Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 8 (2017), pp. 4223-4228

http://www.ripublication.com

Equicontinuity and Sensitivity of Sequence Dynamical Systems

Khundrakpam Binod Mangang
Assistant Professor,
Department of Mathematics,
Manipur University, Canchipur, Imphal, Manipur-795003, India.

Abstract

The concept of sequence dynamical system$(X, f_i)_{i=1}^{\infty}$, which is the generalization ofdynamical system, has been introduced. The notions of transitivity, equicontinuity, sensitivity have been defined and some related results have been proposed and proved. When each member of the sequence $(f_i)_{i=1}^{\infty}$ is equal to*f*, the sequence dynamical system $(X, f_i)_{i=1}^{\infty}$ becomes the dynamical system (X, f) and the definitions and results are also coincided.

Keywords: equicontinuity; sensitivity; transitivity

1. INTRODUCTION

By a dynamical system we mean a pair (X, f) in which X is a compact metrizable space with metric d and f is continuous self map on X. Let (X, f) be a dynamical system, the orbit $O(x, f)$ of a point $x \in X$ is defined by $O(x, f) = \{f^n(x) : n \in N\}$, where f^nmeans the function f is composed n times. A dynamical system (X, f) is said to be transitive if for every pair of nonempty open sets U, V in X, there is a positive integer n such that $f^n(U) \cap V \neq \phi$. A point $x \in X$ is called a transitive point if the orbit of x is dense in X, i.e. $\overline{O(x, f)} = X$. The set of all transitive points of f is denoted by Γ. A point x is said to be periodic if there exists a positive integer n such that $f^n(x) = x$. The set of all periodic points of (X, f) is denoted by $Per(f)$. A dynamical system (X, f) is called a minimal dynamical system if there is no proper closed invariant subset. A set

$Y \subseteq X$ is said to be residual if there exists a sequence $(U_i)_{i \geq 0}$ of open dense sets, such that $\bigcap_{i \geq 0} U_i \subseteq Y$ [5]. It is well known by Baire theorem [5] that, any residual set of a compact metric space is nonempty and dense. In [11], chaos of a sequence of time invariant continuous functions on a general metric space has been investigated. The authors also introduced quite a few new concepts, such as chaos in the successive way in the sense of Devaney, chaos in the iterative way in the sense of Devaney.In [6], the dynamical properties of product dynamical systems have been investigated. In [2], turbulent maps and strongly transitive maps in general metric spaces have been investigated. It has

been proved that if (f_n) is a sequence of continuous functions which is topologically transitive in the strongly iterative way in an infinite compact metric space (X, d) the uniform limit function in the iterative way is topologically mixing.

Several authors have studied the dynamical properties inherited by the uniform limit f of a sequence (f_n) of continuous self-maps. Ithas been shown in [9], that a sequence $(f_n)$of continuous and transitive maps that converge uniformly to f , is not necessarily topologically transitive. Recently many Mathematicians have studied topological transitivity of the uniform limit of a sequence of uniformly convergent transitive system. In [8], there is a sufficient condition so that the uniform limit is transitive. In [3], also there is a sufficient condition for the transitivity of the limit function. In [1], the concept of orbital convergent of a sequence (f_n) is defined. They have shown that if a sequence (f_n) of transitive system is orbitally convergent to f, then f is topologically transitive. They also have shown that if a sequence (f_n) of chain transitive dynamical system is convergent uniformly to f, then f is also chain transitive. In [10], the topological transitivity of sequence of transitive maps under group action has been studied. They also discuss the G-transitivity of limit functions of G-uniformly convergent and G-orbitally convergent sequences. In [4], there is a sufficient condition for the transitivity of the uniform limit of a sequence of transitive system. In [7], the equicontinuity, minimality of the limit function of a sequence of dynamical system have been investigated. An example of orbitally convergent sequence of continuous functions has been given.

Let $F = (f_i)_{i=1}^{\infty}$be a sequence of continuous functions on X, the n compositions of the first n functions is given by $f_n \circ f_{n-1} \circ ... \circ f_1(x)$. In this connection if we have a sequence of dynamical systems $(X, f_i)_{i=1}^{\infty}$ and we define the orbit of a point x as follows

$$O(x) = \{F_n(x) : n \in N\},$$

where$F_n(x) = \bigcup_{i_1, i_2, \dots, i_n \in N} f_{[i_1, i_2, \dots, i_n]}(x)$ and $f_{[i_1, i_2, \dots, i_n]}(x) = f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}(x)$.The sequence (X, F) ,with this definition of orbit, is called Sequence

Dynamical Systems. We use the notation $f^{-1}_{[i_1,i_2,...,i_n]}(x)$to denote the inverse image $f^{-1}_{i_1} \circ f^{-1}_{i_2} \circ ... \circ f^{-1}_{i_n}(x)$. Also $F^{-1}_n(x) = \cup_{i_1,i_2,...,i_n \in N} f^{-1}_{[i_1,i_2,...,i_n]}(x)$. A subset $A \subseteq X$ is said to be invariant in (X, F) if $f_i(A) \subseteq A$ for each $i \in N$. It can be easily verified that the above definitions and terminology of sequence dynamical system (X, F) becomes the usual definitions and terminology of the dynamical systems (X, f) when $f_i = f$ for all $i \in N$. Thus sequence dynamical system (X, F) is generalization of dynamical system (X, f). In section 2, we define various definitions and in section 3, we propose and prove some results of equicontinuity, transitivity and sensitivity.

2. DEFINITIONS

Definition 2.1.*Let* (X, F) *be a sequence dynamical system. We say that it is equicontinuous if for each* $x \in X$, *for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $d(f_{[i_1,i_2,...,i_n]}(x), f_{[i_1,i_2,...,i_n]}(y)) < \epsilon$ *for all* y *with* $d(x, y) < \delta$, *for all* $n \geq 0$ *and for all* $i_1, i_2, ..., i_n \in N$.

A sequence dynamical system (X, F) is said to be equicontinous at a point $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(f_{[i_1,i_2,...,i_n]}(x), f_{[i_1,i_2,...,i_n]}(y)) < \epsilon$ for all y with $d(x, y) < \delta$, for all $n \geq 0$ and for all $i_1, i_2, ..., i_n \in N$.The set of all equicontinuous points is denoted by Σ. If $\Sigma = X$, then we say that (X, F) is equicontinuous. A subset $A \subseteq X$ is said to be inversely invariant with respect to the sequence dynamical system (X, F) if $f^{-1}_i(A) \subseteq A$ for every $i \in N$. The open ball of radius $\delta > 0$ and centre$x$ is denoted by $B_\delta(x)$ and defined by $B_\delta(x) = \{ y: d(x, y) < \delta\}$.

Definition 2.2.*A sequence dynamical system* (X, F) *is said to be sensitive if there exists* $\epsilon > 0$ *such that for every* $x \in X$ *and for every* $\delta > 0$, *there exists* $y \in B_\delta(x)$, $n \geq 0$ *such that* $d(f_{[i_1,i_2,...,i_n]}(y), f_{[i_1,i_2,...,i_n]}(x)) \geq \epsilon$ *for some* $i_1, i_2, ..., i_n \in N$.

Definition 2.3.*A sequence dynamical system* (X, F) *is said to be almost equicontinuous if* Σ*is a residual set.*

Definition 2.4.*A point is transitive if its orbit is dense. i.e.,* $\overline{O(x)} = X \Leftrightarrow x$ *is transitive point.*

Definition 2.5.*A sequence dynamical system* (X, F) *is said to be transitive if for each pair U, V of non-empty open sets, there exists* $n > 0$ *such that* $f_{[i_1,i_2,...,i_n]}(U) \cap V \neq \phi$ *or equivalently* $f^{-1}_{[i_1,i_2,...,i_n]}(V) \cap U \neq \phi$, *for some* $i_1, i_2, ..., i_n \in N$.

3. MAIN RESULTS

For any dynamical system, we have the following result [5].

Theorem 3.1. *Let* (X, f) *be a dynamical system and* $\epsilon > 0$. *Let* $\Sigma_\epsilon = \{x \in X: \exists \delta > 0, \forall y, z \in B_\delta(x), \forall n \geq 0, d(f^n(z), f^n(y)) < \epsilon\}$. *Then* Σ_ϵ *is inversely invariant,*

open and $\Sigma = \bigcap_{m>0} \Sigma_{1/m}$.

It is natural to ask that whether the above result can be generalized in the case of sequence dynamical systems or not. Although, the inversely invariant part of Σ_ϵ is yet to be investigated. There is a similar result in the case of sequence dynamical system.

Theorem 3.2. *Let* (X, F) *be a sequence dynamical system and* $\epsilon > 0$. *Let* $\Sigma_\epsilon = \{ x \in X: \exists \delta > 0, \forall y, z \in B_\delta(x), \forall n \geq 0, \forall i_1, i_2, \ldots, i_n \in N, d(f_{[i_1,i_2,\ldots,i_n]}(z), f_{[i_1,i_2,\ldots,i_n]}(y)) < \epsilon\}$. *Then* Σ_ϵ *is*

open and $\Sigma = \bigcap_{m>0} \Sigma_{1/m}$.

Proof. In order to show the result, for any $x \in \Sigma_\epsilon$ and for δ which satisfies the definition of Σ_ϵ for x, we shall show that $B_{\delta/2}(x) \subseteq \Sigma_\epsilon$. Indeed if $y \in B_{\delta/2}(x)$ and if $z, w \in B_{\delta/2}(x)$, then $z, w \in B_\delta(x)$, so $d(f_{[i_1,i_2,\ldots,i_n]}(z), f_{[i_1,i_2,\ldots,i_n]}(y)) < \epsilon$, for all $n \geq 0$ and for all $i_1, i_2, \ldots, i_n \in N$. This shows that $y \in \Sigma_\epsilon$ and therefore $B_{\delta/2}(x) \subseteq \Sigma_\epsilon$. Hence Σ_ϵ is open. If $x \in \Sigma_{1/m}$ for all $m > 0$, then clearly $x \in \Sigma$. Therefore

$\bigcap_{m>0} \Sigma_{1/m} \subseteq \Sigma_\epsilon$. Conversely, if $x \in \Sigma$ and $m > 0$, then there exists $\delta > 0$ such that $d(f_{[i_1,i_2,\ldots,i_n]}(z), f_{[i_1,i_2,\ldots,i_n]}(y)) < \epsilon/2$, for all $y \in B_\delta(x)$, for all $n \geq 0$ and for all $i_1, i_2, \ldots, i_n \in N$. Thus if $y, z \in B_\delta(x)$, then by triangular inequality of metric we have

$d(f_{[i_1,i_2,\ldots,i_n]}(y), f_{[i_1,i_2,\ldots,i_n]}(z)) \leq d\left(f_{[i_1,i_2,\ldots,i_n]}(y), f_{[i_1,i_2,\ldots,i_n]}(x)\right) + d(f_{[i_1,i_2,\ldots,i_n]}(x), f_{[i_1,i_2,\ldots,i_n]}(z)) < \frac{1}{2m} + \frac{1}{2m}$. i.e., $d\left(f_{[i_1,i_2,\ldots,i_n]}(y), f_{[i_1,i_2,\ldots,i_n]}(z)\right) < \frac{1}{m}$. This implies that $x \in \Sigma_{1/m}$ and therefore $\Sigma = \bigcap_{m>0} \Sigma_{1/m}$. This completes the proof.

Theorem 3.3. *Let* (X, F) *be a sequence dynamical system such that for each* $\epsilon > 0$, Σ_ϵ *is inversely invariant. If* (X, F) *is transitive, then it is either sensitive or almost equicontionuous.*

Proof. Let (X, F) be a transitive sequence dynamical system such that for each $\epsilon > 0$, Σ_ϵ is inversely invariant. Suppose if possible we assume that Σ_ϵ is non-empty and non-dense, then $U = X \setminus \overline{\Sigma}_\epsilon$ is open and non-empty. Since (X, F) is transitive, we have

$U \cap f^{-1}_{[i_1,i_2,\ldots,i_n]}(\Sigma_\epsilon) \neq \phi$,for some $n \geq 0$ and $i_1, i_2, \ldots, i_n \in N$. Also we have $U \cap f^{-1}_{[i_1,i_2,\ldots,i_n]}(\Sigma_\epsilon) \subseteq U \cap \Sigma_\epsilon = \phi$.This is a contradiction. Therefore any $\Sigma_\epsilon$is either empty or dense. If all $\Sigma_\epsilon$are non-empty, then $\Sigma = \bigcap_{m>0} \Sigma_{1/m}$is residual set, so $(X, F)$is almost equicontinuous. If $\Sigma_\epsilon = \phi$, for some $\epsilon > 0$, then for any $x \in X$and for any $\delta > 0$ there exists $y, z \in B_\delta(x)$ and $n \geq 0$ such that $d\left(f_{[i_1,i_2,\ldots,i_n]}(y), f_{[i_1,i_2,\ldots,i_n]}(z)\right) \geq \epsilon$for some $i_1, i_2, \ldots, i_n \in N$. It follows that either $d\left(f_{[i_1,i_2,\ldots,i_n]}(y), f_{[i_1,i_2,\ldots,i_n]}(x)\right) \geq \epsilon/2$or

$d\left(f_{[i_1,i_2,\ldots,i_n]}(z), f_{[i_1,i_2,\ldots,i_n]}(x)\right) \geq \epsilon/2$. Therefore (X, F) is sensitive with sensitive constant $\epsilon/2$. This completes the proof.

The above result is the generalization of the following theorem 3.4 of a dynamical system [5]. For a dynamical system (X, f), the set Σ_ϵ is inversely invariant. The above result is after we assume that Σ_ϵ is inversely invariant in the case of sequence dynamical systems.

Theorem 3.4. *Let* (X, f) *be a dynamical system .If* (X, F) *is transitive, then it is either sensitive or almost equicontionuous.*

REFEENCES

[1] Alessandro Fedeli and Attilio Le Donne, *A note on the uniform limit of transitive dynamical systems*, Bull. Gelg.Math. Soc. Simon Stevin **16** (2009), 59-66.

[2] Bhaumik, Indrani and ChoudhuryS,Binayak , Iterative sequence of maps in a metric space with some chaotic properties, *Bulletin of Mathematical Analysis andApplications* **3** (2011),no.4, 116-122.

[3] Heriberto Roman-Flores, *Uniform convergence and transitivity*, Chaos, Solitons andFractals **38** (2006), 148-153.

[4] Kexongyan, FanpingZeng and Gengrong Zhang, *Devaney's chaos on uniformlimit maps*, Chaos, Soliton and Fractals **44** (2011), 522-525

[5] Kurka,P., "Topological and symbolic Dynamics"*Societemathematique de France*, 2003.

[6] Mangang,KB., "Product dynamical systems", *Far east journal of dynamical systems*, to appear.

[7] Mangang,KB., *Equicontinuity and Minimality of Limit function of a sequence of maps*,J. Ind. Math. Soc. **81** (2014),no.1-2, 115-121.

[8] Raghib Abu-Saris, KifahAlHami,*Uniform convergence and chaoticbehaviour,* Non-linear Analysis **65** (2006), 933-937.

[9] RaghibM.Abu-Saris, Felix Martinez-Gimenez, Alfredo Peris, *Erratum to uniform convergence and chaotic behaviour[Nonlinear Analysis **65 (4)** (2006), 933-937]*, Non-linear Analysis **68** (2008), 1406-1407.

[10] Ruchi Das and Tarun Das, *Topological Transitivity of Uniform Limit Functions on G-spaces*, Int. Journal of Math. Analysis **6**(2012),**no.30**1491-1499.

[11] Tian, Chuanjun and Chen, Guanrong,Chaos of sequence of maps in a metric space, *Chaos, Solitons and Fractals **28**(2006), 1067-1075.*