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(ψ, ϕ, θ)-weak contraction of continuous and discontinuous functions

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Abstract

We prove weak contraction theorem using three control functions in which one is continuous and other two are discontinuous.

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1. Introduction

some definitions and results are as follows:

Definition 1.1. A mapping $T : X \to X$, where (X,d) is a metric space is said to be weakly contractive if x,y ∈ X

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Theorem 1.2. Let (X, d) be a complete metric space, T a weakly contractive mapping then T has a unique fixed point p in x.

Definition 1.3. (Altering distance function) A function $\psi : [0, \infty) \to [0, \infty)$ is said to be altering distance function if the following properties are satisfied

(i) ϕ is monotone nondecreasing and continuous

(ii) $\phi(t) = 0$ if and only if $t = 0$.

Let by Θ we denote the set of all functions $\alpha : [0, \infty) \to [0, \infty)$ such that
(i_α) α is bounded on any bounded interval in $[0, \infty)$
(ii_α) α is continuous at 0 and $\alpha(0) = 0$.

Definition 1.4. Let (X,d) be a metric space, T a self mapping of X, we shall call T a generalised weak contractive mapping if for all $x, y \in X$

$$\psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \phi(max[d(x, y), d(y, Ty)]) \tag{1}$$

where

$$m(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$.
A general weak contraction is more general than that satisfying

$$d(Tx, Ty) \leq km(x, y) \tag{2}$$

for some $0 < k < 1$, and is included in those mapping which satisfies

$$d(Tx, Ty) \leq m(x, y) \tag{3}$$

to verify that (1) is stronger than (2) using (2)

$$d(Tx, Ty) \leq m(x, y) - (1 - k)m(x, y)$$

which is the special case of (1) with ψ the identity function and ϕ defined by $\phi(t) = (1 - k)t$ and (3) is derived from (1) by taking ψ the identity function and $\phi(t) = 0$.

Theorem 1.5. Let (X,d) be a complete metric space, T a genelalised weakly contractive self-mapping of X. Then T has a unique fixed point.

2. Main Results

Theorem 2.1. Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfies

$$\psi(d(Tx, Ty)) \leq \phi(max[d(x, Tx), d((y, Ty)]) - \theta(max[d(x, Tx), d((y, Ty)]) \tag{4}$$

where ψ is a altering distance function and $\phi, \theta \in \Theta$ and for all $x, y \in X$,

$$\psi(x) \leq \phi(y) \iff x \leq y \tag{5}$$

for any sequence $\{x_n\}$ in $[o, \infty)$ with $\{x_n\} \to t > 0$

$$\psi(t) - \overline{\lim}\phi(x_n) + \underline{\lim}\theta(x_n) > 0 \tag{6}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$, we define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ fo all $n \geq 0$. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T, hence we shall assume that $x_N \neq x_{N+1}$ for all $n \geq 0$. Now substituting $x = x_{n+1}$ and $y = x_n$ in (4) we obtain

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) \leq \phi(&max[d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})]) \\ &- \theta(max[d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})]) \end{aligned} \tag{7}$$

Suppose $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some positive integer $n \geq 0$ then

$$\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_{n+1}, x_{n+2})) - \theta(d(x_{n+1}, x_{n+2}))$$

from (5) it follows that $\theta(d((x_{n+1}, x_{n+2})) \leq 0$, which implies that $d((x_{n+1}, x_{n+2}) = 0$, contradicting our assumption that $x_N \neq x_{N+1}$ for each n. Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \geq 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real number. Let $R_n = d(x_n, x_{n+1})$. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = r \tag{8}$$

in view of the above fact, from (7) we have for all $n \geq 0$,

$$\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_n, x_{n+1})) - \theta(d(x_n, x_{n+1}) \tag{9}$$

i.e.

$$\psi(d(Ty, x_{n+2})) \leq \phi(d(y, Ty) - \theta(d(y, Ty) \tag{10}$$

Taking limit supremum on both sides of (9) using (8) and the continuity property of ψ and (i_α) property of ϕ, θ we obtain

$$\psi(r) \leq \overline{\lim}\phi(R_n) + \overline{\lim}(-\theta(R_n))$$

since $\overline{\lim}(-\theta(R_n)) = -\underline{\lim}\theta(R_n)$, it follows that

$$\psi(r) \leq \overline{\lim}\phi(R_n) - \underline{\lim}\theta(R_n)$$

That is

$$\psi(r) - \overline{\lim}\phi(R_n) + \underline{\lim}\theta(R_n) \leq 0$$

Which by (6) is a contraction unless $r = 0$, therefore

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty \tag{11}$$

Next we show that $\{x_n\}$ is a cauchy sequence. Suppose that $\{x_n\}$ is not a cauchy scquence. Then there exits an $\epsilon > 0$ for which we can find two sequences of positive integer $\{m(k)\}$, $\{n(k)\}$ such that for all positive integer k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer we get
$(n(k) > m(k) > k)$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and
$d(x_{m(k)}, x_{n(k)-1}) < \epsilon$
now,

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})$$

That is

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq \epsilon + d(x_{n(k)-1}, x_{n(k)})$$

Letting $k \to \infty$ in the above inequlity and using (11) we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{12}$$

Again,

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})$$

Letting $k \to \infty$ in the above inequality using (11) and (12) we have

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon \tag{13}$$

As $n(k) > m(k)$ so $d(x_{m(k)}, x_{n(k)}) > d(x_{m(k)+1}, x_{n(k)+1})$.

For $x = x_{n(k)}$, $y = x_{m(k)}$, we have from (4)

$$\begin{aligned}\psi(d(x_{n(k)+1}, x_{m(k)+1}) &\leq \phi(max[(d(x_{n(k)}, x_{n(k)+1}), (d(x_{m(k)}, x_{m(k)+1})]) \\ &\quad - \theta(max[(d(x_{n(k)}, x_{n(k)+1}), (d(x_{m(k)}, x_{m(k)+1})])\end{aligned}$$

i.e.

$$\psi(d(x_{n(k)+1}, x_{m(k)+1}) \leq \phi(d(x_{m(k)}, x_{m(k)+1}) - \theta(d(x_{m(k)}, x_{m(k)+1})$$

Taking limit supremum in both sides of the above inequality, using (11) and the continuity property of ψ and (i_α) property of ϕ, θ we get

$$\psi(\epsilon) \leq \overline{\lim}\phi(d(x_{m(k)}, x_{m(k)+1})) + \overline{\lim}(-\theta(d(x_{m(k)}, x_{m(k)+1})))$$

since

$$\overline{\lim}(-\theta(d(x_{m(k)}, x_{m(k)+1}))) = -\underline{\lim}(\theta(d(x_{m(k)}, x_{m(k)+1}))$$

$$\psi(\epsilon) \leq \overline{\lim}\phi(d(x_{m(k)}, x_{m(k)+1})) - \underline{\lim}(\theta(d(x_{m(k)}, x_{m(k)+1}))$$

i.e.

$$\psi(\epsilon) - \overline{\lim}\phi(d(x_{m(k)}, x_{m(k)+1})) + \underline{\lim}(\theta(d(x_{m(k)}, x_{m(k)+1})) \leq 0$$

Which is a contradiction by (6). Therefore $\{x_n\}$ is a cauchy sequence in X.
From the compactness of X, there exist $p \in X$ such that

$x_n \to p$ as $n \to \infty$

Now in (10) We put $y = p$ then we get

$$\psi(d(Tp, x_{n+2})) \leq \phi(d(p, Tp)) - \theta(d(p, Tp))$$

Letting $n \to \infty$, we get

$$\psi(d(Tp, p)) \leq \phi(d(p, Tp)) - \theta(d(p, Tp))$$

from (5) it follows that $\theta(d(p, Tp)) \leq 0$ which implies that

$$d(p, Tp) \leq 0$$

i.e.

$$Tp = p$$

If there exists another point $q \in X$ such that $Tq = q$. Then using an argument similar to the above we get

$$\psi(d(p, q)) = \psi(d(Tp, Tq)) \leq \phi(max[d(p, Tp), d(q, Tq)]) \\ - \theta(max[d(p, Tp), d(q, Tq)])$$

Since both $d(p.Tp) = 0$ and $d(q, Tq) = 0$ we get

$$\psi(d(p, q)) \leq 0$$

i.e.

$$d(p, q) = 0$$

Hence $p = q$. The proof is completed. ∎

3. Example

Let $X = [0, 1, 2, 3, \cdots\cdots)$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space.

Let $T : X \to X$ is given by $Tx = \frac{x}{2}$

and $\psi, \phi, \theta \in [0, \infty) \to [0, \infty)$ is given by

$$\psi(t) = \frac{t}{2}, \quad \theta(t) = \frac{[t]}{5}, \quad \phi(t) = \frac{[t]}{2} \quad \text{for all} \quad t \in X \\ = [t], \quad \text{otherwise},$$

Hence all the condition of theorem 2.1 are satisfied. Here it is seen that 0 is the unique fixed point of T.

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