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An Inverse 1-D Heat conduction problem (IHCP): An Observer Based Approach

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Abstract

The objective of this paper is to estimate the external heat source independent of time added along the length of the bar in backward heat conduction equation using an alternative method based on the concept of observers, which are well known in control theory. However, Tikhonov does not give a recursive (sequential) estimation for the source. Therefore, we propose an observer to estimate the hidden states of a dynamical system using only the available input and output measurements. The conduction governing equations are represented in the form of state space model. The non- iterative nature of solution procedure in this method makes the solution very fast and it only requires the solutions to the direct problems which are in general well posed and well-studied and hence by using this method source is estimated with low computational cost.

Keywords: Inverse problems, Inverse Heat conduction problem, Ill-posedness, observers, state- space presentation, Discretization

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1. INTRODUCTION

1.1 Inverse problems-

Inverse problem is a vibrant and expanding branch of Mathematics that has found numerous applications. Two problems are called inverse to each other if the formulation of each of them requires full or partial knowledge of the other. Hadamard's concept reflected the idea that any mathematical model of a physical problem violating any of these conditions- existence, uniqueness and continuous dependence is called **Ill-posed** Inverse problem. It turns out that many interesting and important inverse problems in science lead to ill-defined problems whereas the corresponding direct problems are well posed [1]. Based on unknowns, the inverse problems can be classified as follows: if the problem is required to estimate any model parameter, then it is called coefficients inverse problem or inverse media problem. If the source is the unknown then the problem is inverse source problem. The inverse problem is called retrospective if the initial conditions are unknowns, and it is called boundary problem if the boundary conditions are unknowns. These are not all the classes; there are some mixed cases e.g. the unknowns are the initial and boundary conditions; for more details in the classification of inverse problems see [6].

1.2 An inverse IHCP-

The problem of deriving an unknown source or boundary conditions from a set of measured temperatures is known as inverse heat conduction problem (IHCP). On the other hand, the more common route of determining the temperature distribution in a body subject to known boundary conditions or source is often referred as the direct problem. Inverse problems are treated as mathematically ill-posed for two reasons [5]. Firstly, they are often unstable, being extremely sensitive to small variations in input like random measurement errors. Secondly, though the existence of the solution of inverse problems can be intuitively argued from physical considerations, it can be formally proved only for a few cases.

Several approaches have been used for solving IHCP problems. The common approaches include variation methods like conjugate gradient methods [4], regularized deterministic methods like quasi-Newton method [5] and hybrid techniques of Laplace transform and finite difference method [3]. The majority of IHCP algorithms are iterative in nature, which requires large computation times. Only a few approaches [9] use direct methods for estimation of thermal boundary conditions. The objective of the present work is to develop and demonstrate a non-iterative technique for estimation of external source using observer based approach.

1.3 Observability

An observer is a dynamical system used to estimate the state or part of the state of an observable dynamical system using the available input and output measurements. One advantage of using an observer solving inverse problems is that it only requires the solution to the direct problems which are in general well-posed and well-studied.

Observability is a structural property of a system, and it means the ability to reconstruct the state vector using system outputs. In other words; it means the possibility to determine the behavior of the state using some measurements

Definition. A linear system is observable at $t_0 \epsilon$ T if it is possible to determine $\xi(t_0)$ from the output $z[t_0;t_1]$ where t_1 is finite time in T. If this condition is satisfied for all t_0 and ξ (t_0), then the system is completely observable.

1.3.1 State-Space Representation

In control engineering, a state –space representation is a mathematical model of a physical system as a set of input, output and state variables related by first- order differential equations. ″state – space refers to the space whose axes are the state variables.This representation can describe the dynamics in physical systems such as biological systems, mechanical systems, and economic systems. Linear continuous-time state-space systems can be written in the following state-space representation [2]:

$$\begin{cases} \dot{\xi} = A(t)\xi(t) + B(t)v(t), \\ z(t) = C(t)\xi(t) + D(t)v(t) \end{cases} \tag{1.3.1}$$

Where $\xi(t) \in R^n$ is the state vector, z(t) $\in R^m$ is the output vector, $v(t) \in R^r$is the input vector , A is the state matrix of dimension $n \times n$, B is the input matrix of dimension n×r, C is the output matrix with dimension m×n, and D is transmission (feed through) matrix from input to output with dimension m×r. Moreover, in the system (4.3.1) the first equation is called the state equation while the second equation is called the output equation. Also, (4.3.1) has the following solution [10]:

$$\xi(t) = \emptyset(t, t_0) + \int_{t_0}^{t} \emptyset(t, \tau)B(\tau)v(\tau)d\tau, \tag{1.3.2}$$

Where $\emptyset(t, \tau) = U(t)Ut^{-1}$ and U(t) is the solution of $\dot{U}(t) = A(t)U(t)$. If the system is invariant then $\emptyset(t, t_0)$ can be defined as:

$$\emptyset(t, t_0) = e^{A(t-t_0)}.$$

2. MATHEMATICAL MODEL

The mathematical formulation of 1 – D Heat conduction problem with Neumann Boundary conditions and initially at a temperature denoted by f(x).

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + h(x) \qquad 0 < x < L, t > 0$$

For simplicity we describe the case where $D = \frac{k}{\rho c}$ is constant and D=1. Therefore, The final mathematical model is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x) \qquad 0 < x < L \qquad (2.1)$$

$$u_x(0,t) \qquad t > 0 \qquad (2.2)$$

$$u_x(L,t) \quad = 0 \qquad t > 0 \; or \; t \in (0,t) \qquad (2.3)$$

$$u(x,0) = f(x) \qquad 0 \le x \le L \qquad (2.4)$$

where u(x,t) (temperature), f(x) (initial condition), x (spatial variable), t (time variable) are dimensionless quantities and h(x) represents the source term which is assumed for simplicity, to be independent of time. Conditions such as 2.2, 2.3 are called Neumann boundary conditions and condition 2.4 is called initial condition.

Direct problem- For the direct problem where the initial condition f(x) is specified, the problem given by equations (2.1- 2.4) is concerned with the determination of the temperature distribution u(x,t) in a one- dimensional slab as a function of time and position heated by external heat source h(x,t).

Inverse problem- The inverse problem involves determination of the source term h(x) from the measured values of the temperature at x = L.

3. ILL- POSEDNESS OF BACKWARD HEAT PROBLEM

Denote by $\{u_n(x)\}_{n=1}^{\infty} \subset L^2(a,b), \{\lambda_n\}_1^{\infty}$ the eigenfunctions and eigenvalues of the problem respectively, where $\{u_n(x)\}_{n=1}^{\infty}$ is the unitary orthogonal functions. Then the solution to (2.1) can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n \; e^{(-\lambda_n{}^2 t)} \qquad (3.1)$$

With the coefficient

$$c_n = \int_0^L f(x)u_n(x)dx \tag{3.2}$$

Where $u_n(x) = \cos(\mu_n x)$, $\mu_n = \frac{n\pi}{L}, \lambda_n = \mu_n$ (3.3)

Now consider the operator A: f(x) →u(x,t) defined by the problem and by using (3.1) and (3.2) has the expression

$$Af(x) = \int_0^L \sum_{n=1}^{\infty} u_n(x)u_n(y)\, e^{(-\lambda_n{}^2 T)} f(y)dy = u(x,T) \tag{3.4}$$

from which the equation Af(x)=u(x,T) can be converted as Fredholm Integral equation $\int_0^L K(x,y)f(y)dy = u(x,T)$ (3.5)

with the kernel function

$$K(x,y) = \sum_{n=1}^{\infty} u_n(x)u_n(y)\, e^{(-\lambda_n{}^2 T)}$$

from (3.3) putting the values of u_n and λ_n, we get

$$K(x,y) = \sum_{n=1}^{\infty} cos\mu_n(x) cos\mu_n(y)\, e^{(-\lambda_n{}^2 T)}$$

$$= \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi y}{L}\right) e^{-\left(n\pi/_L\right)^2 DT} \tag{3.6}$$

It is obvious that A is a self-adjoint compact operator from $L^2(a, b)$ to itself.

On the other hand, the coefficients c_n can also be given by

$$c_n = e^{(-\lambda_n{}^2 T)} \int_0^L u(x,T)u_n(x)dx$$

Therefore,

$$f(x) = A^{-1}u(x,T) = \sum_{n=1}^{\infty} c_n u_n(x) \tag{3.7}$$

The problem (3.7) is ill-posed in the sense that the inverse operator A^{-1} of A exists but it is not continuous. So, the determination of f(x) from this expression is unstable. That is, the small perturbation of u(x, T) can cause large change of f(x) given by (2.7). That is, the backward problem is ill-posed.

4. AN OBSERVER TO SOLVE GIVEN INVERSE HEAT CONDUCTION PROBLEM

In this section, we propose to apply an observer on a one-dimensional Heat equation to estimate the heat source. As known in control theory, a state observer is a system that provides an estimate of its internal state, given measurements of the input and output of the real system.

For this purpose, we propose to write the one dimensional heat equation (2.1-2.4) in a state space representation firstly. Then, the system is discretized in both space and time. Finally, the observer design is presented. Using this method, unknown external heat source is estimated.

4.1 A state- space representation for Heat equation

We first propose to rewrite (2.1) in an appropriate form by introducing two auxiliary variables[8]:

$$\begin{cases} \xi_1(x,t) = u(x,t) \\ \xi_2(x,t) = \dfrac{\partial u(x,t)}{\partial x}, \end{cases}$$

We denote

$$\xi(x,t) = \begin{bmatrix} \xi_1(x,t) \\ \xi_2(x,t) \end{bmatrix} \tag{4.1.1}$$

The one- dimensional heat equation with BCs and IC becomes then:

$$\begin{cases} \frac{\partial \xi(x,t)}{\partial x} = F\xi(x,t) + H(x) \\ \xi_2(0,t) = 0 \\ \xi_2(L,t) = 0 \\ \xi_1(x,0) = f(x) \end{cases} \tag{4.1.2}$$

As introduced above the observer – based approach allows solving the inverse problem by solving direct problems. So, by solving problem (4.1.2) we can find unknown heat source h(x)

Where the operator F is given by

$$F = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}, \; H = \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

The idea is to estimate the state variables ξ_1 and ξ_2 using an observer and hence get H.

4.2 Discretization

There are three known methods for discretization: finite difference method, FDM; finite element method, FEM, and finite volume method, FVM. For simplicity and validation, we propose to apply FDM to discretize system (4.1.2).

Discretized equations using implicit Euler scheme in time and the central finite difference, after application of boundary conditions become

$$\dot{\xi}_1 = \frac{1}{(\Delta x)^2}[\xi_2 - \xi_1] + \frac{h}{\Delta x} \quad (a)$$

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$$\dot{\xi}_i = \frac{1}{(\Delta x)^2}[\xi_{i+1} - 2\xi_i + \xi_{i-1}],\; i = 2, \dots \dots N-1 \quad (b)$$

$$\frac{\dot{\xi}_N}{2} = \frac{1}{(\Delta x)^2}[-\xi_N + \xi_{N-1}] \quad (c)$$

where Δx refers to the space step [7].

Expressed in matrix notations, the above equations become

$$\{\dot{\xi}\} = [A]\{\xi\} + [B]\{H\} \quad \text{(d)}$$

In the above equation, the matrices A and B and the input vector W are as follows:

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & 0 \dots \dots \dots 0 \\ 1 & -2 & 1 \dots \dots \dots 0 \\ & \dots \dots \dots \dots \dots & \\ 0 \dots \dots \dots 1 & -2 & 1 \\ 0 \dots \dots \dots 0 & 1 & -2 \end{bmatrix}; B = \frac{1}{(\Delta x)}\begin{bmatrix} 1 & 0 & 0 \dots \dots 0 \\ 0 & 0 & 0 \dots \dots 0 \\ & \dots \dots \dots . & \\ 0 & 0 & 0 \dots \dots 0 \\ 0 & 0 & 0 \dots \dots 0 \end{bmatrix}; H = \begin{pmatrix} h \\ 0 \\ 0 \\ .. \\ 0 \end{pmatrix}$$

……..(e)

4.3 Observer design

Since sensors are placed at only P of the above N points, the state vector is partitioned into vectors of size P×1 and (N-P) ×1 representing measured states and non-measured states, which need to be estimated. Thus Eq. (d) becomes

$$\begin{pmatrix} \dot{\xi}_m \\ \dot{\xi}_n \end{pmatrix} = [A]\begin{pmatrix} \xi_m \\ \xi_n \end{pmatrix} + [B]\{H\} \qquad (5.1.3)$$

For a given time series of temperatures at the sensor locations, $\dot{\xi}_m$ can be easily obtained. Thus the problem reduces to determining the unknowns ξ_n, $\dot{\xi}_n$ and H. As the problem evolves in time, using the available values of ξ_n estimated at earlier time steps, $\dot{\xi}_n$ can be estimated. Thus starting from an initial estimate of $\dot{\xi}_n$, presumably equal to zero, the following system of algebric equations need to be solved.

$$[C]\binom{\xi_n}{W} = \begin{pmatrix} \dot{\xi}_m \\ \dot{\xi}_n \end{pmatrix} + [D](\xi_m) \qquad (5.1.4)$$

The coefficient matrices, **C** and **D** are obtained from algebraic manipulation of Eq. (a-e).

6. CONCLUSION

This chapter reflects the possibilities of considering a steady state problem from a dynamical theory perspective. A method, based on state space representation of governing energy equation, is developed for estimation of heat source independent of time for heat conduction. Different from standard approaches to tackle this problem, the proposed method, being non-iterative, requires significantly less computational time than common inverse solution techniques like Conjugate Gradient Method and Tikhonov regularization. The results for the estimation of heat source are also quite accurate and more successful in capturing sudden changes in boundary conditions than other approaches unless the noise level is high.

REFERENCES

[1] B. Jagdamba, A. Khan and M. Sama(eds.), In A.H. Siddiqi, R.C. Singh and P.Manchanda, *Inverse problems of parameter identification in partial differential equations,* Mathematics in Science and Tech., World Scientific (2011) (228-258). .

[2] D.Ricerca, " Observation and Control of PDE with Disturbances", in Esame finale anno accademico 2010 – 2011.

[3] H. T. Chen and X.-Y. Wu, "Estimation of surface conditions for nonlinear inverse heat conduction problems using the hybrid inverse scheme", Numerical Heat Transfer, Part B: Fundamentals, vol. 51, pp. 159–178, 2007.

[4] J. Su and G.F. Hewitt, "Inverse heat conduction problem of estimating time-varying heat transfer coefficient", Numerical Heat Transfer, Part A:

Applications, vol. 45, pp. 777–789, 2004.

[5] L.D. Chiwiakowsky and H.F. De Campos Velho, “Different approaches for the solution of a backward heat conduction problem”, Inverse Problems in Engineering, vol. 11, pp. 471–494, 2003.

[6] S. I. Kabanikhin, "Definitions and examples of inverse and ill-posed problems," Journal of Inverse and Ill-posed Problems, vol. 16, pp. 317-357, 2008.

[7] S.M.Asiri, “An Inverse Source Problem for a One-dimensional Wave Equation: An Observer-Based Approach”, Thesis for fulfillment of M.Sc., King Abdullah University of Science and Technology, Thuwal, May, 2013.

[8] T. M. Laleg-Kirati, A. radwan and C. Zayane, “an observer-based method to solve the Cauchy Problem for laplace equation”, King Abduallah University of Science and technology (KAUST), Saudi Arabia, 2010.

[9] V. Weston, \Invariant imbedding for the wave equation in three dimensions and the applications to the direct and inverse problems," Inverse Problems, vol. 6, no. 6, p. 1075, 1990.

[10] W. L. Brogan, Modern Control Theory, 3rd ed. Upper Saddle River, New Jersey 07458: Prentice Hall, 1991.