) = \int_{-\infty}^{\infty} g(t)f(x-t)dt$ is also in $\mathcal{L}^1(\mathbb{R})$. Moreover $\|f * g\| \leq \|f\|\|g\|$ [4].

Clearly $\wp[\psi(t)](\theta)$ is a member of $\mathcal{L}^1(\mathbb{R})$. The Xiao transform of convolution $(\psi_1 * \psi_2)(t)$ for $\psi_1, \psi_2 \in \mathcal{L}^1(\mathbb{R})$ is given by [9]

$$\wp[\psi(t)](\theta) = \wp[(\psi_1 * \psi_2)(t)](\theta) = \sqrt{2\pi}\Xi_1(\theta)\Xi_2(\theta). \tag{1.2}$$

2. Xiao Transform For Integrable Boehmians

Let $\triangle$ be the set of all sequences of continuous real functions $\{\delta_n\}, n \in \mathbb{N}$ from $\mathcal{L}^1(\mathbb{R})$ satisfying the following properties :

(i) $\int_{\mathbb{R}} \delta_n(x)dx = 1$, for all $n \in \mathbb{N}$;

(ii) $\|(\delta_n)\| \leq M$ for some $M > 0$ and $\forall\, k \in \mathbb{N}$;

(iii) $supp(\delta_n) \longrightarrow 0$ as $n \longrightarrow \infty$, where $supp(\delta_n)$ is support of δ_n for $n \in \mathbb{N}$.

The member of $\triangle$ are called a *delta sequences*. For example as below.

Example 2.1. The sequence of functions

$$\delta_n(t) = \begin{cases} 2n^2\left(\dfrac{2}{n} - t\right) & for \quad \dfrac{1}{n} \leq t \leq \dfrac{2}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2. Let $\{\varphi_n\}, \{\psi_n\} \in \triangle$ then the convolution $(\varphi_n * \psi_n) \in \triangle$ for each $n \in \mathbb{N}$.

Proof. Let $\{\varphi_n\}, \{\psi_n\} \in \triangle$ for $n \in \mathbb{N}$ and K compact support of delta sequences. Using the property (i), (ii) and (iii) of $\triangle$ respectively, we get
(i)

$$\begin{aligned}\int_{\mathbb{R}} (\varphi_n * \psi_n)(t)dt &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} \varphi_n(x)\psi_n(t-x)dxdt \\ &= \int_{-\infty}^{\infty} \varphi_n(x)dx \int_{\mathbb{R}} \psi_n(t-x)dt = \int_{-\infty}^{\infty} \varphi_n(x)dx \int_{\mathbb{R}} \psi_n(y)dy = 1.\end{aligned}$$

(ii)

$$\begin{aligned}\|(\varphi_n * \psi_n)\| &= \int_{\mathbb{R}} \left| \int_{-\infty}^{\infty} \varphi_n(x)\psi_n(t-x)dx \right| dt \leq \int_{-\infty}^{\infty} |\varphi_n(x)|dx \int_{\mathbb{R}} |\psi_n(t-x)|dt \\ &\leq \int_{-\infty}^{\infty} |\varphi_n(x)|dx \int_{\mathbb{R}} |\psi_n(y)|dy \leq M_1 M_2,\end{aligned}$$

where $\|\varphi_n(x)\| \leq M_1$ and $\|\psi_n\| \leq M_2$ for some $M_1, M_2 > 0$.
(iii)

$$supp\{\varphi_n * \psi_n\} \subset [supp\{\varphi_n\} + supp\{\psi_n\}] \longrightarrow \{0\} \; as \; n \longrightarrow \infty.$$

■

Lemma 2.3. Let $f \in \mathcal{L}^1(\mathbb{R})$ and $\{\psi_n\}$ is a delta sequence then $f * \psi_n \longrightarrow f$ uniformly on every compact set of $\mathbb{R}$ as $n \longrightarrow \infty$.

Proof. Let $\epsilon > 0$ is given and $f \in \mathcal{L}^1(\mathbb{R}), \{\psi_n\} \in \triangle$ then there exist $M > 0$ such that $\displaystyle\int_{\mathbb{R}} |\psi_n(x)|dx \leq M, \forall\, n \in \mathbb{N}$. Using the uniform continuity of the mapping $x \mapsto f_x$ form $\mathbb{R}$ to $\mathcal{L}^P(\mathbb{R})$ for $1 \leq p < \infty$ (see Theorem 9.5 [8]), choose $\delta > 0$ such that $\displaystyle\int_{\mathbb{R}} |f(t-x) - f(t)|dt < \frac{\epsilon}{M}$ whenever $|x| < \delta$,

$$\begin{aligned}
\|(f * \psi_n) - f\| &= \|(\psi_n * f) - f\| \\
&= \int_{\mathbb{R}} \left| \int_{-\infty}^{\infty} \psi_n(x) f(t-x)dx - \int_{\mathbb{R}} \psi_n(x) f(t)dx \right| dt \\
&\leq \int_{-\infty}^{\infty} |\psi_n(x)| \bigg(\int_{\mathbb{R}} |f(t-x)dt - f(t)|dt \bigg) dx < \epsilon.
\end{aligned}$$

Hence $(f * \psi_n) \longrightarrow f$ uniformly on every compact set of $\mathbb{R}$ as $n \longrightarrow \infty$. ■

A pair of sequences (f_n, φ_n) is called a quotient of the sequences, denoted by f_n/φ_n, where $f_n \in \mathcal{L}^1(\mathbb{R})(n \in \mathbb{N})$, $\{\varphi_n\}$ is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m$ holds $\forall\, m, n \in \mathbb{N}$. Two quotient of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n * \psi_n = g_n * \varphi_n \;\forall\, n \in \mathbb{N}$. The equivalence class of quotient of sequence is called an *integrable Boehmian*, the space of all integrable Boehmians is denoted by $\mathcal{B}_{\mathcal{L}^1}$. Let function $f \in \mathcal{L}^1(\mathbb{R})$ can be identified with the Boehmian $[f * \delta_n/\delta_n]$, where $\{\delta_n\}$ is the delta sequence. If $F = [f_n/\varphi_n]$, then $f * \delta_n = f_n \in \mathcal{L}^1(\mathbb{R}) \;\forall\, n \in \mathbb{N}$.

Definition 2.4. A sequence of Boehmians F_n is called $\Delta-$convergent to a Boehmian F $(\Delta - \lim F_n = F)$ if there exist a delta sequence $\{\delta_n\}$ such that $(F_n - F) * \delta_n \in \mathcal{L}^1(\mathbb{R})$, for every $n \in \mathbb{N}$ and that $\|(F_n - F) * \delta_n\| \to 0$ as $n \to \infty$.

Definition 2.5. A sequence of Boehmiasn F_n is called $\delta-$convergent to a Boehmian F $(\delta - \lim F_n = F)$ if there exist a delta sequence $\{\delta_n\}$ such that $F_n * \delta_k \in \mathcal{L}^1(\mathbb{R})$ and $F * \delta_k \in \mathcal{L}^1(\mathbb{R})$ for every $n, k \in \mathbb{N}$ and that $\|(F_n - F) * \delta_k\| \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$.

Let $\{\delta_n\}$ is a delta sequence, then δ_n/δ_n represents an integrable Boehmian. Since the Boehmian $[\delta_n/\delta_n]$ corresponds to Dirac delta distribution δ, all the derivative of δ are also integrable Boehmian. Further $\{\delta_n\}$ is infinitely differentiable and bounded, then the k^{th} derivative of δ is define by $\delta^{(k)} = [\delta_n^{(k)}/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$, for each $k \in \mathbb{N}$. The k^{th} derivative of Boehmian $F \in \mathcal{B}_{\mathcal{L}^1}$ is define by $F^{(k)} = F * \delta^{(k)}$. The scalar multiplication, addition and convolution in $\mathcal{B}_{\mathcal{L}^1}$ are define as,

$$\begin{aligned} \lambda[f_n/\varphi_n] &= [\lambda f_n/\varphi_n] \\ [f_n/\varphi_n] + [g_n/\psi_n] &= [(f_n * \psi_n + g_n * \varphi_n)/\varphi_n * \psi_n] \\ [f_n/\varphi_n] * [g_n/\psi_n] &= [f_n * g_n/\varphi_n * \psi_n] \end{aligned}$$

Lemma 2.6. Let $\Delta - \lim F_n = F$ in $\mathcal{B}_{\mathcal{L}^1}$ then $\Delta - \lim F_n^{(k)} = F^{(k)}$ for $\forall\, k \in \mathbb{N}$ in $\mathcal{B}_{\mathcal{L}^1}$.

Proof. The convolution is a continuous operation on $\mathcal{B}_{\mathcal{L}^1}$, therefore for each $k \in \mathbb{N}$ we have

$$\|F_n^{(k)} - F^{(k)}\| = \|F_n * \delta^{(k)} - F * \delta^{(k)}\| = \|(F_n - F) * \delta^{(k)}\| \longrightarrow 0 \quad as\ n \longrightarrow \infty.$$

■

Lemma 2.7. Let $\Delta - \lim_{n\to\infty} F_n = F$ and $\Delta - \lim_{n\to\infty} G_n = G$ in $\mathcal{B}_{\mathcal{L}^1}$ then $\Delta - \lim_{n\to\infty} F_n * G_n = F * G$ in $\mathcal{B}_{\mathcal{L}^1}$.

Proof. Let $\Delta - \lim_{n\to\infty} F_n = F$ and $\Delta - \lim_{n\to\infty} G_n = G$ in $\mathcal{B}_{\mathcal{L}^1}$. From the continuity of convolution in $\mathcal{B}_{\mathcal{L}^1}$ we have

$$\begin{aligned} \|(F_n * G_n - F * G) * \delta_n\| &= \|F_n * G_n - F * G_n + F * G_n - F * G\|\|\delta_n\| \\ &\leq M\|F_n - F\|\|G_n\| + M\|F\|\|G_n - G\| \to 0 \quad as\ n \to \infty, \end{aligned}$$

where $\|\delta_n\| \leq M$ for $M > 0$ and δ_n is a delta sequence. ■

Let $\triangle_0 = \{\wp(\delta_n); \{\delta_n\} \in \triangle\}$ be the space of complex valued functions on $\mathbb{R}$, the operation $\cdot$ is pointwise multiplication and $C_0(\mathbb{R})$ be the space of all continuous functions vanishing at infinity on $\mathbb{R}$. Now, we construct the another space of Boehmians, denoted by $\mathcal{B} = (\mathcal{L}^1(\mathbb{R}), C_0(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R}), \cdot, \triangle_0)$. This is range of Xiao transform on $\mathcal{B}_{\mathcal{L}^1}$ and each element of $\mathcal{B}$ is denoted by $\wp(f_n)/\wp(\delta_n)$ for all $n \in \mathbb{N}$, where $\{f_n\} \in \mathcal{L}^1(\mathbb{R})$ for $n \in \mathbb{N}$. ■

Lemma 2.8. Let $f, g \in \mathcal{L}^1(\mathbb{R}); \varphi, \psi \in C_0(\mathbb{R})$ and $\lambda \in \mathbb{C}$ then

(i) $f \cdot \varphi \in \mathcal{L}^1(\mathbb{R})$

(ii) $(f + g) \cdot \varphi = f \cdot \varphi + f \cdot \varphi$

(iii) $(\lambda f) \cdot \varphi = \alpha(f \cdot \varphi)$

(iv) $f \cdot (\varphi \cdot \psi) = (f \cdot \varphi) \cdot \psi$.

Lemma 2.9. Let $f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $\varphi \in C_0(\mathbb{R})$ then $f_n \cdot \varphi \longrightarrow f \cdot \varphi$ in $\mathcal{L}^1(\mathbb{R})$.

The proof of above two lemmas are straightforward.

Lemma 2.10. Let $\{\delta_n\} \in \triangle$ for all $n \in \mathbb{N}$ then $\wp(\delta_n)$ converges uniformly on each compact set to a constant function 1.

Proof. Let $\{\delta_n\} \in \triangle$. Using property (i), (ii) of $\triangle$ we have $supp(\delta_n) \longrightarrow 0$ for $n \longrightarrow \infty$ on each compact set of $\mathbb{R}$ and there exist $M > 0$, $sup_{t\in K}|e^{-i\frac{t}{\theta}} - \sqrt{2\pi}| < M$, where K is compact support of delta sequences in $\mathbb{R}$. Hence

$$\begin{aligned}
\|\big(\wp(\delta_n) - 1\big)\| &= \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{-i\frac{t}{\theta}} - \sqrt{2\pi}]\delta_n(t)dt \right| d\theta \\
&= \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_{K} [e^{-i\frac{t}{\theta}} - \sqrt{2\pi}]\delta_n(t)dt \right| d\theta \\
&\leq \frac{M}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{K} |\delta_n(t)| dt d\theta \longrightarrow 0 \quad as\ n \longrightarrow \infty.
\end{aligned}$$

■

Lemma 2.11. Let $f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $\wp(\varphi_n) \in \triangle_0$ then $f_n \cdot \wp(\varphi_n) \rightarrow f$ in $\mathcal{L}^1(\mathbb{R})$.

Proof. Let $f_n \rightarrow f$ as $n \longrightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $\wp(\varphi_n) \in \triangle_0$. Using lemma (2.10) we get,

$$\begin{aligned}
\|f_n \cdot \wp(\varphi_n) - f\| &= \|f_n \cdot \wp(\varphi_n) - f \cdot \wp(\varphi_n) + f \cdot \wp(\varphi_n) - f\| \\
&\leq \|f_n - f\|\|\wp(\varphi_n)\| + \|f\|\|\wp(\varphi_n) - 1\| \longrightarrow 0 \quad as\ n \longrightarrow \infty.
\end{aligned}$$

■

Lemma 2.12. Let $\wp(\varphi_n), \wp(\psi_n) \in \triangle_0$ then $\wp(\varphi_n) \cdot \wp(\psi_n) \in \triangle_0$.

Proof. Let $\wp(\varphi_n), \wp(\psi_n) \in \triangle_0$, from equation (1.2) and lemma (2.2), we get
$\wp(\varphi_n) \cdot \wp(\psi_n) = \frac{1}{\sqrt{2\pi}} \wp(\varphi_n * \psi_n) \in \triangle_0$. ■

Definition 2.13. Let $\{f_n\} \in \mathcal{L}^1(\mathbb{R})$ and $\{\delta_n\} \in \triangle$, we define the Xiao transform $\wp : \mathcal{B}_{\mathcal{L}^1} \longrightarrow \mathcal{B}$ as

$$\wp[f_n/\delta_n] = \wp(f_n)/\wp(\delta_n) \qquad for \quad [f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}. \tag{2.1}$$

The Xiao transform on $\mathcal{B}_{\mathcal{L}^1}$ is well defined. Indeed if $[f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$, then $f_n * \delta_m = f_m * \delta_n$ for all $m, n \in \mathbb{N}$. Applying the Xiao transform on both sides, we get $\wp(f_n)\wp(\delta_m) = \wp(f_m)\wp(\delta_n)$ for all $m, n \in \mathbb{N}$ and hence $\wp(f_n)/\wp(\delta_n) \in \mathcal{B}$. Further if

$[f_n/\psi_n] = [g_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$ then we have $f_n * \delta_n = g_n * \psi_n$ for all $n \in \mathbb{N}$. Again applying the Xiao transform on both sides, we get $\wp(f_n)\wp(\delta_n) = \wp(g_n)\wp(\psi_n)$ for all $n \in \mathbb{N}$. i.e. $\wp(f_n)/\wp(\psi_n) = \wp(g_n)/\wp(\delta_n)$ in $\mathcal{B}$.

Lemma 2.14. Let $[f_n/\varphi_n] \in \mathcal{B}_{\mathcal{L}^1}$ then the Xiao transform of the sequence

$$\wp[f_n](\theta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\frac{t}{\theta}} f_n(t)dt \tag{2.2}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof. Let $\{\delta_n\}$ is delta sequence, then $\wp(\delta_n)$ converges uniformly on each compact set to a constant function 1. Hence, for each compact set M of $\mathbb{R}$, $\wp(\delta_m) > 0$ on M, for almost all $m \in \mathbb{N}$ Moreover,

$$\begin{aligned}\wp(f_n) = \wp(f_n)\frac{\wp(\delta_m)}{\wp(\delta_m)} &= \frac{1}{\sqrt{2\pi}}\frac{\wp(f_n * \delta_m)}{\wp(\delta_m)}\\ &= \frac{1}{\sqrt{2\pi}}\frac{\wp(f_m * \delta_n)}{\wp(\delta_m)} = \frac{\wp(f_m)}{\wp(\delta_m)}\wp(\delta_n), \qquad on \quad M,\end{aligned} \tag{2.3}$$

as $n \to \infty$ we get $\wp(f_n) \to \dfrac{\wp(f_m)}{\wp(\delta_m)}$, on each compact subset M for $m \in \mathbb{N}$. ■

Definition 2.15. In view of the above lemma, we define the Xiao transform of an integrable Boehmian $F = [f_n/\delta_n]$ in the space of continuous functions on $\mathbb{R}$ as

$$\wp(F)(\mu) = \lim_{n\to\infty} \wp(f_n). \tag{2.4}$$

Now, we show that the above definition is well define. Let two Boehmians $[f_n/\varphi_n]$ and $[g_n/\psi_n]$ are the same but representation is different then $f_n * \psi_n = g_n * \varphi_n$ for all $n \in \mathbb{N}$ taking Xiao transform on both side we have

$$\wp(f_n)(\mu)\wp(\psi_n)(\mu) = \wp(g_n)(\mu)\wp(\varphi_n)(\mu)$$

which implies that

$$\lim_{n\to\infty} \wp(f_n) = \lim_{n\to\infty} \wp(g_n)$$

Hence

$$\wp[f_n/\varphi_n](\mu) = \wp[g_n/\psi_n](\mu).$$

Theorem 2.16. The Xiao transform $\wp : \mathcal{B}_{\mathcal{L}^1} \longrightarrow \mathcal{B}$ is consistent with $\wp : \mathcal{L}^1(\mathbb{R}) \longrightarrow \mathcal{L}^1(\mathbb{R})$.

Proof. Let $f \in \mathcal{L}^1(\mathbb{R})$. The Boehmian representing f in $\mathcal{B}_{\mathcal{L}^1}$ is $[(f * \delta_n)/\delta_n]$ where $\{\delta_n\} \in \triangle$. Then for each $n \in \mathbb{N}$, $\wp((f * \delta_n)/\delta_n) = \wp(f)\wp(\delta_n)/\wp(\delta_n)$, which is a Boehmian in $\mathcal{B}$, which represents $\wp(f)$. ■

Theorem 2.17. The Xiao transform $\wp : \mathcal{B}_{\mathcal{L}^1} \longrightarrow \mathcal{B}$ is a bijection.

Proof. Let $F = [f_n/\varphi_n], G = [g_m/\psi_m] \in \mathcal{B}_{\mathcal{L}^1}$ such that $\wp(F) = \wp(G)$. From this we get $\wp(f_n)\wp(\psi_m) = \wp(g_m)\wp(\varphi_n)$ for all $m, n \in \mathbb{N}$ and hence $\wp(f_n * \psi_m) = \wp(g_m * \varphi_n)$ for all $m, n \in \mathbb{N}$. Since Xiao transform is one-to-one in $\mathcal{L}^1(\mathbb{R})$, we get $(f_n * \psi_m) = (g_m * \varphi_n)$ for all $m, n \in \mathbb{N}$. This implies $F = G$.

Let $G = [g_m/\psi_m] \in \mathcal{B}_{\mathcal{L}^1}$. Since $\wp : \mathcal{L}^1(\mathbb{R}) \longrightarrow \mathcal{L}^1(\mathbb{R})$ is onto. Choose $f_n \in \mathcal{L}^1(\mathbb{R})$ such that $g_n = \wp(f_n)$. Now using the relation $g_n \cdot \wp(\psi_m) = g_m \cdot \wp(\psi_n)$ for all $m, n \in \mathbb{N}$. we obtain $\wp(f_n) \cdot \wp(\psi_m) = \wp(f_m) \cdot \wp(\psi_n) \Rightarrow \wp(f_n * \psi_m) = \wp(f_m * \psi_n)$ Since Xiao transform is one-to-one in $\mathcal{L}^1(\mathbb{R})$, we get $(f_n * \psi_m) = (f_m * \psi_n)$. Now if we take $F = [f_n/\psi_n]$ then $F \in \mathcal{B}_{\mathcal{L}^1}$ and $\wp(F) = G$. Thus the theorem is hold. ∎

Theorem 2.18. Let $F, G \in \mathcal{B}_{\mathcal{L}^1}$ then

(a) $\wp(\lambda F + G) = \lambda\wp(F) + \wp(G)$, for any complex λ,

(b) $\wp(F * G) = \wp(F)\wp(G)$,

(c) $\wp(F(kt)) = \dfrac{1}{a}\wp(F)(a\theta)$

(d) $\wp(e^{iat}F)(\theta) = \wp(F)(\dfrac{\theta}{1 - a\theta})$

(e) $\wp(F(t + \tau)) = e^{\frac{i\tau}{\theta}}\wp(F(x))(\theta)$

(f) $\wp(F^{(n)}) = (\dfrac{i}{\theta})^n\wp(F)(\theta)$,

(g) If $\delta - \lim F_n = F$, then $\wp(F_n) \to \wp(F)$ uniformly on each compact set.

Proof. Properties (a)-(f) can be easily prove from the corresponding properties Xiao transform in $\mathcal{L}^1(\mathbb{R})$. Now we prove (g) Let $\{\delta_m\} \in \triangle$ such that $F_n * \delta_m, F * \delta_m \in \mathcal{L}^1(\mathbb{R})$ for all $n, m \in \mathbb{N}$ and $\|(F_n - F) * \delta_m\| \to 0$ as $n \to \infty$ for each $m \in \mathbb{N}$. Let M be a compact set in $\mathbb{R}$ then $\wp(\delta_m) > 0$ on M for almost all $m \in \mathbb{N}$. Since $\wp(\delta_m)$ is a continuous function and $\wp(F_n) * \wp(\delta_m) - \wp(F) * \wp(\delta_m) = ((\wp(F_n) - \wp(F)) * \wp(\delta_m))$, implies $\|(\wp(F_n) - \wp(F)) * \wp(\delta_m)\| \to 0$ as $n \to \infty$, thus $\wp(F_n) \to \wp(F)$ uniformly on each M. ∎

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