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Generalized α-ψ-Proximal Contraction Mapping and Related Best Proximity Point

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Abstract

We establish best proximity point theorem generalized α-ψ-proximal contraction complete metric spaces. Our results generalize the various results from the literature. We also give an application. As an application we deduce best proximity point and fixed point results in metric spaces.

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1. Introduction

The Banach contraction principle is fundamental result in fixed point theory which has various nontrivial implications in many branches of pure and applied sciences. The Banach contraction principle states that if (X, d) be complete metric space and $x \in X$. A self mapping $T : X \to X$ is a contraction, then T has a unique fixed point. This result was extended to other important classes of mappings and has numerous applications. Important and interesting generalizations of Banach contraction principle can be found in the literature. However, in the case that T is not a self-mapping, the aforesaid equation does not necessarily admit a solution. Eventually, in such scenarios, one probes into the existence of an approximate solution y, designated as an optimal approximate solution, with the least possible error $d(y, Ty)$, where T is a non-self mapping from A to B. Indeed, an approximate solution to the equation $Tx = x$ with the least possible error is a solution of the non-linear programming problem $\min_{x \in A} d(x, Tx)$ and vice versa. In light of the fact that d(x, Tx) is at least d(A,B) for all x in A, the least error that one can think of for an approximate solution of the equation Tx = x is d(A,B). Essentially, if d(y,Ty) = d(A,B), then y is a solution of the preceding nonlinear programming problem with the global minimum value d(A,B) and hence an approximate solution of the equation Tx = x with the least possible error. Such a common solution of the preceding approximation and optimization problems is known as a best proximity point of the mapping T. The results that explore the sufficient conditions for the existence of a best proximity point are called best proximity point theorems. Several best proximity point theorems for various classes of mappings can be found in [1, 2, 3, 4, 5, 7, 8, 9, 12, 11, 13, 15, 17, 21, 18, 22, 23, 24, 25, 26, 27]. The aim of this paper is to focus on best proximity point theorems for proximal contraction of non-self mapping. Also, necessary and sufficient conditions are established for a non-self contraction mapping to have a best proximity point and we also give application to illustrates our results. The results of this paper are extension and generalizations of some results in the literature.

2. Preliminaries

Before proceeding further let us fix the following notations: Let A and B are nonempty subsets of a metric space X,

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x, y) = d(A, B) \quad for\ some\ y \in B\}$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \quad for\ some\ x \in A\}$$

Definition 2.1. An element $x \in A$ is said to be a best proximity point of the non-self mapping $T : A \to B$ if

$$d(x, Tx) = d(A, B).$$

Because of the fact that $d(x, Tx) \geq d(A, B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto (x, Tx)$ is attained at a best proximity point. Moreover, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point.

Definition 2.2. Let the mapping $g : A \to A$ is an isometry if

$$d(gx, gy) = d(x, y)$$

for any $x, y \in A$.

Definition 2.3. [19] Let a mapping $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve the isometric distance with respect to g if for any $x, y \in A$, one has

$$d(T(gx), T(gy)) = d(Tx, Ty).$$

The notion of P-property was introduced in [10] as follows.

Definition 2.4. ([10]) Let (A, B) be a pair of nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if

$$d(x_1, y_1) = dist(A, B) \quad d(x_2, y_2) = dist(A, B) \quad \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

In 2012 Samet et al. [6] first introduced the following concepts.

Definition 2.5. [6] Let (X,d) be a complete metric space and $T : X \to X$ is said to be α-ψ-contractive mapping if

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$, where functions $\alpha : X \times X \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ is nondecreasing function such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$.

Definition 2.6. [6] Let T be a self mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. A mapping T is an α-admissible if

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

In 2013 Salimi et al. [20] generalized the concepts of α-admissible mapping as follows.

Definition 2.7. [20] Let T be a self mapping on a metric space (X,d) and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. A mapping T is an α-admissible with respect to η if

$$x, y \in X, \qquad \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Here, if we take $\eta(x, y) = 1$, then this definition reduced to Definition 2.6.

Definition 2.8. [14] Let (X, d) be a metric space and A and B are two non empty subsets of X. Let $T : A \to B$ be a non-self mapping is called α-proximal admissible if

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases} \Longrightarrow \alpha(u, v) \geq 1$$

for all $x, y, u, v \in A$, where $\alpha : A \times A \to [0, \infty)$.

Here, if we take A = B, then we obtain T is α-admissible mapping. Later, Hussain et al. [16] generalized the concept of α-proximal admissible.

Definition 2.9. [16] Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. We say that T is α-proximal admissible with respect to η if, for all $x, y, u, v \in A$,

$$\begin{cases} \alpha(x, y) \geq \eta(x, y), \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases} \Longrightarrow \alpha(u, v) \geq \eta(u, v)$$

Note that if we take $\eta(x, y) = 1$ for all $x, y \in A$, then this definition reduces to Definition 2.8. In case $\alpha(x, y) = 1$ for all $x, y \in A$, then we shall say that T is η-proximal sub-admissible mapping. Clearly, if A = B, then the definition reduces to Definition 2.7.

3. Main Results

Set $G = \{\psi : [0, \infty) \to [0, \infty) : \psi$ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$.
In this section first we give some generalized α-proximal contraction with respect to η.

Definition 3.1. Let (X,d) be a metric and A, B be two nonempty subsets of X. Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. We say that T is generalized α-ψ-proximal contraction with respect to η if, for all $x, y \in A$ and ψ in G such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)), \tag{1}$$

where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B) \right\}$$

Definition 3.2. Let (X,d) be a metric and A, B be two nonempty subsets of X. Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. We say that T is generalized α-ψ-proximal contraction with respect to η if, for all $x, y, u, v \in A$ and $\psi \in G$ such that

$$\begin{cases} \alpha(x, y) \geq \eta(x, y), \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases} \Longrightarrow d(u, v) \leq \psi(M(x, y)).$$

Following is the main result of this paper.

Theorem 3.3. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction with respect to η;

(ii) T is continuous;

(iii) g is an isometry;

(iv) $A_0 \subseteq g(A_0)$;

(v) $T(A_0) \subseteq B_0$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1)$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \alpha(gy, gz) \geq \eta(gy, gz).$$

Proof. Choose $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, gx_1) \geq \eta(x_0, gx_1) \tag{2}$$

Again, choose a point $x_1 \in A_0$ there exists $x_2 \in A_0$

$$d(gx_2, Tx_1) = d(A, B) \tag{3}$$

From (2), (3) and from the fact that T is an α-proximal admissible with respect to η and $\alpha(x_1, gx_2) \geq \eta(x_0, gx_1)$. By repeating this process, having chosen by induction $\{x_n\} \in A_0$ such that,

$$\alpha(gx_{n-1}, gx_n) \geq \eta(gx_{n-1}, gx_n) \tag{4}$$

$$d(gx_n, Tx_{n-1}) = d(A, B) \tag{5}$$

$$d(gx_{n+1}, Tx_n) = d(A, B) \tag{6}$$

for all $n \in \mathbb{N}$. Since T is a generalized $\alpha - \psi -$ proximal contraction type mapping, we have

$$d(x_{n+1}, x_n) \leq d(gx_{n+1}, gx_n) \leq \psi(M(x_n, x_{n-1})) \tag{7}$$

for all $n \in \mathbb{N}$.

On the other hand, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} - d(A, B), \\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2} - d(A, B)\Big\} \\ &\leq \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} - d(A, B), \\ &\quad \frac{d(x_{n-1}, x_{n+1})}{2} - d(A, B)\Big\} \\ &\leq \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} - d(A, B)\Big\} \\ &\leq \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} \end{aligned}$$

which implies that

$$d(x_{n+1}, x_n) \leq \psi\big(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\big).$$

$$If \ \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}) \ \forall\, n \in \mathbb{N}$$

$$d(x_{n+1}, x_n) \leq \psi\big(d(x_{n+1}, x_n)\big) < d(x_{n+1}, x_n)$$

which is a contradiction. Thus we conclude that

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n) \ \forall\, n \in \mathbb{N}$$

$$d(x_{n+1}, x_n) \leq \psi\big(d(x_{n-1}, x_n)\big) \ \forall\, n \in \mathbb{N}.$$

Since ψ is nondecreasing, we get by induction

$$d(x_{n+1}, x_n) \leq \psi^n\big(d(x_1, x_0)\big) \ \forall\, n \in \mathbb{N}.$$

By the of ψ, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Now we shall prove that $\{x_n\}$ is a cauchy sequence. If $\epsilon > 0$ there exists $h = h(\epsilon) \in \mathbb{N}$ such that

$$d(x_{n+1}, x_n) \leq \epsilon - \psi(\epsilon) \ for\ all\ n \geq h(\epsilon).$$

Let we fix $m \geq h(\epsilon)$

$$d(x_{m+1}, x_m) \leq \epsilon - \psi(\epsilon)$$

for some $n \geq m$ therefore $d(x_m, x_{n+1}) < \epsilon$ we have,

$$\begin{aligned} d(x_m, x_{n+2}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}x_{n+2}) \\ &\leq d(x_m, x_{m+1}) + d(gx_{m+1}, gx_{n+2}) \\ &\leq d(x_m, x_{m+1}) + \psi\big(M(x_m, x_{n+1})\big) \\ &\leq \epsilon - \psi(\epsilon) + \psi\big(d(x_m, x_{n+1})\big) \\ &\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \\ d(x_m, x_{n+2}) &< \epsilon \end{aligned}$$

$$\lim_{n\to\infty} d(x_m, x_{n+1}) = 0.$$

Therefore x_n is a Cauchy sequence. By the completeness of X and A_0 is closed, we have $x \in A_0$ such that $x_n \to x$. Moreover by the continuity of g, we have $gx_n \to gx$. Thus $gx \in A_0$, since $gx_n \in A_0$, for all $n \in \mathbb{N}$. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$ there exists $z \in A_0$ such that

$$d(z, Tx) = d(A, B) \qquad and \qquad \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$ and $x_n \to z$ as $n \to \infty$, we get

$$\alpha(x_n, z) \geq \eta(x_n, z) \; for \; all \; n \in \mathbb{N}$$

$$\begin{aligned} d(gx_{n+1}, z) &\leq \psi(M(x_n, x)) \\ &< M(x_n, x) \end{aligned}$$

for all $n \in \mathbb{N}$ and taking as $n \to \infty$ then $d(z, gx_{n+1}) \to 0$ and $z = gx$ which implies

$$d(gx, Tx) = d(A, B).$$

■

Now, we drop the continuity condition of T and prove the Theorem 3.3 without continuity.

Theorem 3.4. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction with respect to η;

(ii) g is an isometry;

(iii) $A_0 \subseteq g(A_0)$;

(iv) $T(A_0) \subseteq B_0$;

(v) if $\{x_n\}$ is a sequence in A_0 such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq \eta(x_n, gx)$ for all $n \in \mathbb{N}$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1)$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \alpha(gy, gz) \geq \eta(gy, gz).$$

Proof. Following the proof of Theorem 3.3, we can choose a sequence $\{x_n\} \subseteq A_0$ such that

$$\alpha(gx_{n+1}, gx_n) \geq \eta(gx_{n+1}, gx_n)$$
$$d(gx_{n+1}, Tx_n) = d(A, B) \tag{8}$$

for all $n \in \mathbb{N}$. Also, $\{x_n\}$ is a Cauchy sequence and $\{x_n\}$ is closed in complete metric spaces X. There exists $x \in A$ such that $x_n \to x \; as \; n \to \infty$. Since $T(A_0) \subseteq B_0$ and B is closed then we can choose $\{Tx_n\}$ be a sequence in B and $\{Tx_{n_k}\}$ is a convergent subsequence of $\{Tx_n\}$ there exists some $b \in B$ such that

$$\lim_{k\to\infty} d(Tx_{n_k}, b) = 0, \tag{9}$$

we have

$$d(A, B) = \lim_{k\to\infty} d(gx_{n_k+1}, Tx_{n_k}) = d(gx, b). \tag{10}$$

Therefore, we have that

$$d(gz, Tx) = d(A, B) \qquad \text{for some } z \in A_0 \tag{11}$$

here $Tx \in B_0$, g is an isometry. From assumption (v), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(gx_{n_k}, gx) \geq \eta(gx_{n_k}, gx)$ for all $k \in \mathbb{N}$. Since T is a generalized $\alpha - \psi$-proximal contraction type mapping and g is an isometry, we have

$$d(x_{n_k+1}, z) \leq d(gx_{n_k+1}, gz) \leq \psi\big(M(x_{n_k}, x)\big) \tag{12}$$

for all $k \in \mathbb{N}$, On the other hand,

$$\begin{aligned} M(x_{n_k}, x) &= \max\Big\{d(x_{n_k}, x), \frac{d(x_{n_k}, Tx_{n_k}) + d(x, Tx)}{2} - d(A, B), \\ &\quad \frac{d(x_{n_k}, Tx) + d(x, Tx_{n_k})}{2} - d(A, B)\Big\} \\ &= \max\Big\{d(x_{n_k}, x), 0, \frac{d(x_{n_k}, x) + d(x, Tx) + d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k})}{2} \\ &\quad -d(A, B)\Big\} \\ &= \max\Big\{d(x_{n_k}, x), 0, d(x_{n_k}, x)\Big\} \end{aligned}$$

in the above equality, we have

$$M(x_{n_k}, z) = d(x_{n_k}, x). \tag{13}$$

Further, $d(x_{n_k+1}, z) < M(x_{n_k}, x)$, letting $k \to \infty$

$$d(x, z) \leq d(z, x). \tag{14}$$

which is contradiction and this implies that $d(x, z) = 0$ that is $z = x$. From (11), we have $d(gx, Tx) = d(A, B)$, since $A_0 \subseteq g(A_0)$ and g is an isometry. Hence x is a best proximity point of T. ■

4. Corollaries

Taking $\eta(x, y) = 1$ in Theorem 3.3 and Theorem 3.4, we obtain the following results.

Corollary 4.1. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction;

(ii) T is continuous;

(iii) $T(A_0) \subseteq B_0$;

(iv) g is an isometry;

(v) $A_0 \subseteq g(A_0)$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq 1$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \alpha(gy, gz) \geq 1.$$

Corollary 4.2. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction;

(ii) $T(A_0) \subseteq B_0$;

(iii) g is an isometry;

(iv) $A_0 \subseteq g(A_0)$;

(v) if $\{x_n\}$ is a sequence in A_0 such that $\alpha(gx_n, x_{n+1}) \geq 1$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq 1$ for all $n \in \mathbb{N}$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq 1.$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \alpha(gy, gz) \geq 1.$$

Taking $\alpha(x, y) = 1$ in Theorem 3.3 and Theorem 3.4, we obtain the following results.

Corollary 4.3. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is ψ-proximal contraction;

(ii) T is continuous;

(iii) $T(A_0) \subseteq B_0$ and T is η-subadmissible ;

(iv) g is an isometry ;

(v) $A_0 \subseteq g(A_0)$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \eta(x_0, x_1) \leq 1$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \eta(gy, gz) \leq 1.$$

Corollary 4.4. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is ψ-proximal contraction;

(ii) g is an isometry;

(iii) $A_0 \subseteq g(A_0)$;

(iv) $T(A_0) \subseteq B_0$ and T is η-subadmissible;

(v) if $\{x_n\}$ is a sequence in A_0 such that $\eta(gx_n, x_{n+1}) \leq 1$ and $gx_n \to gx \in A$, then $\eta(x_n, gx) \leq 1$ for all $n \in \mathbb{N}$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \eta(x_0, x_1) \leq 1.$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(gy, Ty) = d(A, B) = d(gz, Tz) \qquad \text{and} \qquad \eta(gy, gz) \leq 1.$$

If g is the identity mapping in Theorem 3.3, then we obtain the following.

Corollary 4.5. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction with respect to η;

(ii) T is continuous;

(iii) $T(A_0) \subseteq B_0$;

(iv) there exists $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1)$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(y, Ty) = d(A, B) = d(z, Tz) \qquad \text{and} \qquad \alpha(y, z) \geq \eta(y, z).$$

If g is the identity mapping in Theorem 3.4, then we obtain the following.

Corollary 4.6. Let A and B be a two non-empty closed subsets of a complete metric space (X, d) such that A_0 and B_0 are non-empty. Let the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:

(i) T is α-ψ-proximal contraction with respect to η;

(iv) $T(A_0) \subseteq B_0$;

(v) if $\{x_n\}$ is a sequence in A_0 such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x \in A$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$;

(vi) there exists $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \qquad \text{and} \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1)$$

Then T has a unique best proximity point if for every $y, z \in A$ such that

$$d(y, Ty) = d(A, B) = d(z, Tz) \qquad \text{and} \qquad \alpha(y, z) \geq \eta(y, z).$$

5. Applications

An application of our results, we give some new fixed point theorems which can be deduced from our results. In Theorem 3.3 and Theorem 3.4, if we take A = B then we deduce the following result.

Theorem 5.1. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is continuous α-admissible with respect to η and let $g : A \to A$ is an isometry and $A \subseteq g(A)$ such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)), \tag{15}$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Then T has a unique fixed point.

Theorem 5.2. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is α-admissible with respect to η such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)), \tag{16}$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Let $g : A \to A$ is an isometry and $A \subseteq g(A)$. If $\{x_n\}$ is a sequence in A such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq \eta(x_n, gx)$ for all $n \in \mathbb{N}$. Then T has a unique fixed point.

If we take $\psi(t) = kt$ in Theorem 5.1 and Theorem 5.2, where $0 \leq k < 1$, then we conclude the following theorems.

Theorem 5.3. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is continuous α-admissible with respect to η and let $g : A \to A$ is an isometry and $A \subseteq g(A)$ such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq kM(x, y), \tag{17}$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Then T has a unique fixed point.

Theorem 5.4. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is α-admissible with respect to η such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq kM(x, y), \tag{18}$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Let $g : A \to A$ is an isometry and $A \subseteq g(A)$. If $\{x_n\}$ is a sequence in A such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq \eta(x_n, gx)$ for all $n \in \mathbb{N}$. Then T has a unique fixed point.

If M(x,y) = d(x,y), then Theorem 5.1 and Theorem 5.2 includes the following.

Theorem 5.5. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is continuous α-admissible with respect to η and let $g : A \to A$ is an isometry and $A \subseteq g(A)$ such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(d(x, y)), \tag{19}$$

Then T has a unique fixed point.

Theorem 5.6. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is α-admissible with respect to η such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(d(x, y)). \tag{20}$$

Let $g : A \to A$ is an isometry and $A \subseteq g(A)$. If $\{x_n\}$ is a sequence in A such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq \eta(x_n, gx)$ for all $n \in \mathbb{N}$. Then T has a unique fixed point.

If M(x,y) = d(x,y) in Theorem 5.3 and Theorem 5.4, we obtain the following.

Theorem 5.7. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is continuous α-admissible with respect to η and let $g : A \to A$ is an isometry and $A \subseteq g(A)$ such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq kd(x, y). \tag{21}$$

Then T has a unique fixed point.

Theorem 5.8. Let (X, d) be a complete metric space and $A \subseteq X$. Let the mapping $T : A \to A$ is α-admissible with respect to η such that

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq kd(x, y). \tag{22}$$

Let $g : A \to A$ is an isometry and $A \subseteq g(A)$. If $\{x_n\}$ is a sequence in A such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \to gx \in A$, then $\alpha(x_n, gx) \geq \eta(x_n, gx)$ for all $n \in \mathbb{N}$. Then T has a unique fixed point.

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