Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 8 (2017), pp. 4265–4273

http://www.ripublication.com/gjpam.htm

Some Lacunary Difference Sequence Spaces Defined by a Modulus Function

Mushir A. Khan

Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002 (India).

Abstract

In this paper, we introduce the sequence spaces $[N_\theta, f, p]_1(\Delta^r, E)$, $[N_\theta, f, p]_0(\Delta^r, E)$, $[N_\theta, f, p]_\infty(\Delta^r, E)$ and $S_\theta(\Delta^r, E)$, where E is any Banach space.We discuss some toplogical properties and establish some inclusion relations between these spaces.

AMS subject classification: 40A05, 40C05, 46A45.
Keywords: Laucunary sequence, Difference sequence, Modulus function.

1. Introduction

By a Lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [6] as follows:

$$N_\theta = \{x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} | x_k - L |= 0 \text{ for some } L\}$$

The space N_θ is a BK-space with the norm

$$\|x\|_\theta = \sup_r \left(h_r^{-1} \sum_{k\in I_r} | x_k | \right).$$

N_θ^0 denotes the subset of those sequences in N_θ for which $L = 0$. $(N_\theta^0, \|\cdot\|_\theta)$ is also a BK-space. There is a strong connection [6] between N_θ and the space w of strongly Cesaro summable sequences, which is defined by

$$w = \{x = (x_k) : \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} | x_k - L |= 0 \text{ for some } L\}.$$

In the special case when $\theta = (2^r)$, we have $N_\theta = w$.

The idea of difference sequence spaces was introduced by kizmaz [8]. In 1981, Kizmaz [8] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = l_\infty$, c and c_0, where $\Delta x = (x_k - x_{k+1})$. Then Et and Colak [ ] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\}$$

for $X = l_\infty$, c and c_0, where $r \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = (\Delta^r x_k - \Delta^r x_{k+1})$, and so $\Delta^r x_k = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} x_{k+\nu}$.

Later on difference sequence spaces were studied by Malkowsky and Parashar [12], Et and Basarir [3], Mursaleen, Mushir A. Khan and Qamaruddin [15] and many others.

We recall that a modulus f is a funtion from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

(iii) f is increasing

(iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. A modulus may be bounded or unbounded. Ruckle [16] and Maddox [11], used a modulus f to construct some sequence spaces.

Subsequently modulus function has been discussed in [1], [13], [18] and others have also used a modulus function to construct some sequence spaces.

The following inequality will be used throughout this paper.

$$| a_k + b_k |^{p_k} \leq C\{| a_k |^{p_k} + | b_k |^{p_k}\}, \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = H$, $C = \max(1, 2^{H-1})$ [10].

2. Main Results

In this section we introduce the sequence space $[N_\theta, f, p]_1(\Delta^r, E)$, $[N_\theta, f, p]_0(\Delta^r, E)$, $[N_\theta, f, p]_\infty(\Delta^r, E)$ and $S_\theta(\Delta^r, E)$. We give some topological properties and establish some inclusion relations between these spaces.

Definition 2.1. Let E be a Banach space. We define $w(E)$ to be the vector space of all E-valued sequences that is $w(E) = \{x = (x_k) : x_k \in E\}$. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$[N_\theta, f, p]_1(\Delta^r, E) = \{x \in w(E) : \lim_r h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r x_k - L\|)]^{p_k} = 0, \text{ for some } L\},$$

$$[N_\theta, f, p]_0(\Delta^r, E) = \{x \in w(E) : \lim_r h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r x_k\|)]^{p_k} = 0\},$$

$$[N_\theta, f, p]_\infty(\Delta^r, E) = \{x \in w(E) : \sup_r h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r x_k\|)]^{p_k} < \infty\}.$$

If $x \in [N_\theta, f, p]_1(\Delta^r, E)$ then we will write $x_k \to L[N_\theta, f, p]_1(\Delta^r, E)$ and L will be called Lacunary difference limit of x with respect to the modulus f.

Throughout this paper Z will denote any one of the notation 0, 1 or ∞.

In the case $f(x) = x$, $p_k = 1$ for all $k \in \boldsymbol{N}$, we shall write $[N_\theta]_Z(\Delta^r, E)$ and $[N_\theta, f]_Z(\Delta^r, E)$ instead of $[N_\theta, f, p]_Z(\Delta^r, E)$ respectively.

Theorem 2.2. Let the sequence (p_k) be bounded. Then the sequence spaces $[N_\theta, f, p]_Z(\Delta^r, E)$ are linear spaces.

Proof. We shall prove only for $[N_\theta, f, p]_0(\Delta^r, E)$. The others can be treated similarly. Let $x, y \in [N_\theta, f, p]_0(\Delta^r, E)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers M_α and N_β such that $| \alpha | \leq M_\alpha$ and $| \beta | \leq N_\beta$. Since f is subadditive and Δ^r is linear

$$h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r(\alpha x_k + \beta y_k)\|)]^{p_k}$$

$$\begin{aligned} &\leq h_r^{-1} \sum_{k \in I_r} [f(| \alpha | \, \|\Delta^r x_k\|) + f(| \beta | \, \|\Delta^r y_k\|)]^{p_k} \\ &\leq C(M_\alpha)^H h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r x_k\|)]^{p_k} + C(N_\beta)^H h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r y_k\|)]^{p_k} \to 0 \end{aligned}$$

as $r \to \infty$. This proves that $[N_\theta, f, p]_0(\Delta^r, E)$ is a linear space. ■

Theorem 2.3. Let f be a modulus function, then $[N_\theta, f, p]_0(\Delta^r, E) \subset [N_\theta, f, p]_1(\Delta^r, E) \subset [N_\theta, f, p]_\infty(\Delta^r, E)$.

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x \in [N_\theta, f, p]_1(\Delta^r, E)$. By definition of f we have

$$h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k\|)]^{p_k} = h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k - L + L\|)]^{p_k}$$

$$\leq Ch_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k - L\|)]^{p_k} + Ch_r^{-1}\sum_{k\in I_r}[f(\|L\|)]^{p_k}.$$

There exists a positive integer K_L such that $\|L\| \leq K_L$. Hence we have

$$h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k\|)]^{p_k} = Ch_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k - L\|)]^{p_k} + Ch_r^{-1}[K_L f(1)]^H h_r.$$

Since $x \in [N_\theta, f, p]_1(\Delta^r, E)$ we have $x \in [N_\theta, f, p]_\infty(\Delta^r, E)$ and this completes the proof. ■

Theorem 2.4. $[N_\theta, f, p]_0(\Delta^r, E)$ is a paranormed (need not total paranorm) space with

$$g_\Delta(x) = \sup_r \left(h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k\|)]^{p_k}\right)^{\frac{1}{M}}$$

where $M = \max(1, \sup p_k)$.

Proof. Clearly $g_\Delta(x) = g_\Delta(-x)$. It is trivial that $\Delta^r x_k = 0$ for $x = 0$. Since $f(o) = 0$, we get $g_\Delta(x) = 0$ for $x = 0$. Since $p_k/M \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of f, for each r, we have $g_\Delta(x+y) \leq g_\Delta(x) + g_\Delta(y)$. Finally, to check the continuity of multiplication, let us take any complex number β. By definition of f we have

$$g_\Delta(\beta x) = \sup_r \left(h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r(\beta x_k)\|)]^{p_k}\right)^{\frac{1}{M}}$$

$$\leq K_\beta^{\frac{H}{M}} g_\Delta(x)$$

where K_β is a positive integer such that $|\ \beta\ | < K_\beta$. Now let $\beta \to 0$ for any fixed x with $g_\Delta(x) \neq 0$. By definition of f for $|\ \beta\ | < 1$, we have

$$h_r^{-1}\sum_{k\in I_r}[f(\|\beta\Delta^r x_k\|)]^{p_k} < \epsilon \text{ for } r > r_0(\epsilon). \tag{2}$$

Also, for $1 \leq r \leq r_0$, taking β small enough, since f is continuous we have

$$h_r^{-1}\sum_{k\in I_r}[f(\|\beta\Delta^r x_k\|)]^{p_k} < \epsilon. \tag{3}$$

(2) and (3) together imply that $g_\Delta(\beta x) \to 0$ as $\beta \to 0$. ■

Theorem 2.5. If $r \geq 1$, then the inclusion $[N_\theta, f]_Z(\Delta^{r-1}, E) \subset [N_\theta, f]_Z(\Delta^r, E)$ is strict. In general $[N_\theta, f]_Z(\Delta^i, E) \subset [N_\theta, f]_Z(\Delta^r, E)$ for all $i = 1, 2, \ldots, r-1$ and the inclusion is strict.

Proof. We give the proof for $Z = \infty$ only. Other can be proved in a similar way for $Z = 0$ and $Z = 1$. Let $x \in [N_\theta, f]_\infty(\Delta^{r-1}, E)$. Then we have

$$\sup_r h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^{r-1} x_k\|)] < \infty$$

By definition of f, we have

$$h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^r x_k\|)] \leq h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^{r-1} x_k\|)] + h_r^{-1} \sum_{k \in I_r} [f(\|\Delta^{r-1} x_{k+1}\|)] < \infty$$

Thus $[N_\theta, f]_\infty(\Delta^{r-1}, E) \subset [N_\theta, f]_\infty(\Delta^r, E)$. Proceeding in this way one will have $[N_\theta, f]_\infty(\Delta^r, E) \subset [N_\theta, f]_\infty(\Delta^r, E)$ for $i = 1, 2, \cdots, r-1$. Let $E = \mathbb{C}$, and $h_r = r$ for each $r \in \boldsymbol{N}$. Then the sequence $x = (k^r)$, for example, belongs to $[N_\theta, f]_\infty(\Delta^r, E)$, but does not belong to $[N_\theta, f]_\infty(\Delta^{r-1}, E)$ for $f(x) = x$. (If $x = (k^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^{r+1} r! \left(k + \frac{r-1}{2}\right)$ for all $k \in \boldsymbol{N}$).

The proof of the following result is a routine work. ■

Proposition 2.6. $[N_\theta, f, p]_1(\Delta^{r-1}, E) \subset [N_\theta, f, p]_0(\Delta^r, E)$.

Theorem 2.7. Let f, f_1, f_2 be modulus functions. Then we have

(i) $[N_\theta, f_1, p]_Z(\Delta^r, E) \subset [N_\theta, fof_1, p]_Z(\Delta^r, E)$,

(ii) $[N_\theta, f_1, p]_Z(\Delta^r, E) \cap [N_\theta, f_2, p]_Z(\Delta^r, E) \subset [N_\theta, f_1 + f_2, p]_Z(\Delta^r, E)$.

Proof. (i). We shall only prove (i). Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_k = f_1(\|\Delta^r x_k\|)$ and consider

$$\sum_{k \in I_r} [f(y_k)]^{p_k} = \sum_1 [f(y_k)]^{p_k} + \sum_2 [f(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 [f(y_k)]^{p_k} < h_r \epsilon^H \tag{4}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

By the definition of f we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$h_r^{-1}\sum_2 [f(y_k)]^{p_k} \;\le\; \max(1, (2f(1)\delta^{-1})^H)h_r^{-1}\sum_{k\in I_r} y_k. \tag{5}$$

from (4) and (5), we obtain $[N_\theta, f, p]_0(\Delta^r) \subset [N_\theta, fof_1, p]_0(\Delta^r)$.

The proof of (ii) follows from the following inequality

$$[(f_1 + f_2)(\|\Delta^r x_k\|)]^{p_k} \;\le\; C[f_1(\|\Delta^r x_k\|)]^{p_k} + C[f_2(\|\Delta^r x_k\|)]^{p_k}.$$

The following result is a consequence of Theorem 2.7 (i). ■

Proposition 2.8. Let f be a modulus function. Then
$[N_\theta, p]_Z(\Delta^r, E) \subset [N_\theta, f, p]_Z(\Delta^r, E)$.

3. Statistical Convergence

The idea of statistical convergence was introduced by Fast [5] and studied by various authors ([2],[7],[9],[14] and [17]).

In this section we give some inclusion relations between $S_\theta(\Delta^r, E)$ and
$[N_\theta, f, p](\Delta^r, E)$.

Definition 3.1. A sequence $x = (x_k)$ is said to be $S_\theta(\Delta^r, E)$-statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_r h_r^{-1}\left|\{k \in I_r \;:\; \|\Delta^r x_k - L\| \;\ge\; \epsilon\}\right| \;=\; 0.$$

In this case we write $S_\theta(\Delta^r, E) - \lim x = L$ or $x_k \to LS_\theta(\Delta^r, E)$. In the case $h_r = r$ and $L = 0$ we shall write $S(\Delta^r, E)$ and $S_\theta^0(\Delta^r, E)$ instead of $S_\theta(\Delta^r, E)$.

Theorem 3.2.

(i) If $x_k \to L[N_\theta]_1(\Delta^r, E)$ then $x_k \to LS_\theta(\Delta^r, E)$,

(ii) If $x \in \ell_\infty(\Delta^r, E)$ and $x_k \to LS_\theta(\Delta^r, E)$, then $x_k \to L[N_\theta]_1(\Delta^r, E)$,

(iii) $S_\theta(\Delta^r, E) \cap \ell_\infty(\Delta^r, E) = [N_\theta]_1(\Delta^r, E) \cap \ell_\infty(\Delta^r, E)$.

where $\quad \ell_\infty(\Delta^r, E) = \{x \in w(E) \;:\; \sup_k \|\Delta^r x_k\| < \infty\}$.

Proof.

(i) Let $\epsilon > 0$ and $x_k \to L[N_\theta]_1(\Delta^r, E)$. Then we have

$$\sum_{k \in I_r} \|\Delta^r x_k - L\| \ \geq \ \epsilon \left|\{k \in I_r \ : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\}\right|$$

Hence $x_k \to LS_\theta(\Delta^r, E)$.

(ii) Suppose that $x_k \to LS_\theta(\Delta^r, E)$ and $x \in \ell_\infty(\Delta^r, E)$ say $\|\Delta^r x_k - L\| \leq M$. Given $\epsilon > 0$, we have

$$h_r^{-1} \sum_{k \in I_r} \|\Delta^r x_k - L\| = h_r^{-1} \sum_{\substack{k \in I_r \\ \|\Delta^r x_k - L\| \geq \epsilon}} \|\Delta^r x_k - L\| + h_r^{-1} \sum_{\substack{k \in I_r \\ \|\Delta^r x_k - L\| < \epsilon}} \|\Delta^r x_k - L\|$$

$$\leq \ M h_r^{-1} \{k \in I_r \ : \|\Delta^r x_k - L\| \ \geq \ \epsilon\} + \epsilon$$

Hence x is $S_\theta(\Delta^r, E)$-statistically convergent to the number L.

(iii) This follows from (i) and (ii). ■

Theorem 3.3. If $\liminf \dfrac{h_r}{r} > 0$, then $S(\Delta^r, E) \subseteq S_\theta(\Delta^r, E)$.

Proof. For given $\epsilon > 0$, we get

$$\{k \leq r : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\} \supset \{k \in I_r : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\}.$$

Hence

$$\frac{1}{r} \left|\{k \leq r : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\}\right| \ \geq \ \frac{1}{r} \left|\{k \ \leq \ r : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\}\right|$$

$$\geq \ \frac{h_r}{r} \ h_r^{-1} |\{k \in I_r : \ \|\Delta^r x_k - L\| \ \geq \ \epsilon\}|$$

Hence $x \in S_\theta(\Delta^r, E)$. ■

Theorem 3.4. Let f be a modulus function and $\sup\limits_k p_k = H$. Then $[N_\theta, f, p]_1(\Delta^r, E) \subset S_\theta(\Delta^r, E)$.

Proof. Let $x \in [N_\theta, f, p]_1(\Delta^r, E)$ and $\epsilon > 0$ be given. Let $\sum\limits_1$ denote the sum over

$k \le r$ such that $\|\Delta^r x_k - L\| < \epsilon$. Then

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r}[f(\|\Delta^r x_k - L\|)]^{p_k} &= h_r^{-1}\sum_1[f(\|\Delta^r x_k - L\|)]^{p_k} + h_r^{-1}\sum_2[f(\|\Delta^r x_k - L\|)]^{p_k}\\
&\ge h_r^{-1}\sum_1[f(\|\Delta^r x_k - L\|)]^{p_k}\\
&\ge h_r^{-1}\sum_1[f(\epsilon)]^{p_k}\\
&\ge h_r^{-1}\sum_1 \min([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H)\\
&\ge h_r^{-1}\big|\{k \in I_r : \|\Delta^r x_k - L\|\\
&\ge \epsilon\}|\ \min([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H).
\end{aligned}$$

Hence $x \in S_\theta(\Delta^r, E)$. ■

Theorem 3.5. Let f be bounded and $0 < m = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. Then $S_\theta(\Delta^r, E) \subset [N_\theta, f, p]_1(\Delta^r, E)$.

Proof. Suppose that f is bounded. Let $\epsilon > 0$ be given and $\sum_1$ and $\sum_2$ be in previous theorem. Since f is bounded there exists an integer K such that $f(x) < K$, for all $x \ge 0$. Then

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r}[f(|\ \Delta^r x_k - L\ |)]^{p_k} &= h_r^{-1}\sum_1[f(\|\Delta^r x_k - L\|)]^{p_k} + h_r^{-1}\sum_2[f(\|\Delta^r x_k - L\|)]^{p_k}\\
&\le h_r^{-1}\sum_1 \max(K^m, K^H) + h_r^{-1}\sum_2[f(\epsilon)]^{p_k}\\
&\le \max(K^m, K^H)h_r^{-1}\ |\ \{k \in I_r : \|\Delta^r x_k - L\|\\
&\ge \epsilon\}\ |\ + \max(f(\epsilon)^m, f(\epsilon)^H)
\end{aligned}$$

Hence $x \in [N_\theta, f, p]_1(\Delta^r, E)$. ■

References

[1] T. Bilgin, *On the statistical convergence*, An. Univ. Timisoara Ser. Math. Inform **32**(1) (1994), 3–7.

[2] J.S. Connor, *The statistical and strong p-cesáro convergence of sequences*, Analysis **8**(1988), 47–63.

[3] M. Et. and M. Basarir, *On some new generalized difference sequence spaces*, Periodica Mathematica Hungarica **35**(3)(1997), 169–175.

[4] M. Et. and R. Colak, *On some generalized difference sequence spaces*, Soochow J. Math. **21**(1995), 377–386.

[5] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**(1951), 241–244.

[6] A.R. Freedman, J.J. Sember and M. Raphael, *Some Cesáro type summability spaces*, Proc. London Math. Soc. **37**(1978), 508–520.

[7] J.A. Fridy, *On statistical convergence*, Analysis **5**(1985), 301–313.

[8] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. **24**(1981), 169–176.

[9] E. Kolk, *The statistical convergence in Banach spaces*, Acta. Comment. Univ. Tartu. **928**(1991), 41–52.

[10] I.J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, 1970.

[11] I.J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc. **100**(1986), 161–166.

[12] E. Malkowsky and S.D. Parashar, *Matrix transformations in spaces of bounded and convergent difference sequences of order m*, Analysis **17**(1997), 87–97.

[13] E. Malkowsky and E. Savas, *Some λ-sequence spaces defined by a modulus*, Archivum Mathematicum **36**(2000), 219–228.

[14] Mursaleen, *λ-statistical convergence*, Math. Slovaca **50**(2000), 111–115.

[15] Mursaleen, Mushir A. Khan and Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Mathematica **32** No. 1(1999), 145–150.

[16] W.H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25**(1973), 973–978.

[17] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca **30**(1980), 139–150.

[18] E. Savas, *On some generalized sequence spaces defined by a modulus*, Indian J. Pure and Appl. Math. **30**(5)(1999), 459–464.