(ψ, φ, θ)-weak contraction of continuous and discontinuous functions

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Abstract
We prove weak contraction theorem using three control functions in which one is continuous and other two are discontinuous.

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1. Introduction

some definitions and results are as follows:

Definition 1.1. A mapping \( T : X \rightarrow X \), where \((X,d)\) is a metric space is said to be weakly contractive if \( x, y \in X \)

\[ d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \]

where \( \phi : [0, \infty) \rightarrow [0, \infty) \) is a continuous and nondecreasing function such that \( \phi(t) = 0 \) if and only if \( t = 0 \).

Theorem 1.2. Let \((X, d)\) be a complete metric space, \( T \) a weakly contractive mapping then \( T \) has a unique fixed point \( p \) in \( X \).

Definition 1.3. (Altering distance function) A function \( \psi : [0, \infty) \rightarrow [0, \infty) \) is said to be altering distance function if the following properties are satisfied
(i) \( \phi \) is monotone nondecreasing and continuous

(ii) \( \phi(t) = 0 \) if and only if \( t = 0 \).

Let by \( \Theta \) we denote the set of all functions \( \alpha : [0, \infty) \rightarrow [0, \infty) \) such that

(i) \( \alpha \) is bounded on any bounded interval in \( [0, \infty) \)

(ii) \( \alpha \) is continuous at 0 and \( \alpha(0) = 0 \).

**Definition 1.4.** Let \((X,d)\) be a metric space, \( T \) a self mapping of \( X \), we shall call \( T \) a generalised weak contractive mapping if for all \( x, y \in X \)

\[
\psi(d(Tx, Ty)) \leq \psi(max\{d(x, y), d(y, Ty)\}) - \phi(max\{d(x, y), d(y, Ty)\})
\]

where

\[
m(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}
\]

\( \psi \) is an altering distance function and \( \phi : [0, \infty) \rightarrow [0, \infty) \) is a continuous function

A general weak contraction is more general than that satisfying

\[
d(Tx, Ty) \leq km(x, y)
\]

for some \( 0 < k < 1 \), and is included in those mapping which satisfies

\[
d(Tx, Ty) \leq m(x, y)
\]

to verify that (1) is stronger than (2) using (2)

\[
d(Tx, Ty) \leq m(x, y) - (1 - k)m(x, y)
\]

which is the special case of (1) with \( \psi \) the identity function and \( \phi \) defined by \( \phi(t) = (1 - k)t \) and (3) is derived from (1) by taking \( \psi \) the identity function and \( \phi(t) = 0 \).

**Theorem 1.5.** Let \((X,d)\) be a complete metric space, \( T \) a generalised weakly contractive self-mapping of \( X \). Then \( T \) has a unique fixed point.

2. Main Results

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and \( T : X \rightarrow X \) be a self mapping satisfies

\[
\psi(d(Tx, Ty)) \leq \phi(max\{d(x, Tx), d((y, Ty)\}) - \theta(max\{d(x, Tx), d((y, Ty)\})
\]

where \( \psi \) is a altering distance function and \( \phi, \theta \in \Theta \) and for all \( x, y \in X \),

\[
\psi(x) \leq \phi(y) \iff x \leq y
\]
for any sequence \( \{x_n\} \) in \([0, \infty)\) with \( \{x_n\} \to t > 0 \)

\[
\psi(t) - \limsup \phi(x_n) + \lim \theta(x_n) > 0
\]

(6)

Then \( T \) has a unique fixed point.

**Proof.** Let \( x_0 \in X \), we define a sequence \( \{x_n\} \) in \( X \) such that \( x_{n+1} = Tx_n \) for all \( n \geq 0 \). If there exists a positive integer \( N \) such that \( x_N = x_{N+1} \), then \( x_N \) is a fixed point of \( T \), hence we shall assume that \( x_N \neq x_{N+1} \) for all \( n \geq 0 \). Now substituting \( x = x_{n+1} \) and \( y = x_n \) in (4) we obtain

\[
\psi(d(x_{n+1}, x_{n+2})) \leq \phi(\max[d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})]) - \theta(\max[d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})])
\]

(7)

Suppose \( d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2}) \) for some positive integer \( n \geq 0 \) then

\[
\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_{n+1}, x_{n+2})) - \theta(d(x_{n+1}, x_{n+2}))
\]

from (5) it follows that \( \theta(d((x_{n+1}, x_{n+2})) \leq 0 \), which implies that \( d((x_{n+1}, x_{n+2}) = 0 \), contradicting our assumption that \( x_N \neq x_{N+1} \) for each \( n \). Therefore \( d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \) for all \( n \geq 0 \) and \( \{d(x_n, x_{n+1})\} \) is a monotone decreasing sequence of non-negative real number. Let \( R_n = d(x_n, x_{n+1}) \). Hence there exists an \( r \geq 0 \) such that

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} d(x_n, x_{n+1}) = r
\]

(8)

in view of the above fact, from (7) we have for all \( n \geq 0 \),

\[
\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_n, x_{n+1})) - \theta(d(x_n, x_{n+1})
\]

(9)

i.e.

\[
\psi(d(Ty, x_{n+2})) \leq \phi(d(y, Ty)) - \theta(d(y, Ty))
\]

(10)

Taking limit supremum on both sides of (9) using (8) and the continuity property of \( \psi \) and \((i_{\alpha})\) property of \( \phi, \theta \) we obtain

\[
\psi(r) \leq \limsup \phi(R_n) + \lim \theta(-R_n)
\]

since \( \lim(-\theta(R_n)) = -\lim \theta(R_n) \), it follows that

\[
\psi(r) \leq \lim \phi(R_n) - \lim \theta(R_n)
\]

That is

\[
\psi(r) - \lim \phi(R_n) + \lim \theta(R_n) \leq 0
\]

Which by (6) is a contraction unless \( r = 0 \), therefore

\[
d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty
\]

(11)
Next we show that \( \{x_n\} \) is a cauchy sequence. Suppose that \( \{x_n\} \) is not a cauchy sequence. Then there exists an \( \epsilon > 0 \) for which we can find two sequences of positive integer \( \{m(k)\} \), \( \{n(k)\} \) such that for all positive integer \( k, n(k) > m(k) > k \) and \( d(x_{m(k)}, x_{n(k)}) \geq \epsilon \).

Assuming that \( n(k) \) is the smallest such positive integer we get \( n(k) > m(k) > k \).

Letting \( k \to \infty \) in the above inequality and using \( (11) \) we have
\[
\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{12}
\]

Again,
\[
d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})
\]

Letting \( k \to \infty \) in the above inequality using \( (11) \) and \( (12) \) we have
\[
\lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon \tag{13}
\]

As \( n(k) > m(k) \) so \( d(x_{m(k)}, x_{n(k)}) > d(x_{m(k)+1}, x_{n(k)+1}) \).

For \( x = x_{n(k)}, y = x_{m(k)} \), we have from \( (4) \)
\[
\psi(d(x_{n(k)+1}, x_{m(k)+1}) \leq \phi(max\{d(x_{n(k)}, x_{n(k)+1}), (d(x_{m(k)}, x_{m(k)+1}))
\]
\[
- \theta(max\{d(x_{n(k)}, x_{n(k)+1}), (d(x_{m(k)}, x_{m(k)+1}))
\]

i.e.
\[
\psi(d(x_{n(k)+1}, x_{m(k)+1}) \leq \phi(d(x_{m(k)}, x_{m(k)+1})) - \theta(d(x_{m(k)}, x_{m(k)+1}))
\]

Taking limit supremum in both sides of the above inequality, using \( (11) \) and the continuity property of \( \psi \) and \( (i_\alpha) \) property of \( \phi, \theta \) we get
\[
\psi(\epsilon) \leq \limsup \phi(d(x_{m(k)}, x_{m(k)+1})) + \liminf(-\theta(d(x_{m(k)}, x_{m(k)+1})))
\]

since
\[
\liminf(-\theta(d(x_{m(k)}, x_{m(k)+1}))) = -\limsup(\theta(d(x_{m(k)}, x_{m(k)+1})))
\]
\[
\psi(\epsilon) \leq \limsup \phi(d(x_{m(k)}, x_{m(k)+1})) - \limsup(\theta(d(x_{m(k)}, x_{m(k)+1})))
\]

i.e.
\[
\psi(\epsilon) - \limsup \phi(d(x_{m(k)}, x_{m(k)+1})) + \limsup(\theta(d(x_{m(k)}, x_{m(k)+1}))) \leq 0
\]

Which is a contradiction by \( (6) \). Therefore \( \{x_n\} \) is a cauchy sequence in \( X \).

From the compactness of \( X \), there exist \( p \in X \) such that
$x_n \to p$ as $n \to \infty$

Now in (10) We put $y = p$ then we get

$$\psi(d(Tp, x_{n+2})) \leq \phi(d(p, Tp)) - \theta(d(p, Tp))$$

Letting $n \to \infty$, we get

$$\psi(d(Tp, p)) \leq \phi(d(p, Tp)) - \theta(d(p, Tp))$$

from (5) it follows that $\theta(d(p, Tp)) \leq 0$ which implies that

$$d(p, Tp) \leq 0$$

i.e.

$$Tp = p$$

If there exists another point $q \in X$ such that $Tq = q$. Then using an argument similar to the above we get

$$\psi(d(p, q)) = \psi(d(Tp, Tq)) \leq \phi(\max[d(p, Tp), d(q, Tq)]) - \theta(\max[d(p, Tp), d(q, Tq)])$$

Since both $d(p, Tp) = 0$ and $d(q, Tq) = 0$ we get

$$\psi(d(p, q)) \leq 0$$

i.e.

$$d(p, q) = 0$$

Hence $p = q$. The proof is completed.

3. Example

Let $X = [0, 1, 2, 3, \ldots]$ and $d(x, y) = |x - y|$. Then $(X, d)$ is a complete metric space.

Let $T : X \to X$ is given by $Tx = \frac{x}{2}$ and $\psi, \phi, \theta \in [0, \infty) \to [0, \infty)$ is given by

$$\psi(t) = \frac{t}{2}, \quad \theta(t) = \frac{[t]}{5}, \quad \phi(t) = \frac{[t]}{2} \quad \text{for all } t \in X$$

$$= [t], \quad \text{otherwise},$$

Hence all the condition of theorem 2.1 are satisfied. Here it is seen that 0 is the unique fixed point of $T$. 

$\blacksquare$
References


