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A new class of generalized Laguerre-based poly-Bernoulli polynomials

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Abstract

A new class of generalized Laguerre-based poly-Bernoulli polynomials are discussed with an attempt to generate new and interesting identities, some are in relation with Stirling number of the second kind. Different analytical means and generating function method is incorporated to derive implicit summation formulae and symmetry identities for generalized Laguerre poly-Bernoulli polynomials. It is worthy to mention that these results are extension to a number of formally proved identities of generalized Laguerre-based Appell polynomials [8] and generalized poly-Bernoulli polynomials [4, 5].

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1. Introduction

Two variable Laguerre polynomials (sometimes called 2VLP) $L_n(x,y)$ are considered by means of the generating function [2]:

$$\frac{1}{(1-yt)}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}Ł_n(x,y)t^n,\ (|\ yt\ |<1) \tag{1.1}$$

and can also be stated by

$$\exp(yt)C_0(xt)=\sum_{n=0}^{\infty}Ł_n(x,y)\frac{t^n}{n!}, \tag{1.2}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function (for details see [3]). The n^{th} order Tricomi functions $C_n(x)$ are presented by the series:

$$C_n(x)=\sum_{n=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!},\ (n\in\mathbb{N}_0), \tag{1.3}$$

with the following generating function:

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n, \tag{1.4}$$

for $t\neq 0$ and for all finite x. The Tricomi functions $C_n(x)$ are characterized by the following link with the Bessel function $J_n(x)$:

$$C_n(x)=x^{\frac{n}{2}}J_n(2\sqrt{x}). \tag{1.5}$$

The 2VLP are specified explicitly by the series [2]:

$$Ł_n(x,y)=n!\sum_{s=0}^{n}\frac{(-1)^s x^s y^{n-s}}{(s!)^2(n-s)!}=y^nŁ_n(x/y), \tag{1.6}$$

and satisfy the properties

$$Ł_n(x,y)=\frac{(-1)^n x^n}{n!},\ Ł_n(0,y)=y^n,\ Ł_n(x,1)=Ł_n(x), \tag{1.7}$$

where $Ł_n(x)$ are the ordinary Laguerre polynomials [1]:

$$Ł_n(x,y)=n!\sum_{s=0}^{n}\frac{(-1)^s x^s}{(s!)^2(n-s)!}=y^nŁ_n(x/y). \tag{1.8}$$

Kaneko [9] introduced the poly-Bernoulli numbers $B_n^{(k)}$ by means of the following exponential generating function $B_n^{(k)}$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, appear in the following power series:

$$\frac{\mathrm{Li_k(1-e^{-t})}}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1.9}$$

where poly-logarithm $\mathrm{Li_k(z)}$ is depicted by

$$\mathrm{Li_k(z)} = \sum_{\mathrm{m=1}}^{\infty} \frac{\mathrm{z^m}}{\mathrm{m^k}},\ |\mathrm{z}| < 1$$

so for $k \le 1$,

$$\mathrm{Li_1(z)} = -\ln(1-\mathrm{z}), \mathrm{Li_0(z)} = \frac{\mathrm{z}}{1-\mathrm{z}}, \mathrm{Li_{-1}(z)} = \frac{\mathrm{z}}{(1-\mathrm{z})^2}, \dots \tag{1.10}$$

Moreover when $k \geq 1$, the left hand side of (1.9) can be edited in the form of "interated integrals"

$$e^t \underbrace{\frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1} \cdots \frac{1}{e^t-1}}_{(k-1)-\text{times}} \int_0^t \frac{t}{e^t-1} dt \cdots dt = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}. \tag{1.11}$$

In the special case, one can see $B_n^{(1)} = B_n$.

In [4, 5], Jolany et al. brought in the generalized poly-Bernoulli polynomials with three parameter a; b; c as follows:

$$\frac{\mathrm{Li_k(1-(ab)^{-t})}}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x,a,b,c) \frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}. \tag{1.12}$$

Very recently, Khan et al. [8; p.8(2.3a, 2.3b)] introduced Laguerre-based Appell polynomials ${}_LA_n(x,y)$ which are defined through the generating function:

$$A(t)\exp(yt)C_0(xt) = \sum_{n=0}^{\infty} {}_LA_n(x,y)\frac{t^n}{n!}, \tag{1.13}$$

or, equivalently

$$A(t)\left[\frac{1}{(1-yt)}\exp\left(\frac{-xt}{1-yt}\right)\right] = \sum_{n=0}^{\infty} {}_LA_n(x,y)t^n, (|\ yt\ |< 1). \tag{1.14}$$

The Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0) \tag{1.15}$$

and the Stirling number of the second kind is defined by generating function

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}. \tag{1.16}$$

As an extention to the above discussion, in the current paper, we have considered Laguerre-based poly-Bernoulli polynomials and discuss their properties and applications.

A further motivation for the present paper is the vast application of generalized laguerre polynomials in mathematical physics, for example, in evaluation of electron integrals which is interesting from both classical and quantum mechanics point of view. In this regard some expected features of generalized Laguerre-based poly-Bernoulli polynomials are derived in the form of implicit summation formulae and general symmetry identities by using different analytical means and applying generating functions.

It is worthy to mention that these results are extension to a number of formally proved identities of generalized Laguerre-based Appell polynomials [8] and generalized poly-Bernoulli polynomials [4, 5].

2. A new class of generalized Laguerre-based poly-Bernoulli polynomials

Present section of the paper deals with the the generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ for nonnegative integer n defined by

$$\frac{\mathrm{Li_k}(1 - (\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} \exp(yt) C_0(xt)$$
$$= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b) \frac{t^n}{n!}, \mid t \mid < 2\pi/(\mid \ln a + \ln b \mid), x \in \mathbb{R} \tag{2.1}$$

so that

$${}_LB_n^{(k)}(x, y; a, b) = \sum_{m=0}^{n} \binom{n}{m} B_n^{(k)}(a, b) L_m(x, y). \tag{2.2}$$

Adjusting $x = y = 0$ in (2.1), $B_n^{(k)}(a, b) = {}_LB_n^{(k)}(0, 0; a, b)$ are called the poly-Bernoulli numbers.

Setting $a = 1, b = e$ in (2.1), we have

$$\frac{\mathrm{Li_k}(1 - \mathrm{e}^{-\mathrm{t}})}{e^t - 1} \exp(yt) C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y) \frac{t^n}{n!}, (x, y \in \mathbb{R}). \tag{2.3}$$

Now, we showcase some properties of generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ in the following theorems

Theorem 2.1. Let $x \in \mathbb{R}$ and $n \geq 0$. Then

$$_{L}B_n^{(k)}(x, y; e, 1,) = {}_{L}B_n^{(k)}(x, y), {}_{L}B_n^{(k)}(0, 0, ; a, b) = B_n^{(k)}(a, b),$$

$$_{L}B_n^{(k)}(0, y; a, b, 1) = B_n^{(k)}(y; a, b) \tag{2.4}$$

$$_{L}B_n^{(k)}(x, y + z; a, b) = \sum_{m=0}^{n} \binom{n}{m} {}_{L}B_{n-m}^{(k)}(x, y; a, b)z^m. \tag{2.5}$$

Proof. The formula in (2.4) are trivial. Applying Definition (2.1), we have

$$\frac{\mathrm{Li_k}(1 - (\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} \exp((y + z)t)C_0(xt) = \sum_{n=0}^{\infty} {}_{L}B_n^{(k)}(x, y + z; a, b)\frac{t^n}{n!}, \tag{2.6}$$

which can be written as

$$\frac{\mathrm{Li_k}(1 - (\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} e^{yt} e^{zt} C_0(xt) = \sum_{n=0}^{\infty} {}_{L}B_n^{(k)}(x, y; a, b)\frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{z^m t^m}{m!}. \tag{2.7}$$

Replacing n by $n - m$ in (2.7) and comparing the coefficients of $\frac{t^n}{n!}$ leads to formula (2.5). ■

Theorem 2.2. For $n \geq 0$, we have

$$_{L}B_n^{(2)}(x, y) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m + 1} {}_{L}B_{n-m}(x, y). \tag{2.8}$$

Proof. Put in use the Definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_{L}B_n^{(k)}(x, y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1 - e^{-t})}{e^t - 1} \exp(yt)C_0(xt)$$

$$= \frac{e^{yt}C_0(xt)}{e^t - 1} \underbrace{\int_0^t \frac{1}{e^z - 1} \int_0^t \frac{1}{e^z - 1} \cdots \frac{1}{e^z - 1}}_{(k-1)-\text{times}} \int_0^t \frac{z}{e^z - 1} dz \cdots dz.$$

In particular for $k = 2$, we have

$$\sum_{n=0}^{\infty} {}_{L}B_n^{(2)}(x, y)\frac{t^n}{n!} = \frac{e^{yt}C_0(xt)}{e^t - 1} \int_0^t \frac{z}{e^z - 1} dz = \left(\sum_{m=0}^{\infty} \frac{t^m B_m}{m + 1}\right) \frac{t}{e^t - 1} e^{yt} C_0(xt),$$

$$= \left(\sum_{m=0}^{\infty} \frac{B_m m!}{m + 1} \frac{t^m}{m!}\right) \left(\sum_{n=0}^{\infty} {}_{L}B_n(x, y)\frac{t^n}{n!}\right),$$

Replacing n by $n-m$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(2)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{B_m m!}{m+1}{}_LB_{n-m}(x,y)\right)\frac{t^n}{n!}.$$

Finally on identifying the coefficients of the like powers of t^n in the above equation, the result (2.8) is established. ■

Theorem 2.3. For $n \geq 0$, we have

$$_LB_n^{(k)}(x,y) = \sum_{p=0}^{n}\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k(p+1)}\binom{n}{p}{}_LB_{n-p}(x,y). \tag{2.9}$$

Proof. Put in use the Definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y)\frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{t}{e^t-1}e^{yt}C_0(xt)\right). \tag{2.10}$$

Now

$$\frac{1}{t}\mathrm{Li}_k(1-e^{-t}) = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{l^k} = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}(1-e^{-t})^l,$$

$$= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}l!\sum_{p=l}^{\infty}(-1)^pS_2(p,l)\frac{t^p}{p!} = \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\frac{t^p}{p!},$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}, \tag{2.11}$$

From equations (2.10) and (2.11), we get

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y)\frac{t^n}{n!} = \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\left(\sum_{n=0}^{\infty} {}_LB_n(x,y)\frac{t^n}{n!}\right).$$

On replacing n by $n-p$ in the R.H.S of above equation and comparing the coefficients of $\frac{t^n}{n!}$, the result (2.9) is established. ■

Theorem 2.4. For $n \geq 1$, we have

$$_LB_n^{(k)}(x+1,y) - {}_LB_n^{(k)}(x,y) = \sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\binom{n}{p}L_{n-p}(x,y). \tag{2.12}$$

where $S_2(p,l)$ is the Stirling number of the second kind given in (1.16).

Proof. Put in use the Definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$= \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{(y+1)t}C_0(xt) - \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{yt}C_0(xt),$$

$$= \mathrm{Li}_k(1-e^{-t})e^{xt+yt^2}$$

$$= \sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}e^{yt}C_0(xt)$$

$$= \left(\sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}\right).$$

On replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$, the result (2.12) is established. ■

Theorem 2.5. For $d \in \mathbb{N}$ with $d \equiv 1(mod2)$, we have

$$ {}_LB_n^{(k)}(x,y)$$
$$= \sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a{}_LB_{n-p}\left(\frac{a+y}{d},x\right). \tag{2.13}$$

where $S_2(p,l)$ is the Stirling number of the second kind given in (1.16).

Proof. Put in use the equation (2.3), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{yt}C_0(xt),$$

$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{t}{e^{dt}-1}\sum_{a=0}^{d-1}(-1)^a e^{(a+y)t}C_0(xt)\right),$$

$$= \left(\sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}(-1)^a{}_LB_n(\frac{a+y}{d},x)\frac{t^n}{n!}\right).$$

On replacing n by $n-p$ in above equation and comparing the coefficient of $\frac{t^n}{n!}$, the result (2.13) is established. ■

3. Implicit summation formulae involving generalized Laguerre-based poly-Bernoulli polynomials

In this section, we define implicit formulae involving generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ and their basic properties as follows:

Theorem 3.1. The following implicit summation formulae for generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ holds true:

$$ {}_LB_{m+l}^{(k)}(x, z; a, b) = \sum_{n,p=0}^{m,l} \binom{l}{p}\binom{m}{n}(z-y)^{n+p}\,{}_LB_{m+l-n-p}^{(k)}(x, y; a, b). \quad (3.1)$$

Proof. We replace t by $t + u$ and reconstitute the generating function (2.1) as

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-(\mathrm{t+u})})}{b^{t+u}-a^{-(t+u)}}C_0(x(t+u))$$

$$= e^{-y(t+u)}\sum_{k,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, y; a, b)\frac{t^m}{m!},\frac{u^l}{l!}, \text{ (see [6, 7, 10, 11])} \quad (3.2)$$

Replacing y by z in the above equation and equalize the resulting equation to the above equation, we get

$$e^{(z-y)(t+u)}\sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, y; a, b)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, z; a, b)\frac{t^m}{m!}\frac{u^l}{l!}, \quad (3.3)$$

On expanding exponential function, (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-y)(t+u)]^N}{N!}\sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, y; a, b)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, z; a, b)\frac{t^m}{m!}\frac{u^l}{l!}, \quad (3.4)$$

which on using formula [12, p. 52(2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \quad (3.5)$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-y)^{n+p}t^n u^p}{n!p!}\sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, y; a, b, c)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_LB_{m+l}^{(k)}(x, z; a, b)\frac{t^m}{m!}\frac{u^l}{l!}. \quad (3.6)$$

Now replacing m by $m-n$, l by $l-p$ and using the lemma [12, p.100(1)] in the left hand side of (3.6), we get

$$\sum_{n,p=0}^{\infty}\sum_{m,l=0}^{\infty}\frac{(z-y)^{n+p}}{n!p!}{}_LB^{(k)}_{m+l-n-p}(x,y;a,b)\frac{t^m}{(m-n)!}\frac{u^l}{(l-p)!}$$

$$=\sum_{m,l=0}^{\infty}{}_LB^{(k)}_{m+l}(x,z;a,b)\frac{t^m}{m!}\frac{u^l}{l!}. \tag{3.7}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, the required result of the section is established. ■

Remark 3.2. By substituting $l = 0$ in equation (3.1), we straight away deduce the following result.

Corollary 3.3. The following implicit summation formula for Laguerre poly-Bernoulli polynomials ${}_LB^{(k)}_n(x,z;a,b)$ holds true:

$${}_LB^{(k)}_m(x,z;a,b)=\sum_{n=0}^{m}\binom{m}{n}(z-y)^n\,{}_LB^{(k)}_{m-n}(x,y;a,b). \tag{3.8}$$

Remark 3.4. On replacing z by $z+y$ and setting $x=0$ in Theorem (3.1), we infer the following result involving generalized Laguerre poly-Bernoulli polynomials of one variable

$${}_LB^{(k)}_{m+l}(z+y;a,b)=\sum_{n,p=0}^{m,l}\binom{l}{p}\binom{m}{n}z^{n+p}\,{}_LB^{(k)}_{m+l-p-n}(y;a,b). \tag{3.9}$$

whereas by setting $z=0$ in Theorem 3.1, we get another result involving generalized Laguerre poly-Bernoulli polynomials of one and two variables

$${}_LB^{(k)}_{m+l}(y;a,b)=\sum_{n,p=0}^{m,l}\binom{l}{m}\binom{k}{p}(-y)^{n+p}\,{}_LB^{(k)}_{m+l-p-n}(x,y;a,b). \tag{3.10}$$

Remark 3.5. In conjunction with the above results we will exploit extended forms of generalized Laguerre poly-Bernoulli polynomials ${}_LB^{(k)}_{m+l}(z,x;a,b)$ by setting $x=0$ in Theorem (3.1) to get

$${}_LB^{(k)}_{m+l}(z;a,b)=\sum_{n,p=0}^{m,l}\binom{l}{p}\binom{m}{n}(z-y)^{n+p}\,{}_LB^{(k)}_{m+l-p-n}(y;a,b). \tag{3.11}$$

Theorem 3.6. The following implicit summation formulae for generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ holds true:

$$ {}_LB_n^{(k)}(x, y; a, b) = \sum_{m=0}^{n-j} \sum_{j=0}^{n} x^j y^{n-m-j} B_m^{(k)}(a, b) \frac{n!}{m!(j!)^2(n-j-m)!}. \tag{3.12} $$

Proof. Applying definition (2.1) to the term $\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}}$ and expanding the exponential function $e^{yt}C_0(xt)$ at $t = 0$ yields

$$ \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} e^{yt} C_0(xt) = \left(\sum_{m=0}^{\infty} B_m^{(k)}(a, b) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{y^n t^n}{n!} \right) \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j t^j}{(j!)^2} \right), $$

$$ = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} B_m^{(k)}(a, b) y^{n-m} \right) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j t^j}{(j!)^2} \right), $$

Replacing n by $n - j$, we have

$$ \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-j} \sum_{j=0}^{n} \binom{n-j}{m} B_m^{(k)}(a, b) y^{n-m-j} x^j \right) \frac{t^n}{(n-j)!(j!)^2}. $$

Equating their coefficients of t^n produce the formula (3.12). ■

Theorem 3.7. The following implicit summation formulae for generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ holds true:

$$ {}_LB_n^{(k)}(x+1, y; a, b) = \sum_{m=0}^{n} \binom{n}{m} {}_LB_m^{(k)}(x, y; a, b) \tag{3.13} $$

Proof. From the definition of generalized Laguerre poly-Bernoulli polynomials, we have

$$ \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x+1, y; a, b) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b) \frac{t^n}{n!} $$

$$ = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} e^{yt} C_0(xt)(e^t - 1), $$

$$ = \left(\sum_{m=0}^{\infty} {}_LB_m^{(k)}(x, y; a, b) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right) - \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b) \frac{t^n}{n!}, $$

$$ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}_LB_m^{(k)}(x, y; a, b) \frac{t^n}{(n-m)!m!} - \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b) \frac{t^n}{n!}. $$

Finally, equating the coefficients of the like powers of t^n, the result (3.13) is established. ■

Theorem 3.8. The following implicit summation formulae for generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ holds true:

$$\sum_{m=0}^{n} \binom{n}{m} (\ln ab)^m {}_LB_{n-m}^{(k)}(x, -y; a, b) = (-1)^n {}_LB_n^{(k)}(x, y; a, b). \tag{3.14}$$

Proof. We replace t by $-t$ in (2.1) and then subtract the result from (2.1) itself finding

$$C_0(xt)\left[\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}}(e^{yt} - (ab)^t e^{-yt})\right] = \sum_{n=0}^{\infty}[1-(-1)^n]{}_LB_n^{(k)}(x, y; a, b)\frac{t^n}{n!},$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b)\frac{t^n}{n!} - \left(\sum_{m=0}^{\infty}(\ln ab)^m \frac{t^m}{m!}\right)\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, -y; a, b)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y; a, b)\frac{t^n}{n!} - \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\ln ab)^m\right) {}_LB_{n-m}^{(k)}(x, -y; a, b)\frac{t^n}{(n-m)!}\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}[1-(-1)^n]{}_LB_n^{(k)}(x, y; a, b)\frac{t^n}{n!}.$$

Replacing n by $n - m$ in L.H.S. of above equation and thus by equating coefficients of like powers of t^n, we get (3.14). ■

4. Symmetry Identities for Generalized Laguerre poly-Bernoulli polynomials

This portion of the paper deals with establishment of general symmetry identities for the generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y; a, b)$ by applying the generating function (1.11) and (2.1).

Theorem 4.1. For $x, y \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m {}_LB_{n-m}^{(k)}(bx, by; A, B) {}_LB_m^{(k)}(az, aw; A, B)$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^m {}_LB_{n-m}^{(k)}(ax, ay; A, B) {}_LB_m^{(k)}(bz, bw; A, B). \tag{4.1}$$

Proof. Proceeding with

$$g(t)=\left(\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(A^{at}-B^{-at})(A^{bt}-B^{-bt})}\right)\exp(ab(y+z)t)C_0(abxt)C_0(abwt) \tag{4.2}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t)=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(bx,by;A,B)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_LB_m^{(k)}(az,aw;A,B)\frac{(bt)^m}{m!}$$

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_LB_{n-m}^{(k)}(bx,by;A,B){}_LB_m^{(k)}(az,aw;A,B)\right)\frac{t^n}{n!}, \tag{4.3}$$

Following the similar lines, we can show that

$$g(t)=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(ax,ay;A,B)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_LB_m^{(k)}(bz,bw;A,B)\frac{(at)^m}{m!}$$

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m{}_LB_{n-m}^{(k)}(ax,ay;A,B){}_LB_m^{(k)}(bz,bw;A,B)\right)\frac{t^n}{n!}. \tag{4.4}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, the desired result is established. ■

Theorem 4.2. Let $x, y \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HB_{n-m}^{(k)}\left(bx+\frac{b}{a}i+j,b^2z;A,B,c\right)B_m^{(k)}(ay;A,B,c)b^ma^{n-m}$$

$$=\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_HB_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z;A,B,c\right)B_m^{(k)}(by;A,B,c)a^mb^{n-m}. \tag{4.5}$$

Proof. We assume

$$g(t)=\left(\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right)\frac{(e^{abt}-1)^2\exp(ab(y+z)t)C_0(abxt)C_0(abwt)}{(e^{at}-1)(e^{bt}-1)}.$$

which can be expressed as

$$g(t)=\left(\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{(B^{at}-A^{-at}}\right)\exp(abyt)C_0(abxt)\left(\frac{e^{abt}-1}{e^{bt}-1}\right)$$

$$\times \left(\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{B^{bt}-A^{-bt}} \right) \exp(abzt) C_0(abwt) \left(\frac{e^{abt}-1}{e^{at}-1} \right),$$

$$= \left(\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{(B^{at}-A^{-at}} \right) \exp(abyt) C_0(abxt)$$

$$\times \sum_{i=0}^{a-1} e^{bti} \left(\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{B^{bt}-A^{-bt}} \right) \exp(abzt) C_0(abwt) \sum_{j=0}^{b-1} e^{atj},$$

$$= \left(\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{B^{at}-A^{-at}} \right) C_0(abxt) \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} e^{(bx+\frac{b}{a}i+j)at} \sum_{m=0}^{\infty} {}_LB_m^{(k)}(az, aw; A, B) \frac{(bt)^m}{m!},$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_LB_n^{(k)} \left(bx + \frac{b}{a}i + j, by; A, B \right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_LB_m^{(k)}(az, aw; A, B) \frac{(bt)^m}{(m)!},$$

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_LB_{n-m}^{(k)} \left(bx + \frac{b}{a}i + j, by; A, B \right) \right.$$

$$\left. \times {}_LB_m^{(k)}(az, aw; A, B) b^m a^{n-m} \right) \frac{t^n}{n!}. \tag{4.6}$$

On the other hand, we have

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_LB_{n-m}^{(k)} \left(ax + \frac{a}{b}i + j, ay; A, B \right) \right.$$

$$\left. \times {}_LB_m^{(k)}(bz, bw; A, B) a^m b^{n-m} \right) \frac{t^n}{n!}. \tag{4.7}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result. ■

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