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Fixed Point Theorems with Implicit Relations in Fuzzy Metric Space

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Abstract

The aim of this paper is to prove fixed point theorems in a complete fuzzy metric space employing implicit relations besides furnishing illustrative examples. As an application to our main result, we develop a fixed point theorem for six finite families of mappings which can be exerted to deduce common fixed point theorems for any number of mappings.

Keywords: Fuzzy metric space, Compatible mappings, Weak compatible mappings, Semicompatible mappings, Fixed point, Reciprocal continuity, Implicit relation.

MSC(2010): 47H10, 54H25.

1. INTRODUCTION

The notion of fuzzy sets that was introduced by Zadeh[1] in the year 1965 with a view to represent the ambiguity in daily life exclusively led to the evolution of fuzzy mathematics. The use of fuzzy sets has been evolved as a viable method for executing many real - life problems that contain multiple objectives. In the last few years, there has been a colossal development in fuzzy mathematics due to its richness of applications. Fuzzy set theory has its applications in mathematical programming, image processing, coding theory, etc.

In 1975, Kramosil and Michalek[2] introduced the concept of fuzzy metric spaces which was later modified by George and Veeramani[3] with the help of continuous t - norms. In recent years, many authors proved fixed point theorems in fuzzy metric spaces([4] - [8]). Grabiec[9] gave the fuzzy version of Banach Contraction principle. Jungck[10] enlarged the concept of weakly commuting pair of maps to compatible mappings. Further, Jungck and Rhoades[11] embarked the study of weakly compatible maps in metric space. Pant[12] introduced the conception of reciprocal continuity in metric spaces. Pant and Jha[13] proved a cognate of the result given by Balasubrahmaniam[14] by obtaining connection between the continuity and reciprocal continuity for four maps in fuzzy metric spaces. B. Singh and S. Jain[15] initiated semicompatibility in fuzzy metric spaces. Recently, B. Singh and S. Jain[16] and Singh and Chauhan[17] established semicompatibility, compatibility and weak compatibility in fuzzy metric spaces.

In fixed point theory, Implicit relations are employed to embrace several contractive conditions at a stroke instead of proving a separate theorem for each contraction. B. Singh and S. Jain[15] proved a common fixed point theorem for semicompatible mappings in fuzzy metric space using implicit relation. In recent times, many fixed point and common fixed point theorems were proved in the the frame of fuzzy metric space via implicit relations([18] - [23]). In this paper, using the implicit relations of R. K. Saini, B. Singh and A. Jain[19], we obtain a common fixed point to six self maps in a fuzzy metric space. In the sequel, some related results are also deduced besides equipping examples.

2. PRELIMINARIES

Definition 2.1. [24] A binary operation $*$: $[0,1] \times [0,1] \to [0,1]$ is said to be a continuous t - norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $p * q \leq r * s$ whenever $p \leq r$ and $q \leq s$ $(p, q, r, s \in [0,1])$.

Examples of t - norm are $p * q = \min\{p, q\}$ and $p * q = pq$.

Definition 2.2. [25] Let X be any non - empty set, $*$ is a continuous t - norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying

- $M(x, y, t) > 0$
- $M(x, y, t) = 1 \Leftrightarrow x = y$
- $M(x, y, t) = M(y, x, t)$
- $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$
- $M(x, y, .) : (0, \infty) \to (0,1]$ is continuous where $x, y, z \in X$, $s, t > 0$.

Here, $M(x, y, t)$ denotes the degree of nearness between x , y with respect to t. Then, the 3 - tuple $(X, M, *)$ is called a fuzzy metric space.

$M(x, y, .)$ is monotonic in third variable $\forall\ x, y \in X$.

Definition 2.3. [9] If $\{x_n\}$ is a sequence in a fuzzy metric space such that $M(x_n, x, t) \to 1$ whenever $n \to \infty$, then $\{x_n\}$ is said to converge to $x \in X$.

Definition 2.4. [9] A sequence $\{x_n\}$ in a fuzzy metric space X is said to be a Cauchy sequence in X if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$, $p > 0$.

Definition 2.5. [9] If every cauchy sequence in a fuzzy metric space X is convergent, then X is said to be complete.

Remark 2.6. [16] Limit of a sequence in a fuzzy metric space is unique since $*$ is continuous.

Lemma 2.7. [26] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition $\lim_{n\to\infty} M(x, y, t) = 1$ $\forall$ $x, y \in X$. If $\exists$ $k \in (0, 1) \ni M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$ for all $t > 0$, n is any natural number, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.8. [26] If for any two points x, y in a fuzzy metric space X and for a positive number $k < 1$, $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Definition 2.9. [26] A pair of self maps (A, B) of a fuzzy metric space X is said to be compatible if $M(ABx_n, BAx_n, t) \to 1$ as $n \to \infty$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \in X$.

Definition 2.10. [11] Two self maps A, B of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if they commute at their coincidence points, that is, if $Az = Bz$ for some $z \in X$ then $ABz = BAz$.

Remark 2.11. [16] Compatible mappings in a fuzzy metric space are weakly compatible but converse is not true.

Definition 2.12. [16] Two self maps A, B of a fuzzy metric space $(X, M, *)$ are said to be semicompatible if $\lim_{n\to\infty} M(ABx_n, Bz, t) = 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Remark 2.13. [16] Semicompatible mappings in a fuzzy metric space are weakly compatible but the converse is not true.

Remark 2.14. [16] Let A and B be self-maps on a fuzzy metric space $(X, M, *)$. If B is continuous, then (A, B) is semicompatible if and only if (A, B) is compatible.

Remark 2.15. [16] Semicompatibility of the pair (A,B) need not imply the semicompatibility of (B, A).

Definition 2.16. [14] Two self maps A and B of a fuzzy metric space $(X, M, *)$ are said to be reciprocally continuous if $\lim_{n\to\infty} ABx_n = Az$ and $\lim_{n\to\infty} BAx_n = Bz$ whenever

$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \in X$.

If both A and B are continuous then they are obviously reciprocally continuous but the converse is not true.

Definition 2.17. [20] Two finite families of self mappings $\{A_i\}$ and $\{B_j\}$ are said to be pairwise commuting if

a). $A_iA_j = A_jA_i$; $i, j \in \{1, 2, ..., m\}$

b). $B_iB_j = B_jB_i$; $i, j \in \{1, 2, ..., n\}$

c). $A_iB_j = B_jA_i$; $i \in \{1, 2, ..., m\}$; $j \in \{1, 2, ..., n\}$

3. IMPLICIT FUNCTIONS

Let $\boldsymbol{\Phi}$ be the set of all real continuous functions φ: $[0, 1]^6 \to R$ satisfying (R_1). φ is non - increasing in fifth and sixth variable.

(R_2). If for some $k \in (0, 1)$, we have

$$(R_a)\text{: } \varphi(u(kt), v(t), v(t), u(t), u(\tfrac{t}{2}) * v(\tfrac{t}{2}), 1) \geq 1$$

(or)

$$(R_b)\text{: } \varphi(u(kt), v(t), v(t), u(t), 1, u(\tfrac{t}{2}) * v(\tfrac{t}{2})) \geq 1$$

for any fixed $t > 0$ & any non - decreasing functions u, v: $(0, \infty) \to [0,1]$, with $0 < u(t), v(t) \leq 1$ then $\exists\, q \in (0, 1)$ with $u(qt) \geq u(t) * v(t)$.

(R_3). If for some $k \in (0, 1)$, $\varphi\{u(kt), u(t), 1, 1, u(t), u(t)\} \geq 1$ (or) $\varphi\{u(kt), 1, u(t), 1, u(t), 1\} \geq 1$ (or) $\varphi\{u(kt), 1, 1, u(t), 1, u(t)\} \geq 1$ for some $t > 0$ and any non - decreasing function u: $(0, \infty) \to [0, 1]$, then $u(kt) \geq u(t)$.

Example 3.1. Let φ: $[0, 1]^6 \to R$ be defined by $\varphi\{t_1, t_2, t_3, t_4, t_5, t_6\} = \frac{t_1}{\min\{t_2,t_3,t_4,t_5,t_6\}}$ and $*$ be defined by $a * b = \min\{a, b\}$.

For any fixed $t > 0$ and any non-decreasing functions u, v: $(0, \infty) \to [0, 1]$, with $0 < u(t), v(t) \leq 1$ and $\varphi\{u(kt), v(t), v(t), u(t), u(\tfrac{t}{2}) * v(\tfrac{t}{2}), 1\} \geq 1$

We have $\frac{u(kt)}{\min\{v(t),v(t),u(t),u(\frac{t}{2}) * v(\frac{t}{2}),1\}} \geq 1 \Rightarrow \frac{u(kt)}{u(\frac{t}{2}) * v(\frac{t}{2})} \geq 1 \Rightarrow u(kt) \geq u(\tfrac{t}{2}) * v(\tfrac{t}{2})$

$$\therefore \quad u(qt) \geq u(t) * v(t) \text{ for } q = 2k \in (0, 1).$$

Also, $\varphi\{u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})\} \geq 1$

$$\frac{u(kt)}{\min\{v(t),v(t),u(t),1,u(\frac{t}{2}) * v(\frac{t}{2})\}} \geq 1 \Rightarrow \frac{u(kt)}{u(\frac{t}{2}) * v(\frac{t}{2})} \geq 1 \Rightarrow u(kt) \geq u(\frac{t}{2}) * v(\frac{t}{2})$$

$$\therefore \quad u(qt) \geq u(t) * v(t) \text{ for } q = 2k \in (0, 1).$$

Again, for $\varphi\{u(kt), u(t), 1, 1, u(t), u(t)\} \geq 1$ (or) $\varphi\{u(kt), 1, u(t), 1, u(t), 1\} \geq 1$ (or) $\varphi\{u(kt), 1, 1, u(t), 1, u(t)\} \geq 1$

we have, $\frac{u(kt)}{\min\{u(t),1,1,u(t),u(t)\}} \geq 1$ (or) $\frac{u(kt)}{\min\{1,u(t),1,u(t),1\}} \geq 1$ (or) $\frac{u(kt)}{\min\{1,1,u(t),1,u(t)\}} \geq 1$;

$$\frac{u(kt)}{u(t)} \geq 1 \text{ (or) } \frac{u(kt)}{u(t)} \geq 1 \text{ (or) } \frac{u(kt)}{u(t)} \geq 1$$

$$\therefore \quad u(kt) \geq u(t) \text{ for } k \in (0, 1). \text{ Thus, } \varphi \in \boldsymbol{\Phi}.$$

Example 3.2. Let φ: $[0, 1]^6 \to R$ be defined by $\varphi\{t_1, t_2, t_3, t_4, t_5, t_6\} = \frac{t_1}{\min\{t_2,t_3,t_4\}*\max\{t_5,t_6\}}$

and $*$ be defined by $a * b = \min\{a, b\}$.

For any fixed $t > 0$ & any non-decreasing functions u, v: $(0, \infty) \to [0, 1]$, with $0 < u(t), v(t) \leq 1$ and

For $\varphi\{u(kt), v(t), v(t), u(t), u(\frac{t}{2}) * v(\frac{t}{2}), 1\} \geq 1$

We have $\frac{u(kt)}{\min\{v(t),v(t),u(t)\}*\max\{u(\frac{t}{2}) * v(\frac{t}{2}),1\}} \geq 1 \Rightarrow \frac{u(kt)}{\{u(t) * v(t)\}*1} \geq 1 \Rightarrow u(kt) \geq u(t) * v(t)$

$$\therefore \quad u(qt) \geq u(t) * v(t) \text{ for } q = k \in (0, 1).$$

For $\varphi\{u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})\} \geq 1$

We have $\frac{u(kt)}{\min\{v(t),v(t),u(t)\}*\max\{1,u(\frac{t}{2}) * v(\frac{t}{2})\}} \geq 1 \Rightarrow \frac{u(kt)}{\{u(t) * v(t)\}*1} \geq 1 \Rightarrow u(kt) \geq u(t) * v(t)$

$$\therefore \quad u(qt) \geq u(t) * v(t) \text{ for } q = k \in (0, 1).$$

Again, for $\varphi\{u(kt), u(t), 1, 1, u(t), u(t)\} \geq 1$ (or) $\varphi\{u(kt), 1, u(t), 1, u(t), 1\} \geq 1$ (or) $\varphi\{u(kt), 1, 1, u(t), 1, u(t)\} \geq 1$

$\frac{u(kt)}{\min\{u(t),1,1\} * \max\{u(t),u(t)\}} \geq 1$ (or) $\frac{u(kt)}{\min\{1,u(t),1\} * \max\{u(t),1\}} \geq 1$ (or)

$$\frac{u(kt)}{\min\{1,1,u(t)\} * \max\{1,u(t)\}} \geq 1$$

$$\Rightarrow \qquad \frac{u(kt)}{u(t)} \geq 1 \text{ (or) } \frac{u(kt)}{u(t)} \geq 1 \text{ (or) } \frac{u(kt)}{u(t)} \geq 1$$

$\therefore$ $u(kt) \geq u(t)$ for $k \in (0, 1)$. Thus, $\varphi \in \boldsymbol{\Phi}$.

4. MAIN RESULTS

Theorem 4.1. Let A, B, P, Q, R, S be six self - mappings of a complete fuzzy metric space $(X, M, *)$ where $*$ is defined by $a * b = \min\{a, b\}$ satisfying

(a). *$AB(X) \subseteq S(X)$, $PQ(X) \subseteq R(X)$;*

(b). the pair (AB, R) is semicompatible and (PQ, S) is weak compatible;

(c). the pair (AB, R) is reciprocally continuous.

For some $\varphi \in \boldsymbol{\Phi}$, $\exists$ $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\varphi\left\{\begin{matrix} M(ABx, PQy, kt), M(Rx, Sy, t), M(ABx, Rx, t), M(PQy, Sy, t), M(Rx, PQy, t), \\ M(ABx, Sy, t) \end{matrix}\right\} \geq 1 \qquad \textbf{(4.1)}$$

Then AB, PQ, R and S have a unique common fixed point. Furthermore, if the pairs (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) are commuting maps then, A, B, P, Q, R, S have a unique common fixed point in X.

Proof: Let x_0 be any arbitrary point in X.

Since $AB(X) \subseteq S(X)$ and $PQ(X) \subseteq R(X)$ $\exists$ x_1, $x_2 \in X \ni ABx_0 = Sx_1$ and $PQx_1 = Rx_2$.

Let us construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = ABx_{2n} = Sx_{2n+1};\ y_{2n+2} = PQx_{2n+1} = Rx_{2n+2} \qquad \text{for } n = 0, 1, 2, \ldots \qquad \textbf{(4.2)}$$

Let $y_n \neq y_{n+1}$ for all $n = 0, 1, 2, \ldots$

Put $x = x_{2n}$; $y = x_{2n+1}$ in **(4.1)** we get

$$\varphi\left\{\begin{matrix} M(ABx_{2n}, PQx_{2n+1}, kt), M(Rx_{2n}, Sx_{2n+1}, t), M(ABx_{2n}, Rx_{2n}, t), \\ M(PQx_{2n+1}, Sx_{2n+1}, t), M(Rx_{2n}, PQx_{2n+1}, t), M(ABx_{2n}, Sx_{2n+1}, t) \end{matrix}\right\} \geq 1$$

$$\Rightarrow \varphi\left\{\begin{matrix} M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, t), \\ M(y_{2n}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+1}, t) \end{matrix}\right\} \geq 1$$

$$\Rightarrow \varphi\left\{\begin{matrix} M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), \\ M(y_{2n}, y_{2n+1}, \frac{t}{2}) * M(y_{2n+1}, y_{2n+2}, \frac{t}{2}), 1 \end{matrix}\right\} \geq 1$$

from (R_a) in (R_2) $\exists$ $q \in (0, 1)$ such that

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \qquad \textbf{(4.3)}$$

Since $a * b = \min\{a, b\}$, it implies

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) \quad \textbf{(4.4)}$$

Analogously, substituting $x = x_{2n+1}$; $y = x_{2n+2}$ in **(4.1)** and using (R_a) in (R_2) we get

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t) \quad \textbf{(4.5)}$$

Thus, from **(4.4)** and **(4.5)**, for any n = 1, 2, ... and t > 0 we have

$$M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \quad \textbf{(4.6)}$$

Hence, from lemma**(2.7)** we see that $\{y_n\}$ is a cauchy sequence in X. Since X is complete, there exists a point z in X such that $\{y_n\} \to z$ as $n \to \infty$. Moreover, the sequences $\{ABx_{2n}\}$, $\{PQx_{2n+1}\}$, $\{Rx_{2n}\}$, $\{Sx_{2n+1}\}$, being subsequences of $\{y_n\}$ also converges to z.

$$\{ABx_{2n}\} \to z; \{PQx_{2n+1}\} \to z; \quad \{Rx_{2n}\} \to z; \{Sx_{2n+1}\} \to z \quad \textbf{(4.7)}$$

The pair (AB, R) is reciprocally continuous

$$\Rightarrow ABRx_{2n} \to ABz \ \&\ RABx_{2n} \to Rz \quad \textbf{(4.8)}$$

The pair (AB, R) is semicompatible

$$\Rightarrow \lim_{n\to\infty} ABRx_{2n} = Rz \quad \textbf{(4.9)}$$

Following the uniqueness of the limit of a sequence in a fuzzy metric space, from **(4.8)** and **(4.9)**, we see that

$$ABz = Rz \quad \textbf{(4.10)}$$

Step I: Put x = z and $y = x_{2n+1}$ in **(4.1)**, we have

$$\varphi\begin{Bmatrix} M(ABz, PQx_{2n+1}, kt), M(Rz, Sx_{2n+1}, t), M(ABz, Rz, t), M(PQx_{2n+1}, Sx_{2n+1}, t), \\ M(Rz, PQx_{2n+1}, t), M(ABz, Sx_{2n+1}, t) \end{Bmatrix} \geq 1$$

Letting $n \to \infty$ and using **(4.7)**, **(4.10)**, we get

$$\varphi\{M(ABz,z,kt), M(Rz,z,t), M(ABz, Rz,t), M(z,z,t), M(Rz,z,t), M(ABz,z,t)\} \geq 1$$

$$\Rightarrow \varphi\{M(Rz,z,kt), M(Rz,z,t), M(Rz,Rz,t), M(z,z,t), M(Rz,z,t), M(Rz,z,t)\} \geq 1$$

$$\Rightarrow \varphi\{M(Rz,z,kt), M(Rz,z,t), 1, 1, M(Rz, z,t), M(Rz,z,t)\} \geq 1$$

Using (R_3), we obtain

$$M(Rz, z, kt) \geq M(Rz, z, t)$$

Using Lemma**(2.8)** and **(4.10)**

$$Rz = z = ABz \quad \textbf{(4.11)}$$

Step II:

$$AB(X) \subseteq S(X) \quad \Rightarrow \quad \exists\ u \in X \ni Su = ABz = Rz = z \quad \textbf{(4.12)}$$

Put $x = x_{2n}$ and y = u in **(4.1)** $\Rightarrow$

$$\varphi\left\{\begin{matrix} M(ABx_{2n}, PQu, kt), M(Rx_{2n}, Su, t), M(ABx_{2n}, Rx_{2n}, t), M(PQu, Su, t), \\ M(Rx_{2n}, PQu, t), M(ABx_{2n}, Su, t) \end{matrix}\right\} \geq 1$$

Letting n → ∞ and using **(4.7)**, we get

$$\varphi\{M(z,PQu,kt), M(z,Su,t), M(z,z,t), M(PQu,Su,t), M(z,PQu,t), M(z,Su,t)\} \geq 1$$

from **(4.12)** we have

$$\varphi\{M(PQu,z,kt),\ M(z,z,t), M(z,z,t),\ M(PQu,z,t),\ M(z,PQu,t),\ M(z,z,t)\} \geq 1$$

$$\Rightarrow \varphi\{M(PQu, z, kt), M(z, z, t), M(z, z, t), M(PQu, z, t), M(PQu, z, \tfrac{t}{2}) * M(z, z, \tfrac{t}{2}), 1\} \geq 1$$

Using (R_a) in (R_2), ∃ q ∈ (0, 1) ∋

$$M(PQu, z, qt) \geq M(PQu, z, t) * M(z, z, t)$$

$$\Rightarrow M(PQu, z, qt) \geq M(PQu, z, t)$$

Lemma**(2.8)** ⇒ PQu = z

$$\therefore PQu = Su = z. \qquad \textbf{(4.13)}$$

Since (PQ, S) is weakly compatible, we have

$$PQSu = SPQu \text{ i.e., } PQz = Sz \qquad \textbf{(4.14)}$$

Step III: Put x = z and y = z in **(4.1)**, ⇒

$$\varphi\left\{\begin{matrix} M(ABz, PQz, kt), M(Rz, Sz, t), M(ABz, Rz, t), M(PQz, Sz, t), M(Rz, PQz, t), \\ M(ABz, Sz, t) \end{matrix}\right\} \geq 1$$

$$\Rightarrow \varphi\left\{\begin{matrix} M(ABz, PQz, kt), M(ABz, PQz, t), M(ABz, ABz, t), M(PQz, PQz, t), \\ M(ABz, PQz, t), M(ABz, PQz, t) \end{matrix}\right\} \geq 1$$

$$\Rightarrow \varphi\{M(ABz,PQz,kt), M(ABz,PQz,t),\ 1, 1, M(ABz,PQz,t), M(ABz,\ PQz,\ t)\} \geq 1$$

Using (R_3) and from lemma**(2.8)** , we get

$$ABz = PQz \qquad \textbf{(4.15)}$$

Therefore, z = ABz = PQz = Rz = Sz, that is, z is a common fixed point of AB, PQ, R and S.

Step IV: Let w be another common fixed point of AB, PQ, R, and S then, w = ABw = PQw = Rw = Sw.

Substituting x = z and y = w in **(4.1)**, we can easily see that z = w, proving z as the unique common fixed point of AB, PQ, R and S.

Step V: Now let z be the unique common fixed point of both the pairs (AB, R) and (PQ, S). Using commutativity of the pair (A, B), (A, R) and (B, R), we obtain

$$Az = A(ABz) = A(BAz) = AB(Az);$$

$$Az = A(Rz) = R(Az);$$

$$Bz = B(ABz) = B\ (A(Bz)) = BA(Bz) = AB(Bz);$$

$$Bz = B(Rz) = R(Bz) \qquad \textbf{(4.16)}$$

showing that Az and Bz are common fixed points of (AB, R) , yielding thereby Az = Bz. which implies

$$Az = Bz = ABz = Rz = z \qquad \textbf{(4.17)}$$

in view of uniqueness of the common fixed point of the pair (AB, R). Similarly, using commutativity of the pairs (P, Q), (P, S), (Q, S) we can see that

$$Pz = Qz = PQz = Sz = z \qquad \textbf{(4.18)}$$

Step VI: Put x = z and y = z in **(4.1)** and using **(4.17)** and **(4.18)**, we have

$$\varphi\left\{\begin{matrix} M(ABz, PQz, kt), M(Rz, Sz, t), M(ABz, Rz, t), M(PQz, Sz, t), M(Rz, PQz, t), \\ M(ABz, Sz, t) \end{matrix}\right\} \geq 1$$

$$\varphi\{M(Az,Pz,kt),M(Az,Pz,t),M(Az,Az,t),M(Pz,Pz,t),M(Az,Pz,t),M(Az,Pz,t)\} \geq 1$$

$$\Rightarrow \varphi\{M(Az, Pz, kt), M(Az, Pz, t), 1,1, M(Az, Pz, t), M(Az, Pz, t)\} \geq 1$$

from (R_3) and lemma**(2.8)**, we have M(Az, Pz, kt) ≥ M(Az, Pz, t) ⇒ Az = Pz & similarly, we can see that Bz = Qz i.e., Az = Pz(Bz = Qz) also remains the common fixed point of the both the pairs (AB, R) and (PQ, S). Thus, z is the unique common fixed point of A, B, P, Q, R, S.

Example 4.2. Let X = [0, 1], $*$ be the continuous t-norm defined by $a * b = \min\{a, b\}$.

Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y ∈ X and t > 0. Then, Clearly (X, M, $*$) is a complete fuzzy metric space.

Define φ: $[0, 1]^6 \rightarrow R$ by $\varphi\{t_1, t_2, t_3, t_4, t_5, t_6\} = \frac{t_1}{\min\{t_2, t_3, t_4, t_5, t_6,\}}$. Here $\varphi \in \Phi$.

Let Ax = Px = 1, Bx = Qx = x, $Rx = Sx = \begin{cases} \frac{x+1}{2} & if\ 0 \leq x < 1 \\ 1 & if\ x = 1 \end{cases}$ be the self maps in X.

Let $\{x_n\}$ be a sequence in X defined by $x_n = 1 - \frac{1}{n}$. Then, $\lim_{n\to\infty} x_n = 1$

$\lim_{n\to\infty} ABx_n = 1$; $\lim_{n\to\infty} Rx_n = 1$; $\lim_{n\to\infty} ABR\, x_n = 1 = R(1)$.

Thus, the pair (AB, R) is semicompatible

$\lim_{n\to\infty} ABRx_n = 1 = AB(1)$ and $\lim_{n\to\infty} RABx_n = 1 = R(1)$.

Thus, the pair (AB, R) is Reciprocally Continuous.

PQ(1) = S(1) = 1 ⇒ PQS(1) = SPQ(1) = 1.

Thus, the pair (PQ, S) is weak compatible.

∴ A, B, P, Q, R, S satisfies all the hypothesis of theorem**(4.1)** and hence possess a unique common fixed point in X i.e., at x = 1.

Corollary 4.3. Let A, B, P, Q, R, S be six self-mappings of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} satisfying the conditions (a), (4.1) of the theorem **(4.1)** and the pairs (AB, R) and (PQ, S) are semicompatible and one of the pairs (AB, R) or (PQ, S) is reciprocally continuous. Then AB, PQ, R and S have a unique common fixed point. Furthermore, if the pairs (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) are commuting maps then, A, B, P, Q, R, S have a unique common fixed point in X.

Proof. As semicompatibility implies weak compatibility, the proof follows from Theorem **(4.1)**.

On taking R = S in theorem**(4.1)** , we have following result

Corollary 4.4. Let A, B, P, Q, R be self - maps of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} satisfying

(a). $AB(X) \cap PQ(X) \subseteq R(X)$

(b). the pair (AB, R) is semicompatible and (PQ, R) is weak compatible;

For some $\varphi \in \Phi$, there exists k ∈ (0, 1) such that for all x, y ∈ X and t > 0,

$$\varphi\begin{Bmatrix} M(ABx,PQy,kt), M(Rx,Ry,t), M(ABx,Rx,t), M(PQy,Ry,t), M(Rx,PQy,t), \\ M(ABx,Ry,t) \end{Bmatrix} \geq 1$$

Then AB, PQ, R have a unique common fixed point. Furthermore, if the pairs (A, B), (A, R), (B, R), (P, Q), (P, R), (Q, R) are commuting maps then, A, B, P, Q, R have a unique common fixed point in X.

If we take B = Q = identity map in theorem**(4.1)**, then the result follows.

Corollary 4.5. Let A, P, R, S be four self - mappings of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} satisfying

(a). $A(X) \subseteq S(X)$, $P(X) \subseteq R(X)$

(b). the pairs (A, R), (P, S) are commuting maps;

(c). the pair (A, R) is semicompatible and (P, S) is weak compatible;

(d). the pair (A, R) or (P, S) is reciprocally continuous.

For some $\varphi \in \Phi$, there exists k ∈ (0, 1) such that for all x, y ∈ X and t > 0,

$$\varphi\{M(Ax,Py,kt), M(Rx,Sy,t), M(Ax,Rx,t), M(Py,Sy,t), M(Rx,Py,t), M(Ax,Sy,t)\} \geq 1$$

Then A, P, R and S have a unique common fixed point in X.

Since semicompatibility implies weak compatibility, the above corollary holds even if (A, R) and (P, S) are semicompatible commuting pair of maps.

If A = P in the above corollary, then we get the following result.

Corollary 4.6. Let A, R, S be self maps of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} satisfying

(a). $A(X) \subseteq S(X) \cap R(X)$;

(b). the pair (A, R) is semicompatible and (A, S) is weak compatible;

(c). the pair (A, R) or (A, S) is reciprocally continuous.

For some $\varphi \in \Phi$ there exists k ∈ (0, 1) such that for all x, y ∈ X and t > 0,

$$\varphi\{M(Ax,Ay,kt), M(Rx,Sy,t), M(Ax,Rx,t), M(Ay,Sy,t), M(Rx,Ay,t), M(Ax,Sy,t)\} \geq 1$$

Then A, R and S have a unique common fixed point in X.

If we take B = Q = R = S = identity mapping in theorem**(4.1)** then the conditions (a), (b) and (c) are satisfied trivially and we get the following result.

Corollary 4.7. Let A and P be self-mappings of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} such that for some $\varphi \in \Phi$, there exists k ∈ (0, 1) such that for all x, y ∈ X and t > 0,

$$\varphi\{M(Ax, Py, kt), M(x, y, t), M(Ax, x, t), M(Py, y, t), M(x, Py, t), M(Ax, y, t)\} \geq 1$$

Then A, P have a unique common fixed point in X.

Theorem 4.8. Let A, B, P, Q, R, S be six self - mappings of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b} satisfying conditions (a), (c), (4.1) of Theorem**(4.1)** and the pair (AB, R) is compatible and (PQ, S) is weak compatible. Then AB, PQ, R and S have a unique common fixed point. Furthermore, if the pairs (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) are commuting maps then, A, B, P, Q, R, S have a unique common fixed point in X.

Proof:. As seen in the proof of theorem **(4.1)**, the sequence $\{y_n\}$ converges to z ∈ X and (4.7) is satisfied.

The pair (AB, R) is reciprocally continuous

$$\Rightarrow ABRx_{2n} \to ABz \ \& \ RABx_{2n} \to Rz$$

The pair (AB, R) are compatible

$$\lim_{n\to\infty} ABRx_{2n} = \lim_{n\to\infty} RABx_{2n}$$

Since limit of a sequence in a fuzzy metric space is unique.

$$\Rightarrow ABz = Rz \qquad \textbf{(4.19)}$$

Step I: put x = ABx_{2n} and y = x_{2n+1} in **(4.1)**, we have

$$\varphi\begin{Bmatrix} M(ABABx_{2n}, PQx_{2n+1}, kt), M(RABx_{2n}, Sx_{2n+1}, t), M(ABABx_{2n}, RABx_{2n}, t), \\ M(PQx_{2n+1}, Sx_{2n+1}, t), M(RABx_{2n}, PQx_{2n+1}, t), M(ABABx_{2n}, Sx_{2n+1}, t) \end{Bmatrix} \geq 1$$

Letting n → ∞ and using **(4.7)**, **(4.19)**, we get

$\varphi\{M(Rz, z, kt), M(Rz, z, t), M(Rz,Rz, t), M(z,z,t), M(Rz,z,t), M(Rz, z,t)\} \geq 1$

$$\Rightarrow \varphi\{M(Rz, z, kt), M(Rz,z,t), 1, 1, M(Rz, z,t), M(Rz,z,t)\} \geq 1$$

from (R_3) , lemma**(2.8)** and **(4.19)**, we have z = Rz = ABz.

Step II:

$$AB(X) \subseteq S(X) \Rightarrow \exists\, u \in X \ni Su = ABz = Rz = z$$

Put x = x_{2n} and y = u in **(4.1)**

$$\Rightarrow \varphi\begin{Bmatrix} M(ABx_{2n}, PQu, kt), M(Rx_{2n}, Su, t), M(ABx_{2n}, Rx_{2n}, t), M(PQu, Su, t), \\ M(Rx_{2n}, PQu, t), M(ABx_{2n}, Su, t) \end{Bmatrix} \geq 1$$

Letting n → ∞ and using **(4.7)**, **(4.12)** we get

$\Rightarrow \varphi\{M(PQu, z, kt), M(z, z, t), M(z, z, t), M(PQu, z, t), M(z, PQu, t), M(z, z, t)\} \geq 1$

$\Rightarrow \varphi\{M(PQu, z, kt), M(z, z, t), M(z, z, t), M(PQu, z, t), M(PQu, z, \frac{t}{2}) * M(z, z, \frac{t}{2}), 1\} \geq 1$

Using (R_a) in (R_2) we obtain

$$M(PQu, z, qt) \geq M(PQu, z, t) * M(z, z, t) \text{ for some } q \in (0,1)$$
$$\Rightarrow M(PQu, z, qt) \geq M(PQu, z, t)$$

$\Rightarrow$ PQu = z (since from lemma**(2.8)**)

$$\therefore \quad PQu = Su = z.$$

Since (PQ, S) is weakly compatible, we have PQSu = SPQu i.e., PQz = Sz

Step III: Put x = z and y = z in **(4.1)**,

$$\Rightarrow \varphi\begin{Bmatrix} M(ABz, PQz, kt), M(Rz, Sz, t), M(ABz, Rz, t), M(PQz, Sz, t), M(Rz, PQz, t), \\ M(ABz, Sz, t) \end{Bmatrix} \geq 1$$

$$\Rightarrow \varphi\begin{Bmatrix} M(ABz, PQz, kt), M(ABz, PQz, t), M(ABz, ABz, t), M(PQz, PQz, t), \\ M(ABz, PQz, t), M(ABz, PQz, t) \end{Bmatrix} \geq 1$$

$\Rightarrow \varphi\{M(ABz, PQz, kt), M(ABz, PQz, t), 1, 1, M(ABz, PQz, t), M(ABz, PQz, t)\} \geq 1$

Using (R_3) and from lemma**(2.8)**, we get

$$ABz = PQz \qquad \textbf{(4.20)}$$

∴ z = ABz = PQz = Rz = Sz, that is, z is a common fixed point of AB, PQ, R and S. The rest of the proof is same as in the proof of Theorem**(4.1).**

Corollary 4.9. Let A, B, P, Q, R, S be six self-mappings of a complete fuzzy metric space (X, M, ∗) where ∗ is defined by a ∗ b = min{a, b} satisfying the conditions (a), (4.1) of the theorem**(4.1)** and the pairs (AB, R) and (PQ, S) are compatible and one of the pairs (AB, R) or (PQ, S) is reciprocally continuous. Then AB, PQ, R and S have a unique common fixed point. Furthermore, if the pairs (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) are commuting maps then, A, B, P, Q, R, S have a unique common fixed point in X.

Proof. As compatibility implies weak compatibility, the proof follows from Theorem**(4.8)**

As an application of theorem**(4.1)**, we derive the following result for six finite families of mappings.

Theorem 4.10. Let $\{A_\alpha\}_{\alpha=1}^{m}$, $\{B_\beta\}_{\beta=1}^{n}$, $\{P_\gamma\}_{\gamma=1}^{p}$, $\{Q_\delta\}_{\delta=1}^{q}$, $\{R_\vartheta\}_{\vartheta=1}^{r}$, $\{S_\eta\}_{\eta=1}^{s}$ be six finite families of self maps of a fuzzy metric space (X, M, ∗) where ∗ is defined by a ∗ b = min{a, b} such that the mappings $A = A_1A_2 \ldots A_m$; $B = B_1B_2 \ldots B_n$; $P = P_1P_2 \ldots P_p$; $Q = Q_1Q_2 \ldots Q_q$; $R = R_1R_2 \ldots R_r$; $S = S_1S_2 \ldots S_s$ satisfying all the hypothesis of theorem**(4.1)** then, AB, PQ, R and S have a unique common fixed point. Furthermore, if (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) commute pairwise, then $\{A_\alpha\}_{\alpha=1}^{m}$, $\{B_\beta\}_{\beta=1}^{n}$, $\{P_\gamma\}_{\gamma=1}^{p}$, $\{Q_\delta\}_{\delta=1}^{q}$, $\{R_\vartheta\}_{\vartheta=1}^{r}$, $\{S_\eta\}_{\eta=1}^{s}$ have a unique common fixed point in X.

Proof: Since A, B, P, Q, R, S satisfies all the hypothesis of theorem**(4.1)**, following the proof of theorem(4.1) we can conclude that A, B, P, Q, R, S have a unique common fixed point in X say, z. Now, we need to show that z remains the common fixed point of all the component maps. Since (A, B), (A, R), (B, R) commute pairwise, we have

$$AB(A_\alpha z) = A(BA_\alpha z) = A(A_\alpha Bz) = A_\alpha(ABz) = A_\alpha z;$$

$$R(A_\alpha z) = A_\alpha(Rz) = A_\alpha z;$$

$$AB(B_\beta z) = A(BB_\beta z) = A(B_\beta Bz) = B_\beta(ABz) = B_\beta z;$$

$$R(B_\beta z) = B_\beta(Rz) = B_\beta z;$$

$$AB(R_\vartheta z) = A(BR_\vartheta z) = A(R_\vartheta Bz) = (AR_\vartheta)(Bz) = R_\vartheta(ABz) = R_\vartheta z;$$

$$R(R_\vartheta z) = R_\vartheta(Rz) = R_\vartheta z \qquad \textbf{(4.21)}$$

showing that $A_\alpha z$; $B_\beta z$; $R_\vartheta z$ are common fixed points of (AB, R), yielding thereby $A_\alpha z = B_\beta z = R_\vartheta z$ which implies

$$A_\alpha z = B_\beta z = R_\vartheta z = ABz = Rz = z \quad \textbf{(4.22)}$$

in view of uniqueness of the common fixed point of the pair (AB, R). Similarly, since (P, Q) , (P, S) , (Q, S) commute pairwise we can see that

$$P_\gamma z = Q_\delta z = S_\eta z = PQz = Sz = z \quad \textbf{(4.23)}$$

Substituting x = A_αz and y = P_γz in **(4.1)** and using **(4.22)** and **(4.23)**, we obtain A_αz = P_γz. Similarly, we can obtain B_βz = Q_δz and R_ϑz = S_ηz which shows that (for all $\alpha, \beta, \gamma, \delta$) A_αz = P_γz(B_βz = Q_δz) remains the common fixed points of (AB, R) and (PQ, S). Also, we have R_ϑz = S_ηz = z(for all ϑ, η) (since from **(4.22)** and **(4.23)**). Thus, appealing to the uniqueness of fixed points of A, B, P, Q, R, S we obtain A_αz = B_βz = P_γz = Q_δz = R_ϑz = S_ηz = z which shows that z is the unique common fixed point of A_α; B_β; P_γ; Q_δ; R_ϑ; S_η $\forall\ \alpha, \beta, \gamma, \delta, \vartheta, \eta$.

By setting the values of m, n, p, q, r, s suitably, one can deduce the results for any number of self maps in X.

Now, we conclude this note by deriving the following result of integral type.

Theorem 4.11. Let A, B, P, Q, R, S be six self - maps of a complete fuzzy metric space (X, M, $*$) where $*$ is defined by a $*$ b = min{a, b}. Assume that there exists a Lebesgue integrable function ψ: R → R and a function φ: $[0, 1]^6$ → R non - increasing in fifth and sixth variable such that

$$\int_0^{\varphi\{u(kt),v(t),v(t),u(t),u(\frac{t}{2}) * v(\frac{t}{2}),1\}} \psi(s)ds \geq 1 \quad \text{(or)}$$
$$\int_0^{\varphi\{u(kt),v(t),v(t),u(t),1,u(\frac{t}{2}) * v(\frac{t}{2})\}} \psi(s)ds \geq 1 \quad \textbf{(4.24)}$$

Then there exists q ∈ (0, 1) such that u(qt) $\geq$ u(t) $*$ v(t).

Also, if

$$\int_0^{\varphi\{u(kt),u(t),1,1,u(t),u(t)\}} \psi(s)\, ds \geq 1 \text{ (or) } \int_0^{\varphi\{u(kt),1,u(t),1,u(t),1\}} \psi(s)ds \geq 1 \text{ (or) }$$
$$\int_0^{\varphi\{u(kt),1,1,u(t),1,u(t)\}} \psi(s)ds \geq 1 \quad \textbf{(4.25)}$$

then u(kt) ≥ u(t) where u, v: (0, ∞) → [0, 1] are any non-decreasing functions with 0 < u(t), v(t) ≤ 1. Suppose, AB(X) ⊆ S(X) and PQ(X) ⊆ R(X); the pair (AB, R) is semicompatible; the pair (PQ, S) is weak compatible; the pair (AB, R) or (PQ, S) is reciprocally continuous and satisfying the inequality,

$$\int_0^{\varphi\{M(ABx,PQy,kt),M(Rx,Sy,t),M(ABx,Rx,t),M(PQy,Sy,t),M(Rx,PQy,t),M(ABx,Sy,t)\}} \psi(s)ds \geq 1$$

(4.26)

$\forall$ x, y $\in$ X, t > 0, k $\in$ (0, 1). Then A, B, P, Q, R, S have a unique common fixed point in X provided (A, B), (A, R), (B, R), (P, Q), (P, S), (Q, S) are commuting maps.

Proof: Let x_0 be any arbitrary point in X.

Since AB(X) $\subseteq$ S(X) and PQ(X) $\subseteq$ R(X) $\exists$ x_1 , x_2 $\in$ X $\ni$ $ABx_0 = Sx_1$ and $PQx_1 = Rx_2$.

Let us construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = ABx_{2n} = Sx_{2n+1};\ y_{2n+2} = PQx_{2n+1} = Rx_{2n+2} \qquad \text{for } n = 0, 1, 2, ... \tag{4.27}$$

Let $y_n \neq y_{n+1}$ for all n = 0, 1, 2,...

Put $x = x_{2n}$, $y = x_{2n+1}$ in **(4.26)** and using **(4.27)**, we get

$$\int_0^{\varphi\left\{\substack{M(y_{2n+1},y_{2n+2},kt),M(y_{2n},y_{2n+1},t),M(y_{2n+1},y_{2n},t),M(y_{2n+2},y_{2n+1},t),M(y_{2n},y_{2n+2},t),\\ M(y_{2n+1},y_{2n+1},t)}\right\}} \psi(s)ds \geq 1$$

$$\int_0^{\varphi\left\{\substack{M(y_{2n+1},y_{2n+2},kt),M(y_{2n},y_{2n+1},t),M(y_{2n+1},y_{2n},t),M(y_{2n+2},y_{2n+1},t),\\ M(y_{2n},y_{2n+1},\frac{t}{2})\,*\,M(y_{2n+1},y_{2n+2},\frac{t}{2}),1}\right\}} \psi(s)ds \geq 1$$

Using **(4.24)**, $\exists$ 0 < q < 1 $\ni$

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$$

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) \tag{4.28}$$

Analogously, taking $x = x_{2n+1}$, $y = x_{2n+2}$ in **(4.26)** and using **(4.24),** we get

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t) \tag{4.29}$$

Thus, from **(4.28)** and **(4.29)**, for any n = 1, 2, 3, ... and t > 0

$$M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \tag{4.30}$$

Hence, from lemma**(2.7)** we see that $\{y_n\}$ is a cauchy sequence in X. Since X is complete, there exists a point z in X such that $\{y_n\} \to z$ as $n \to \infty$. Moreover, the sequences $\{ABx_{2n}\}$, $\{PQx_{2n+1}\}$, $\{Rx_{2n}\}$, $\{Sx_{2n+1}\}$, being subsequences of $\{y_n\}$ also converges to z.

$$\{ABx_{2n}\} \to z;\ \{PQx_{2n+1}\} \to z;\ \{Rx_{2n}\} \to z;\ \{Sx_{2n+1}\} \to z \tag{4.31}$$

The pair (AB, R) is reciprocally continuous

$$\Rightarrow ABRx_{2n} \to ABz\ \&\ RABx_{2n} \to Rz \tag{4.32}$$

The pair (AB, R) is semicompatible

$$\lim_{n\to\infty} \mathrm{ABRx}_{2n} = \mathrm{Rz} \qquad \textbf{(4.33)}$$

Following the uniqueness of the limit of a sequence in a fuzzy metric space, from **(4.32)** and **(4.33)**, we obtain that

$$\mathrm{ABz} = \mathrm{Rz} \qquad \textbf{(4.34)}$$

Step I: put x = z and y = x_{2n+1} in **(4.26)**, we have

$$\int_0^{\varphi\left\{\substack{M(ABz,PQx_{2n+1},kt),M(Rz,Sx_{2n+1},t),M(ABz,Rz,t),M(PQx_{2n+1},Sx_{2n+1},t),\\ M(Rz,PQx_{2n+1},t),M(ABz,Sx_{2n+1},t)}\right\}} \psi(s)\mathrm{ds} \geq 1$$

Letting n → ∞ and using **(4.31)** and **(4.34)**, we get

$$\int_0^{\varphi\{M(Rz,z,kt),M(Rz,z,t),1,1,M(Rz,z,t),M(Rz,z,t)\}} \psi(s)\mathrm{ds} \geq 1$$

from **(4.25)**, we get

$$M(Rz, z, kt) \geq M(Rz, z, t)$$

$$\therefore \mathrm{Rz} = \mathrm{z} = \mathrm{ABz} \qquad \textbf{(4.35)}$$

Step II: $AB(X) \subseteq S(X) \Rightarrow \exists\, u \in X \ni Su = ABz = Rz = z$ **(4.36)**

Put x = x_{2n} and y = u in **(4.26)**, we have

$$\int_0^{\varphi\{M(ABx_{2n},PQu,kt),M(Rx_{2n},Su,t),M(ABx_{2n},Rx_{2n},t),M(PQu,Su,t),M(Rx_{2n},PQu,t),M(ABx_{2n},Su,t)\}} \psi(s)\mathrm{ds} \geq 1$$

Letting n → ∞ and using **(4.31)**, we get

$$\int_0^{\varphi\{M(z,PQu,kt),M(z,Su,t),M(z,z,t),M(PQu,Su,t),M(z,PQu,t),M(z,Su,t)\}} \psi(s)\mathrm{ds} \geq 1$$

From **(4.36)** we have

$$\int_0^{\varphi\{M(PQu,z,kt),M(z,z,t),M(z,z,t),M(PQu,z,t),M(z,PQu,t),M(z,z,t)\}} \psi(s)\mathrm{ds} \geq 1$$

$$\int_0^{\varphi\{M(PQu,z,kt),M(z,z,t),M(z,z,t),M(PQu,z,t),M(PQu,z,\frac{t}{2})*M(z,z,\frac{t}{2}),1\}} \psi(s)\mathrm{ds} \geq 1$$

From **(4.24)**, we obtain

$$M(PQu, z, qt) \geq M(PQu, z, t) * M(z, z, t) \text{ for some } q \in (0, 1)$$

$$\Rightarrow M(PQu, z, qt) \geq M(PQu, z, t)$$

$$\Rightarrow PQu = z$$

$$\therefore \quad PQu = Su = z \qquad \textbf{(4.37)}$$

Since (PQ, S) is weakly compatible, we have

$$PQSu = SPQu \quad \text{i.e.,} \quad PQz = Sz \qquad \textbf{(4.38)}$$

Step III: Put x = z and y = z in **(4.26)** and using **(4.38)**

$$\int_0^{\varphi\{M(ABz,PQz,kt),M(ABz,PQz,t),M(ABz,ABz,t),M(PQz,PQz,t),M(ABz,PQz,t),M(ABz,PQz,t)\}} \psi(s)ds \geq 1$$

$$\int_0^{\varphi\{M(ABz,PQz,kt),M(ABz,PQz,t),1,1,M(ABz,PQz,t),M(ABz,PQz,t)\}} \psi(s)ds \geq 1$$

Using **(4.25)** and from lemma**(2.8)**, we get

$$ABz = PQz \qquad \textbf{(4.39)}$$

Therefore, z = ABz = PQz = Rz = Sz, that is, z is a common fixed point of AB, PQ, R, and S.

Let w be another common fixed point of AB, PQ, R & S then, w = ABw = PQw = Rw = Sw. By substituting x = z and y = w in **(4.26)**, the uniqueness of z can be proved easily.

Step IV: Let z be the unique common fixed point of both the pairs (AB, R) and (PQ, S). Using commutativity of the pair (A, B), (A, R) and (B, R), we obtain

$$Az = A(ABz) = A(BAz) = AB(Az);$$

$$Az = A(Rz) = R(Az);$$

$$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz);$$

$$Bz = B(Iz) = I(Bz) \qquad \textbf{(4.40)}$$

showing that Az and Bz are common fixed points of (AB, R) , yielding thereby Az = Bz which implies

$$Az = Bz = ABz = Rz = z \qquad \textbf{(4.41)}$$

in the view of uniqueness of the common fixed point of the pair (AB, R).

Similarly, using commutativity of the pairs (P, Q) , (P, S) , (Q, S) we can see that

$$Pz = Qz = PQz = Sz = z \qquad \textbf{(4.42)}$$

Step V: Put x = z and y = z in **(4.26)** and using **(4.41)** and **(4.42),** we have

$$\int_0^{\varphi\{M(ABz,PQz,kt),M(Rz,Sz,t),M(ABz,Rz,t),M(PQz,Sz,t),M(Rz,PQz,t),M(ABz,Sz,t)\}} \psi(s)ds \geq 1$$

$$\Rightarrow \int_0^{\varphi\{M(Az,Pz,kt),M(Az,Pz,t),M(Az,Az,t),M(Pz,Pz,t),M(Az,Pz,t),M(Az,Pz,t)\}} \psi(s)ds \geq 1$$

$$\Rightarrow \int_0^{\varphi\{M(Az,Pz,kt),M(Az,Pz,t),1,1,M(Az,Pz,t),M(Az,Pz,t)\}} \psi(s)ds \geq 1$$

from **(4.25)** and lemma**(2.8)**, we have M(Az, Pz, kt) ≥ M(Az, Pz, t) ⇒ Az = Pz and similarly, we can see that Bz = Qz i.e., Az = Pz(Bz = Qz) also remains the common fixed point of both the pairs (AB, R) and (PQ, S). Thus, z is the unique common fixed point of A, B, P, Q, R, S.

Similar to the theorem**(4.10)** that is proved for six finite families of self maps, we can also extend the result theorem**(4.11)** of integral type to six finite families of mappings.

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