Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 8 (2017), pp. 4323-4332
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Estimation Under Multivariate Inverse Weibull Distribution

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Abstract

In this paper, a multivariate version of inverse Weibull distribution, denoted by $MVIWD$, has been constructed and its properties have been studied. Then, the maximum likelihood estimates ($MLE's$) and Bayes estimates ($BE's$), under squared error loss (SEL) function, are obtained in case of the trivariate inverse Weibull distribution ($TVIWD$) as illustrative example. Finally, a simulation study has been carried carrying out to study the goodness of the results and also to compare between the estimation methods using the mean squared errors ($MSE's$) criterion.

Keywords: Inverse Weibull distribution, Continuous multivariate distributions, Maximum likelihood estimation, Bayes estimation, Monte Carlo simulation.

AMS Subject Classification 2010: 62F10, 62F15, 62N01, 62N02

1. INTRODUCTION

The inverse Weibull distribution (*IWD*) is used to model degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid to mention just a few areas. Keller and Kamath[1] studied the shapes of the density and failure rate functions for the basic inverse model and Keller et al.[2]

applied the model for the reliability analysis of commercial vehicle engines. Erto[3] introduced further properties and identification of the model. Additional results on the IWD including work on reliability and tolerance limits, Bayes 2-sample prediction, maximum likelihood and least squares estimation are given by Calabria and Pulcini[4-6]. Ateya [7] studied the estimation problem under *IWD* based on Balakrishnan's unified hybrid censored scheme. For more details about *IWD*, some of its generalizations and related distributions with applications, see Oluyede and Yang[8].

Multivariate distributions are important both on theoretical and applied grounds. Their uses in multivariate analysis that have been applied to a variety of disciplines are numerous. Factor, cluster and discrimination analysis and multidimensional scaling are sometimes grouped as multivariate analysis. Regression analysis, variance components, experimental design and generally linear models are examples of the domains of applications. Multivariate normal distributions have probably been studied more than any other multivariate distribution. However, multivariate non-normal distributions are no less important as they may be needed in situations where a multivariate normal distribution is probably not the proper model to use. It is well-known that multivariate distributions of given marginals are not unique. Some methods of constructing bivariate and multivariate distributions are the multivariate generalization of Pearson system, multivariate linear exponential-type distributions, Sarmanov and Linnik multivariate distributions, Fréchet, Plackett and Mardia's systems and Farlie-Gumbel-Morgenstern multivariate distributions. For details on such methods of construction and other bivariate and multivariate distributions (see Kotz et al.[9]). AL-Hussaini and Ateya[10-12] suggested the compound technique to construct a general class of multivariate distributions, studied the members of this class, estimated the parameters of these members using the maximum likelihood and Bayes methods and studied the one- and two-sample prediction problem. Using the compound technique suggested by AL-Hussaini and Ateya[10-12], Ateya and Madhagi[13] constructed the multivariate truncated generalized Cauchy distribution and studied its properties and estimated its parameters.

A random variable X is said to have an *IWD* distribution with vector of parameters $\theta = (\alpha, \beta)$ if its *PDF* is given by

$$f(x;\boldsymbol{\theta}) = \alpha\,\beta\,x^{-(\alpha+1)}\exp[-\beta\,x^{-\alpha}],\;\; x > 0, (\alpha > 0, \beta > 0). \qquad (1.1)$$

The cumulative distribution function (*CDF*) of this random variable can be written as

$$F(x;\boldsymbol{\theta}) = \exp[-\beta\,x^{-\alpha}],\;\; x > 0, (\alpha > 0, \beta > 0). \qquad (1.2)$$

Copula:

A two -dimensional copula is a function C from I^2 to I, $I = [0,1]$, with the following properties:

(1) For every u and v in I, $C(u,0) = 0 = C(0,v)$, $C(u,1) = u$ and

$$C(1,v) = v,$$

(2) For every u_1, u_2, v_1 and v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_1,v_1) - C(u_2,v_1) - C(u_1,v_2) + C(u_2,v_2) \geq 0.$$

Let $F_{X,Y}(x,y)$ be a joint distribution function with marginals $F_X(x)$ and $F_Y(y)$, then there exists a copula $C(.,.)$ for all x ,y in R, such that

$$F_{X,Y}(x,y) = C\big(F_X(x), F_Y(y)\big).$$

Remarks:

From the definition of the copula we can see that:

1. If we have the joint distribution function and the marginal distribution functions we can construct the corresponding copula to be

$$c(u,v) = F_{X,Y}\big(F_X^{-1}(u), F_X^{-1}(v)\big),$$

2. If we have a copula and the marginal distribution functions we can construct the bivariate distribution function

3. If we have a multivariate copula $C(u_1,u_2,\dots,u_k)$, $u_i \in [0,1]$, $i = 1, 2, \dots, k$ and the marginal distribution functions are $F_{X_1}(x_1)$, $F_{X_2}(x_2), \dots, F_{X_n}(x_k)$ then the multivariate distribution function will be in the form

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C\left(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_k}(x_k)\right), \boldsymbol{X} = (X_1, X_2, \dots, X_k),$$

$$\boldsymbol{x} = (x_1, x_2, \dots, x_k)$$

4. If we have a multivariate distribution function $F_X(x)$ and the marginal distribution functions are $F_{X_1}(x_1)$, $F_{X_2}(x_2)$, … , $F_{X_k}(x_k)$ then the multivariate copula will be

$$C(u_1,u_2,\dots,u_k) = F_X\left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2), \dots, F_{X_k}^{-1}(u_k)\right),$$

$$u_i \in [0,1], i = 1,2,\dots,k.$$

For more details on copulas, see, Nelsen [14].

2- CONSTRUCTION OF MVIWD

In this section, a multivariate version of *IWD* with vector of parameters $\boldsymbol{\theta} = (\alpha, \beta)$ is constructed using the copula introduced by AL-Hussaini and Ateya [11] which of the form

$$C(\boldsymbol{u}) = \left[1 - k + \sum_{i=1}^{k} u_i^{-1/\gamma}\right]^{-\gamma}, 0 \le u_i \le 1, i = 1, 2, 3, \dots, k, \gamma > 0, \quad (2.1)$$

where $\boldsymbol{u} = (u_1, u_2, \dots, u_k)$.

Then, the conditional distribution functions are constructed from the following theorems.

Theorem 2.1:

Suppose that $\boldsymbol{X} = (X_1, X_2, \dots, X_k)$ is a random vector of random variables such that $X_i \sim IWD(\alpha_i, \beta_i)$, $i=1, 2, \dots, k$,with *PDF* shown in (1.2) after replacing α by α_i and β by β_i , respectively. The joint *CDF* of the random vector $\boldsymbol{X}$ is given by

$$F(x_1, \dots, x_k; \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k, \gamma) = \left[1 - k + \sum_{i=1}^{k} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right]^{-\gamma}, \quad (2.2)$$

and the corresponding *PDF* will be of the form

$$f(x_1, \dots, x_k; \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k, \gamma) =$$

$$\frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \left(\prod_{i=1}^{k} \frac{\beta_i \alpha_i}{\gamma} x_i^{-\alpha_i - 1} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right) \left[1 - k + \sum_{i=1}^{k} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right]^{-\gamma - k},$$

$$x_i > 0, (\alpha, \beta_i, \gamma > 0), i = 1, 2, \dots, k. \quad (2.3)$$

Theorem 2.2:

If $\boldsymbol{X}^{(1)} = (X_1, \dots, X_r)$ and $\boldsymbol{X}^{(2)} = (X_{r+1}, \dots, X_k)$ are subvectors of $\boldsymbol{X}$, then the conditional *PDF's* and *CDF's* of the $MVIWD$ are given in the following forms:

$$\mathrm{f}(\boldsymbol{x}^1 | \boldsymbol{x}^2) = \frac{\Gamma(\gamma^* + \mathrm{r})}{\Gamma(\gamma^*)} \left(\prod_{\mathrm{i}=1}^{\mathrm{r}} \frac{\beta_\mathrm{i}\, \alpha_\mathrm{i}}{\gamma\, \mathrm{c}_1} x_i^{-\alpha_i - 1} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right) \left[1 - \frac{\mathrm{r}}{\mathrm{c}_1} + \sum_{\mathrm{i}=1}^{\mathrm{r}} \frac{1}{\mathrm{c}_1} (e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}})\right]^{-\gamma^* - \mathrm{r}},$$

(2.4)

$$\mathrm{f}(\boldsymbol{x}^2|\boldsymbol{x}^1) = \frac{\Gamma(\gamma^{**}+\mathrm{k}-\mathrm{r})}{\Gamma(\gamma^{**})}\left(\prod_{\mathrm{i=r+1}}^{\mathrm{k}} \frac{\beta_\mathrm{i}\,\alpha_\mathrm{i}}{\gamma\,\mathrm{c}_2}\; x_i^{-\alpha_i-1}\;\; e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right)\left[1-\frac{\mathrm{k-r}}{\mathrm{c}_2}+\sum_{\mathrm{i=r+1}}^{\mathrm{k}} \frac{1}{\mathrm{c}_2}\left(e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right)\right]^{-\gamma^{**}-\mathrm{k}+\mathrm{r}}, \qquad (2.5)$$

$$\mathrm{F}(\boldsymbol{x}^1|\boldsymbol{x}^2) = \left[1-\frac{\mathrm{r}}{\mathrm{c}_1}+\sum_{\mathrm{i=1}}^{\mathrm{r}} \frac{1}{\mathrm{c}_1}\left(e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right)\right]^{-\gamma^{*}}, \qquad (2.6)$$

and

$$\mathrm{F}(\boldsymbol{x}^2|\boldsymbol{x}^1) = \left[1-\frac{\mathrm{k-r}}{\mathrm{c}_2}+\sum_{\mathrm{i=r+1}}^{\mathrm{k}} \frac{1}{\mathrm{c}_2}\left(e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}\right)\right]^{-\gamma^{**}}, \qquad (2.7)$$

where

$$\mathrm{c}_1 = 1-(\mathrm{k}-\mathrm{r})+\sum_{\mathrm{i=r+1}}^{\mathrm{k}} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}}, \quad \gamma^{*} = \gamma+\mathrm{k}-\mathrm{r},$$

$$\mathrm{c}_2 = 1-\mathrm{r}+\sum_{\mathrm{i=1}}^{\mathrm{r}} e^{\frac{\beta_i}{\gamma} x_i^{-\alpha_i}} \text{ and } \gamma^{**} = \gamma+\mathrm{r}\,.$$

3- MAXIMUM LIKELIHOOD ESTIMATION

In this section, the maximum likelihood estimate of the vector of parameters $\boldsymbol{\theta}$**,** where $\boldsymbol{\theta} = (\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma)$ has been obtained. First, the likelihood function of the vector of parameters $\boldsymbol{\theta}$, given the vector of observations $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = (x_i,y_i,z_i)$, $i = 1,2,\dots,n$, is given in the form

$$L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}|\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i,y_i,z_i;\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma) \qquad (3.1)$$

where

$$f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}|\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma) = \frac{\Gamma(\gamma+3)}{\Gamma(\gamma)}\left(\frac{\beta_1\,\beta_2\,\beta_3\,\alpha_1\;\alpha_2\alpha_3}{\gamma^3}\;\; x^{-\alpha_1-1}y^{-\alpha_2-1}\,z^{-\alpha_3-1}\;\; e^{\frac{\beta_1}{\gamma}x^{-\alpha_1}}\; e^{\frac{\beta_2}{\gamma}y^{-\alpha_2}}\; e^{\frac{\beta_3}{\gamma}z^{-\alpha_3}}\right)$$

$$\left[-2+\; e^{\frac{\beta_1}{\gamma}x^{-\alpha_1}}+e^{\frac{\beta_2}{\gamma}y^{-\alpha_2}}+e^{\frac{\beta_3}{\gamma}z^{-\alpha_3}}\right]^{-\gamma-3}, \qquad (3.2)$$

The $MLE's$ of all parameters are the simultaneous solutions of the following equations

$$\frac{\partial \ln L}{\partial \alpha_j} = 0, \ \frac{\partial \ln L}{\partial \beta_j} = 0, \ \ j = 1,2,3 \quad \text{and} \quad \frac{\partial \ln L}{\partial \gamma} = 0. \tag{3.3}$$

4- BAYES ESTIMATION

Using the bivariate prior PDF suggested by Ateya[15] for the independent sets of the parameters $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ and γ which of the forms

$$\pi_1(\alpha_1, \beta_1) \propto \alpha_1^{c_1+c_3-1} \beta_1^{c_3-1} \exp[-\alpha_1 (c_2 + \beta_1)], \quad \alpha_1 > 0, \ \beta_1 > 0, (c_1 > 0, c_2 > 0, c_3 > 0), \tag{4.1}$$

$$\pi_2(\alpha_2, \beta_2) \propto \alpha_2^{c_4+c_6-1} \beta_2^{c_6-1} \exp[-\alpha_2 (c_5 + \beta_2)], \quad \alpha_2 > 0, \ \beta_2 > 0, (c_4 > 0, c_5 > 0, c_6 > 0), \tag{4.2}$$

$$\pi_3(\alpha_3, \beta_3) \propto \alpha_3^{c_7+c_9-1} \beta_3^{c_9-1} \exp[-\alpha_3 (c_8 + \beta_3)], \quad \alpha_3 > 0, \ \beta_3 > 0, (c_7 > 0, c_8 > 0, c_9 > 0), \tag{4.3}$$

and

$$\pi_4(\gamma) \propto \gamma^{c_{10}-1} \exp[-c_{11}\gamma], \ \gamma > 0, (c_{10} > 0, c_{11} > 0). \tag{4.4}$$

where $c_i, i = 1,2,...,11$ are the prior parameters (also known as hyper parameters).

Then, the posterior PDF can be written in the form

$$\pi^*(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma \mid data) = A\, \pi_1(\alpha_1, \beta_1)\, \pi_2(\alpha_2, \beta_2) \pi_3(\alpha_3, \beta_3) \pi_4(\gamma) \times L\left(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \mid \boldsymbol{\theta}\right) \tag{4.5}$$

where $\boldsymbol{A}$ is a normalizing constant.

Using the previous posterior PDF and using the $MCMC$ technique, the $BE's$ of all parameters can be obtained using SEL function.

5. RESULTS AND DISCUSSIONS

In the following, the $MLE's$ and $BE's$ have been computed by applying the following steps:

1- For a given vector of prior parameters (c_1, c_2, c_3) the vector of population parameters α_1 and β_1 have been generated from the joint prior (4.1).

2- For a given vector of prior parameters (c_4, c_5, c_6) the vector of population parameters α_2 and β_2 have been generated from the joint prior (4.2).

3- For a given vector of prior parameters (c_7, c_8, c_9) the vector of population parameters α_3 and β_3 have been generated from the joint prior (4.3).

4- For a given vector of prior parameters (c_{10}, c_{11}) the population parameter γ has been generated from the prior (4.4).

5- Making use of the generated population parameters, samples from the $BVIWD$ with PDF (2.3) have been generated by solving the following equations simultaneously:

a) $F(x) = u_1,\ F(y|x) = u_2 \quad F(z|x,y) = u_3$

where u_1, u_2 and u_3 are random variates from $U(0,1)$ and the conditional $CDF's$ can be obtained from (2.7).

6- The $MLE's$ of all parameters have been obtained as shown in section 3 using the software *Mathematica*8 for solving the resulting nonlinear equations.

7- The $BE's$ for the same parameters under SEL function using $MCMC$ algorithm have been obtained as shown in section 4.

8- The above steps (5-7) are repeated 500 times.

9- If $\hat{\theta}_j$ is an estimate of θ, based on sample $j, j = 1,2,...,m$, then the average estimate over the m samples is given by

$$\bar{\hat{\theta}} = \frac{1}{m}\sum_{j=1}^{m}\hat{\theta}_j .$$

10- The $MSE's$ of $\hat{\theta}$ over the m samples is given by

$$MSE(\hat{\theta}) = \frac{1}{m}\sum_{j=1}^{m}(\hat{\theta}_j - \theta)^2 .$$

11- From 10 , the $MSE's$ for all parameters will be computed.

The computations are shown in Tables 5.1, 5.2 and 5.3.

Table 5.1:

MSE's of the *MLE's* based on different sample sizes n and m=500 repetitions., ($\alpha_1 = 2.74402$, $\alpha_2 = 0.521092, \alpha_3 = 1.99879,\ \beta_1 = 1.32086,\ \beta_2 = 9.23853,\ \beta_3 = 1.21851,\ \gamma = 1.62523$)

N	*MSE*($\hat{\gamma}$)	*MSE*($\hat{\alpha}_1$)	*MSE*($\hat{\alpha}_2$)	*MSE*($\hat{\alpha}_3$)	*MSE*($\hat{\beta}_1$)	*MSE*($\hat{\beta}_2$)	*MSE*($\hat{\beta}_3$)
5	42.590792	0.772836	31.701020	0.608496	17.781111	0.290913	8.343900
10	40.974925	0.291222	24.875111	0.201862	11.588740	0.183438	1.008946
15	39.283149	0.236193	23.484420	0.199309	5.331802	0.138931	2.532172
20	37.929418	0.219486	22.566115	0.014738	3.021589	0.136887	1.566204
25	31.173801	0.207941	20.502352	0.008201	2.162408	0.129336	0.973522

Table 5.2:-

MSE's of the *BE's* under *SEL* function of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and α for different sample sizes n and m= 500 repetitions. ($c_1 = 1.2, c_2 = 2.3$, $c_3 = 3.2$, $c_4 = 1.5$, $c_5 = 1.3$, $c_6 = 4.2$, $c_7 1.2$, $c_8 = 2.3$, $c_9 = 3.0, c_{10} = 2.0$, $c_{11} = 3.0$), ($\alpha_1 = 2.74402$, $\alpha_2 = 0.521092, \alpha_3 = 1.99879,\ \beta_1 = 1.32086,\ \beta_2 = 9.23853,\ \beta_3 =$

n	MSE($\hat{\gamma}$)	MSE($\hat{\alpha}_1$)	MSE($\hat{\alpha}_2$)	MSE($\hat{\alpha}_3$)	MSE($\hat{\beta}_1$)	MSE($\hat{\beta}_2$)	MSE($\hat{\beta}_3$)
5	0.918287	0.189052	12.477917	0.215158	1.139372	0.027276	0.370350
10	0.889039	0.082959	7.494687	0.061473	1.049406	0.021071	0.338808
15	0.777156	0.062949	6.323961	0.057516	1.014403	0.019643	0.319292
20	0.687702	0.049756	5.598059	0.035709	1.009853	0.017352	0.306275
25	0.577034	0.037007	4.857134	0.007846	0.904748	0.012865	0.216062

$1.21851,\ \gamma = 1.62523$)

Table 5.3:

MSE's of the *MLE's* and *BE's* under *SEL* function based on different sample sizes n and m=500 repetitions. ($c_1 = 1.2, c_2 = 2.3$, $c_3 = 3.2$, $c_4 = 1.5$, $c_5 = 1.3$, $c_6 = 4.2$, $c_7 = 1.2$, $c_8 = 2.3$, $c_9 = 3.0, c_{10} = 2.0$, $c_{11} = 3.0$), ($\alpha_1 = 2.74402$, $\alpha_2 =$

$0.521092, \alpha_3 = 1.99879,\ \beta_1 = 1.32086,\ \beta_2 = 9.23853,\ \beta_3 = 1.21851,\ \gamma = 1.62523)$

n	MSE	MSE($\hat{\gamma}$)	MSE($\hat{\alpha}_1$)	MSE($\hat{\alpha}_2$)	MSE($\hat{\alpha}_3$)	MSE($\hat{\beta}_1$)	MSE($\hat{\beta}_2$)	MSE($\hat{\beta}_3$)
5	MLE	42.590792	0.772836	31.701020	0.608496	17.781111	0.290913	8.343900
	SEL	0.918287	0.189052	12.477917	0.215158	1.139372	0.027276	0.370350
10	MLE	40.974925	0.291222	24.875111	0.201862	11.588740	0.183438	1.008946
	SEL	0.889039	0.082959	7.494687	0.061473	1.049406	0.021071	0.338808
15	MLE	39.283149	0.236193	23.484420	0.199309	5.331802	0.138931	2.532172
	SEL	0.777156	0.062949	6.323961	0.057516	1.014403	0.019643	0.319292
20	MLE	37.929418	0.219486	22.566115	0.014738	3.021589	0.136887	1.566204
	SEL	0.687702	0.049756	5.598059	0.035709	1.009853	0.017352	0.306275
25	MLE	31.173801	0.207941	20.502352	0.008201	2.162408	0.129336	0.973522
	SEL	0.577034	0.037007	4.857134	0.007846	0.904748	0.012865	0.216062

6. CONCLUSIONS

In this project, $MLE's$ and $BE's$ of the parameters of $TVIWD$ have been obtained. A simulation study is carried out to examine and compare the performance of the proposed methods for different sample sizes. From the results which are summarized in tables 5.1, 5.2 and 5.3, we observe the following.

1- The $MSE's$ of the $BE's$ based on SEL function are less than that obtained for the $MLE's$ which means that the $BE's$ are better than the $MLE's$.
2- The $MSE's$ of the $MLE's$ and $BE's$ decrease by increasing the sample size n.

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