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Positive solutions of singular multi-point boundary value problem for discrete system with a parameter

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Abstract

This paper is concerned with the existence of one or more positive solutions for a second order discrete system

$$\Delta^2 y_i(k-1) + \lambda a_i(k) f_i(y_1(k), y_2(k), \cdots, y_n(k)) + \lambda e_i(k) = 0,$$
$$k \in \mathbb{N}_{1,T}, \quad i = 1, 2, \cdots, n,$$
$$y_i(0) = 0, \quad y_i(T+1) = \alpha \sum_{l=1}^{m} y(l), \; i = 1, 2, \cdots, n.$$

By using the Krasnoselskii's fixed point theorem in cone, sufficient conditions are obtained for the existence for the above system.

AMS subject classification: 34B16, 39A10.
Keywords: Krasnosel'skii Fixed point theorem, positive solutions, m-point boundary value problem.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \prod_{i=1}^{n} \mathbb{R}_+$, $\mathbb{N}$ be the nonnegative integers, we let $\mathbb{N}_{a,b} := [a, b] \cap \mathbb{N}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. In this paper we consider the following second-order discrete system

$$\Delta^2 y_i(k-1) + \lambda a_i(k) f_i(y_1(k), y_2(k), \cdots, y_n(k)) + \lambda e_i(k) = 0, \quad (1.1)$$
$$k \in \mathbb{N}_{1,T}, \quad i = 1, 2, \cdots, n,$$
$$y_i(0) = 0, \quad y_i(T+1) = \alpha \sum_{l=1}^{m} y(l), \ i = 1, 2, \cdots, n. \quad (1.2)$$

where f is singular at zero, $\lambda > 0$ is a positive parameter, $T \geq 3$ is a fixed positive integers, $m \in \{1, 2, \cdots, T-1\}$, $0 < \alpha < \frac{2T+2}{m(m+1)}$, $f_i \in C(\mathbb{R}_+^n \setminus \{0\}, (0, +\infty))$ is continuous for $i = 1, \cdots, n$, $\Delta y(k-1) = y(k) - y(k-1)$ and $\Delta^2 y(k-1) = \Delta(\Delta y(k-1))$.

Owing to its importance in physics and engineering, the existence of solutions to this problem has been studied by many authors. However, in practice, only its positive solution is significant see [1, 2, 4, 3, 6, 12, 15, 16, 13, 14].

In [3], Cheung et al. studied the second-order difference equation

$$\nabla \Delta u(k) + f(k, u(k)) = 0, \quad k \in \mathbb{N}_{1,T}, \quad (1.3)$$

subject to one of the following discrete nonlocal boundary conditions:

$$\text{(i)} \quad u(0) - \beta \Delta u(0) = 0, \quad u(T+1) = \alpha u(l), \quad (1.4)$$

or

$$\text{(ii)} \quad \Delta u(0) = 0, \quad u(T+1) = \alpha u(l). \quad (1.5)$$

By using the fixed point theorem, they established sufficient conditions for the existence of multiple positive solutions for the boundary value problems (1.3) - (1.4) and (1.3) - (1.5).

Du [4] considered the following second-order discrete boundary value problem

$$\Delta^2 y(k-1) + f(k, y(k)) = 0, \ k \in \mathbb{N}_{1,T} \quad (1.6)$$
$$y(0) = 0, \ y(T+1) = \alpha y(n), \quad (1.7)$$

where f is continuous, $T \geq 3$ and $n \in \{2, \cdots, T-1\}$ are two fixed positive integers, constant $\alpha > 0$ such that $\alpha n < T + 1$. Under suitable conditions, the existence of at least three positive solutions for the discrete boundary value problem (1.6) - (1.7) by using the property of the associate Green's function and Leggett- Williams fixed point theorem has been established.

Sitthiwirattham and Reunsumrit [14] investigated the following second-order difference equation

$$\Delta^2 u(k-1) + a(k)f(k) = 0,\ k \in \mathbb{N}_{1,T}, \tag{1.8}$$

with difference- summation boundary condition

$$u(0) = \beta \Delta u(0), \quad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{1.9}$$

where f is continuous. They proved the existence of positive solutions to the problem (1.8) - (1.9) by using the Krasnoselskii fixed point theorem in cones. However, no associated Green's function was established.

The nonlinearity in the above results are nonsingular. What would happen if the nonlinearity term is singular? The existence and multiplicity of positive periodic solutions for second order differential equations has been established in an interesting paper of Wang [15] for the following problem

$$x_i'' + a_i(t)x_i = \lambda g_i(t) f_i(x) + \lambda e_i(t), \tag{1.10}$$

where $\lambda > 0$ is a positive parameter. On the existence of positive solutions for difference equations, several contributions are available in [9, 10, 11, 16]. Mohamed and Ismail [11] obtained the existence of one and more positive solutions for the following second order discrete system

$$\begin{aligned} &\Delta^2 y_i(k) + \lambda a_i(k) f_i(y_1(k), y_2(k), \cdots, y_n(k)) + \lambda e_i(k) = 0, \\ &\qquad k \in [0, T], \quad i = 1, 2, \cdots, n, \end{aligned} \tag{1.11}$$

$$y_i(0) = y_i(T+2) = 0,\ i = 1, 2, \cdots, n. \tag{1.12}$$

where $\lambda > 0$ is a positive parameter, T and $n \geq 2$ are two fixed positive integers, $f_i \in C(\mathbb{R}_+^n \setminus \{0\}, (0, +\infty))$ is continuous for $i = 1, \cdots, n$.

Motivated by the above work, we fill the gap in the literature, by considering the existence and multiplicity of positive solutions to the problem (1.1) - (1.2) via the well-known Krasnoselskii's fixed point theorem in a cone.

Throughout this paper, we suppose the following conditions hold:

(A1) $a_i : \mathbb{N}_{1,T} \to (0, +\infty)$, $e_i : \mathbb{N}_{1,T} \to (-\infty, +\infty)$ are continuous, $i = 1, 2, \cdots, n$.

(A2) $f_i : \mathbb{R}_+^n \setminus \{0\} \to (0, +\infty)$ is continuous, $i = 1, \cdots, n$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone [17].

Lemma 1.1. Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$; or

(ii) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $T : K \cap (\bar{\Omega_2} \setminus \Omega_1) \to K$.

2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. Let $\alpha \neq \dfrac{2T+2}{m(m+1)}$. Then for $y \in C(\mathbb{N}_{T+1}, [0,\infty))$, the problem

$$\Delta^2 y(k-1) + u(k) = 0, \quad k \in \mathbb{N}_{1,T}, \tag{2.1}$$

$$y_i(0) = 0, \quad y_i(T+1) = \alpha \sum_{l=1}^{m} y(n). \tag{2.2}$$

has the unique solution

$$y(k) = \sum_{l=1}^{T} G(k,l)u(l),$$

where the Green's function $G(k,l)$ is given by

$$G(k,l) = \frac{1}{H}\begin{cases} l[2T+2-2k-\alpha(m-k)(m-k+1)], & l \in \mathbb{N}_{1,k-1} \cap \mathbb{N}_{1,m-1}, \\ l(2T+2-2k) + \alpha k(k-l), & l \in \mathbb{N}_{m,k-1}, \\ k[2T+2-2l-\alpha(m-l)(m-l+1)], & l \in \mathbb{N}_{k,m-1}, \\ 2k(T+1-l), & l \in \mathbb{N}_{k,T} \cap \mathbb{N}_{m,T}. \end{cases}$$

Proof. We get

$$\begin{aligned} \Delta y(k) - \Delta y(k-1) &= -u(k), \\ \Delta y(k-1) - \Delta y(k-2) &= -u(k-1), \\ &\vdots \\ \Delta y(1) - \Delta y(0) &= -u(1), \end{aligned}$$

we sum the above equations to obtain

$$\Delta y(k) = \Delta y(0) - \sum_{l=0}^{k} u(l), \tag{2.3}$$

where and the back, we denote $\sum_{l=p}^{q} u(l) = 0$, if $p > q$. Similarly, we sum (2.3) from 0 to k and by using the boundary condition $y_i(0) = 0$ in (2.2) , we obtain

$$y(k+1) = (k+1)\Delta y(0) - \sum_{l=1}^{k}(k+1-l)u(l) \;\; k \in \mathbb{N}_T.$$

Then by changing the variable from $k+1$ to k, we have

$$y(k) = k\Delta y(0) - \sum_{l=1}^{k-1}(k-l)u(l), \quad k \in \mathbb{N}_{T+1}. \tag{2.4}$$

From (2.4),

$$\begin{aligned}\sum_{l=1}^{m} y(l) &= \frac{m(m+1)}{2}\Delta y(0) - \sum_{l=1}^{m-1}\sum_{j=1}^{m-l} ju(l)\\ &= \frac{m(m+1)}{2}\Delta y(0) - \frac{1}{2}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l).\end{aligned}$$

Again, using the boundary condition $y_i(T+1) = \alpha \sum_{l=1}^{m} y(l)$ in (2.2), we obtain

$$\begin{aligned}(T+1)\Delta y(0) - \sum_{l=1}^{T}(T+1-l)u(l) &= \frac{\alpha m(m+1)}{2}\Delta y(0)\\ &\quad - \frac{\alpha}{2}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l)\\ (T+1)\Delta y(0) - \frac{\alpha m(m+1)}{2}\Delta y(0) &= \sum_{l=1}^{T}(T+1-l)u(l)\\ &\quad - \frac{\alpha}{2}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l)\\ \Delta y(0)\left[\frac{2T+2-\alpha m(m+1)}{2}\right] &= \sum_{l=1}^{T}(T+1-l)u(l)\\ &\quad - \frac{\alpha}{2}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l)\end{aligned}$$

Thus,

$$\Delta y(0) = \frac{2}{2T+2-\alpha m(m+1)}\sum_{l=1}^{T}(T+1-l)u(l) - \frac{\alpha}{2T+2-\alpha m(m+1)}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l). \tag{2.5}$$

Let H be defined by $H = 2T+2-\alpha m(m+1)$, and substituting (2.5) into (2.4), we can show that

$$y(k) = \frac{2k}{H}\sum_{l=1}^{T}(T+1-l)u(l) - \frac{\alpha k}{H}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l) - \sum_{l=1}^{k-1}(k-l)u(l). \tag{2.6}$$

Suppose $k < m$. Then the unique solution of the problem (2.1) - (2.2) can be written as

$$\begin{aligned} y(k) &= -\sum_{l=1}^{k-1}(k-l)u(l) + \frac{2k}{H}\sum_{l=1}^{k-1}(T+1-l)u(l) \\ &\quad + \frac{2k}{H}\Bigg[\sum_{l=k}^{m-1}(T+1-l)u(l) + \sum_{l=m}^{T}(T+1-l)u(l)\Bigg] \\ &\quad - \frac{\alpha k}{H}\Bigg[\sum_{l=1}^{k-1}(m-l)(m-l+1)u(l) + \sum_{l=k}^{m-1}(m-l)(m-l+1)u(l)\Bigg] \\ &= \frac{2k}{H}\sum_{l=1}^{k-1}(T+1-l)u(l) - \frac{\alpha k}{H}\sum_{l=1}^{k-1}(m-l)(m-l+1)u(l) - \sum_{l=1}^{k-1}(k-l)u(l) \\ &\quad + \frac{2k}{H}\sum_{l=k}^{m-1}(T+1-l)u(l) - \frac{\alpha k}{H}\sum_{l=k}^{m-1}(m-l)(m-l+1)u(l) \\ &\quad + \frac{2k}{H}\sum_{l=m}^{T}(T+1-l)u(l) \end{aligned}$$

$$= \sum_{l=1}^{k-1} \frac{2k(T+1-l) - \alpha k(m-k)(m-l+1) - [2(T+1) - \alpha m(m+1)](k-l)}{H} u(l)$$
$$+ \sum_{l=k}^{m-1} \frac{2k(T+1-l) - \alpha k(m-l)(m-l+1)]}{H} u(l)$$
$$+ \sum_{l=m}^{T} \frac{2k(T+1-l)}{H} u(l)$$
$$= \sum_{l=1}^{T} G(k,l)u(l).$$

Suppose $k \geq m$. Then the unique solution of the problem (2.1) - (2.2) can be written as

$$\begin{aligned} y(k) = &-\Big[\sum_{l=1}^{m-1}(k-l)u(l) + \sum_{l=m}^{k-1}(k-l)u(l)\Big] \\ &+ \frac{2k}{H}\Big[\sum_{l=1}^{m-1}(T+1-l)u(l) + \sum_{l=m}^{k-1}(T+1-l)u(l) + \sum_{l=k}^{T}(T+1-l)u(l)\Big] \\ &- \frac{\alpha k}{H}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l) \\ = &-\sum_{l=1}^{m-1}(k-l)u(l) - \sum_{l=m}^{k-1}(k-l)u(l) + \frac{2k}{H}\sum_{l=1}^{m-1}(T+1-l)u(l) \\ &+ \frac{2k}{H}\sum_{l=m}^{k-1}(T+1-l)u(l) + \frac{2k}{H}\sum_{l=k}^{T}(T+1-l)u(l) \\ &- \frac{\alpha k}{H}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l) \\ = &\frac{2k}{H}\sum_{l=1}^{m-1}(T+1-l)u(l) - \frac{\alpha k}{H}\sum_{l=1}^{m-1}(m-l)(m-l+1)u(l) - \sum_{l=1}^{m-1}(k-l)u(l) \\ &+ \frac{2k}{H}\sum_{l=m}^{k-1}(T+1-l)u(l) - \sum_{l=m}^{k-1}(k-l)u(l) \\ &+ \frac{2k}{H}\sum_{l=k}^{T}(T+1-l)u(l) \end{aligned}$$

$$= \sum_{l=1}^{m-1} \frac{2l(T+1-k) - \alpha[k(m-l)(m-l+1) - km(m+1) - m(m+1)l]}{H} u(l)$$
$$+ \sum_{l=m}^{k-l} \frac{2k(T+1-l) - 2(T+1)(k-l) + \alpha m(m+l)(k-l)}{H} u(l)$$
$$+ \sum_{l=k}^{T} \frac{(T+1-l)}{H} u(l) = \sum_{l=1}^{T} G(k,l)u(l).$$

Then the unique solution of the problem (2.1) - (2.2) can be written as $y(k) = \sum_{l=1}^{T} G(k,l)u(l)$. The proof is completed. ■

Remark 2.2. We note that $H > 0,\ 2T+2-\alpha m(m+1) > 0$ implies $G(k,l)$ is positive on $\mathbb{N}_{T+1} \times \mathbb{N}_{1,T}$.

Lemma 2.3. Let $(k,l) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. Then we have

$$G(k,l) \geq M_1 G(k,k), \tag{2.7}$$

where $0 < M_1 < 1$ and $\alpha \geq 0$ is a constant given by

$$M_1 = \min\left\{\frac{1}{m-1}, \frac{1}{T+1-m-\alpha}, \frac{1}{T(T+1-m)}, \frac{2(T-1)+\alpha m}{T(T+1-m)}\right\}.$$

Proof. In order that (2.7) holds, it is sufficient that M_1 satisfies

$$M_1 \leq \min_{(k,l)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}.$$

Then we may choose

$$M_1 \leq \min\left\{\min_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}, \min_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}\right\},$$

since

$$\begin{aligned}
\min_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \min_{k\in\mathbb{N}_{1,m-1}} \left\{\min_{l\in\mathbb{N}_{1,k-1}} \frac{l[2T+2-2k-\alpha(m-k)(m-k+1)]}{k[2T+2-2k-\alpha(m-k)(m-k+1)]},\right. \\
&\quad \min_{l\in\mathbb{N}_{k,m-1}} \frac{k[2T+2-2l-\alpha(m-l)(m-l+1)]}{k[2T+2-2k-\alpha(m-k)(m-k+1)]}, \\
&\quad \left.\min_{l\in\mathbb{N}_{m,T}} \frac{2k(T+1-l)}{k[2T+2-2k-\alpha(m-k)(m-k+1)]}\right\} \\
&= \min_{k\in\mathbb{N}_{1,m-1}} \left\{\frac{1}{k}, \min_{l\in\mathbb{N}_{k,m-1}} \frac{[2T+2-2l-\alpha(m-l)(m-l+1)]}{[2T+2-2k-\alpha(m-k)(m-k+1)]},\right. \\
&\quad \left.\frac{2}{[2T+2-2k-\alpha(m-k)(m-k+1)]}\right\}.
\end{aligned} \tag{2.8}$$

If $\alpha \geq 0$, from (2.8) we get

$$\begin{aligned}
\min_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \min_{k\in\mathbb{N}_{1,m-1}} \left\{\frac{1}{k}, \frac{[2T+2-2k-\alpha(m-k)(m-k+1)]}{[2T+2-2k-\alpha(m-k)(m-k+1)]},\right. \\
&\quad \left.\frac{2}{[2T+2-2k-\alpha(m-k)(m-k+1)]}\right\}, \\
&\geq \min\left\{\frac{1}{m-1}, 1,\right. \\
&\quad \left.\frac{2}{2(T+1-m+1)-\alpha(m-m+1)(m-m+1+1)}\right\} \\
&= \min\left\{\frac{1}{m-1}, 1, \frac{2}{2(T+2-m)-2\alpha}\right\} \\
&= \min\left\{\frac{1}{m-1}, \frac{1}{(T+2-m-\alpha)}\right\}. \qquad (2.9)
\end{aligned}$$

Similarly for $\min_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}$, we have

$$\begin{aligned}
\min_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \min_{k\in\mathbb{N}_{m,T}} \left\{\min_{l\in\mathbb{N}_{1,m-1}} \frac{l[2T+2-2k-\alpha(m-k)(m-k+1)]}{2k(T+1-k)},\right. \\
&\quad \min_{l\in\mathbb{N}_{m,k-1}} \frac{2l[T+1-k+\alpha k(k-l)]}{2k(T+1-k)}, \\
&\quad \left.\min_{l\in\mathbb{N}_{k,T}} \frac{2k(T+1-l)}{2k(T+1-k)}\right\} \\
&= \min_{k\in\mathbb{N}_{m,T}} \left\{\frac{2(T+1-k)-\alpha(m-k)(m-k+1)]}{2k(T+1-k)},\right. \\
&\quad \left.\frac{2(k-1)(T+1-k)+\alpha k(k-k+1)]}{2k(T+1-k)}, 1\right\}. \qquad (2.10)
\end{aligned}$$

If $\alpha \geq 0$, from (2.10) we get

$$\begin{aligned}
\min_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \min_{k\in\mathbb{N}_{m,T}} \left\{\frac{2(T+1-T)-\alpha(m-m)(m-m+1)]}{2T(T+1-m)},\right. \\
&\quad \left.\frac{2(T-1)(T+1-T)+\alpha m}{2T(T+1-m)}, 1\right\} \\
&\geq \min\left\{\frac{2}{2T(T+1-m)}, \frac{2(T-1)+\alpha m}{2T(T+1-m)}, 1\right\} \\
&= \min\left\{\frac{2}{2T(T+1-m)}, \frac{2(T-1)+\alpha m}{2T(T+1-m)}\right\}.
\end{aligned}$$

■

Lemma 2.4. Let $(k,l) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. Then we have

$$G(k,l) \leq M_2 G(k,k), \tag{2.11}$$

where $M_2 \geq 1$ and $\alpha \geq 0$ is a constant given by

$$M_2 = \max\left\{\frac{2(T-m-\alpha+2)}{2T-\alpha(m-1)m}, \frac{2(T+1-m)}{2T-\alpha(m-1)m}, \frac{(m-1)(2T+2-2m)}{2m(T+1-m)}, \frac{2m(T+1-m)+\alpha T(T-m)]}{2m(T+1-m)}\right\}. \tag{2.12}$$

Proof. From (2.11), it is sufficient that M_2 satisfies

$$M_2 \geq \max_{(k,l)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}.$$

We may choose

$$M_2 \geq \max\left\{\max_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}, \max_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}\right\},$$

since

$$\begin{aligned}\max_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \max_{k\in\mathbb{N}_{1,m-1}}\left\{\max_{l\in\mathbb{N}_{1,k-1}} \frac{l[2T+2-2k-\alpha(m-k)(m-k+1)]}{k[2T+2-2k-\alpha(m-k)(m-k+1)]},\right.\\ &\quad \max_{l\in\mathbb{N}_{k,m-1}} \frac{k[2T+2-2l-\alpha(m-l)(m-l+1)]}{k[2T+2-2k-\alpha(m-k)(m-k+1)]},\\ &\quad \left.\max_{l\in\mathbb{N}_{m,T}} \frac{2k(T+1-l)}{k[2T+2-2k-\alpha(m-k)(m-k+1)]}\right\}\\ &= \max_{k\in\mathbb{N}_{1,m-1}}\left\{\frac{k-1}{k}, \max_{l\in\mathbb{N}_{k,m-1}} \frac{[2T+2-2l-\alpha(m-l)(m-l+1)]}{[2T+2-2k-\alpha(m-k)(m-k+1)]},\right.\\ &\quad \left.\frac{2(T+1-m)}{[2T+2-2k-\alpha(m-k)(m-k+1)]}\right\}. \end{aligned} \tag{2.13}$$

If $\alpha \geq 0$, from (2.13) we get

$$\begin{aligned}\max_{(k,l)\in\mathbb{N}_{1,m-1}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \max_{k\in\mathbb{N}_{1,m-1}}\left\{1,\right.\\ &\quad \frac{[2T+2-2(m-1)-\alpha(m-m+1)(m-m+1+1)]}{[2T+2-2(1)-\alpha(m-1)(m-1+1)]},\\ &\quad \left.\frac{2(T+1-m)}{[2T+2-2(1)-\alpha(m-1)(m-1+1)]}\right\},\\ &\leq \max\left\{1, \frac{2(T-m-\alpha+2)}{2T-\alpha(m-1)m}, \frac{2(T+1-m)}{2T-\alpha(m-1)m]}\right\}\\ &= \max\left\{\frac{2(T-m-\alpha+2)}{2T-\alpha(m-1)m}, \frac{2(T+1-m)}{2T-\alpha(m-1)m}\right\}. \end{aligned} \tag{2.14}$$

Similarly for $\max_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)}$, we have

$$\begin{aligned}\max_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \max_{k\in\mathbb{N}_{m,T}} \Bigg\{ \max_{l\in\mathbb{N}_{1,m-1}} \frac{l[2T+2-2k-\alpha(m-k)(m-k+1)]}{2k(T+1-k)}, \\ &\quad \max_{l\in\mathbb{N}_{m,k-1}} \frac{2l[T+1-k+\alpha k(k-l)]}{2k(T+1-k)}, \\ &\quad \max_{l\in\mathbb{N}_{k,T}} \frac{2k(T+1-l)}{2k(T+1-k)} \Bigg\} \\ &= \max_{k\in\mathbb{N}_{m,T}} \Bigg\{ \frac{(m-1)(2T+2-2k)-\alpha(m-k)(m-k+1)]}{2k(T+1-k)}, \\ &\quad \frac{2m(T+1-k)+\alpha k(k-m)]}{2k(T+1-k)}, 1 \Bigg\}. \end{aligned} \tag{2.15}$$

If $\alpha \geq 0$, from (2.15) we get

$$\begin{aligned}\max_{(k,l)\in\mathbb{N}_{m,T}\times\mathbb{N}_{1,T}} \frac{G(k,l)}{G(k,k)} &= \max_{k\in\mathbb{N}_{m,T}} \Bigg\{ \frac{(m-1)(2T+2-2m)-\alpha(m-m)(m-m+1)]}{2m(T+1-m)}, \\ &\quad \frac{2m(T+1-m)+\alpha T(T-m)]}{2m(T+1-m)}, 1 \Bigg\} \\ &\leq \max_{k\in\mathbb{N}_{m,T}} \Bigg\{ \frac{(m-1)(2T+2-2m)}{2m(T+1-m)}, \\ &\quad \frac{2m(T+1-m)+\alpha T(T-m)]}{2m(T+1-m)}, 1 \Bigg\} \\ &= \max_{k\in\mathbb{N}_{m,T}} \Bigg\{ \frac{(m-1)(2T+2-2m)}{2m(T+1-m)}, \\ &\quad \frac{2m(T+1-m)+\alpha T(T-m)]}{2m(T+1-m)} \Bigg\}. \end{aligned}$$

The proof is complete. ■

In the following, we denote

$$m = \min_{k\in[0,T]} \sum_{l=n}^{T} G(k,l) \sum_{i=1}^{n} a_i(l), \quad M = \max_{k\in[0,T+1]} \sum_{l=1}^{T} G(k,l) \sum_{i=1}^{n} a_i(l),$$

$$\tilde{m} = \min_{k\in[0,T]} G(k,k) \sum_{i=1}^{n} a_i(l), \quad \tilde{M} = \max_{k\in[0,T+1]} G(k,k) \sum_{i=1}^{n} a_i(l),$$

then $0 < m < M, 0 < \tilde{m} < \tilde{M}$.

Let Banach space $B = \{y \mid y : [0, T+1] \to \mathbb{R}^+\}$ for $y \in B$, $|y|_0 = \max_{k\in[0,T+1]} |y(k)|$ and $E = B^n$ for $y = (y_1, \cdots, y_n) \in E$ or $\mathbb{R}^n_+$, $\|y\| = \sum_{i=1}^{n} |y_i|_0$. For $y \in E$ or $\mathbb{R}^n_+$, $\|y\|$ denotes the norm of $y \in E$ or $\mathbb{R}^n_+$, respectively.

We denote the cone in E by

$$K = \Big\{ y = (y_1, \cdots, y_n) \in E \; |y_i(k)| \geq 0, \; k \in [0, T+1], i = 1, \cdots, n, \quad \text{and } \min_{k\in[1,T]} \sum_{i=1}^{n} y_i(k)) \geq \sigma \parallel y \parallel \Big\}$$

where $\sigma = \dfrac{M_1 \tilde{m}}{M_2 \tilde{M}}$. It is clear that K is a cone in E.

Now, it is not difficult to show that the problem (1.1) - (1.2) is equivalent to the fixed-point equation,

$$T_\lambda(y) = y \in K \setminus \{0\}.$$

For $y = (y_1, y_2, \cdots, y_n) \in K \setminus \{0\}$, we define the operator $T_\lambda : K \setminus \{0\} \to E$ by

$$\begin{aligned} T_\lambda(y) &= \lambda\Big(\sum_{l=0}^{T} G(k,l)a_1(l) f_1(y_1(l), \cdots, y_n(l)) + e_1(l), \\ &\quad \cdots, \sum_{l=0}^{T} G(k,l)a_n(l) f_n(y_1(l), \cdots, y_n(l)) + e_n(l)\Big) \\ &= (T_1(y), \cdots, T_n(y)), \quad k \in [0, T+1]. \end{aligned}$$

Lemma 2.5. Assume that A1 and A2 hold.

(i) If $lim_{\|y\|\to 0} f_i(y) = \infty$ uniformly with respect to the $k \in [1, T]$ for $i = 1, \cdots, n$, then there is a $\delta > 0$, such that for $r \in (0, \delta)$, $T_\lambda : \bar{\Omega}_r \setminus \{0\} \to K$ is completely continuous.

(ii) If $lim_{\|y\|\to\infty} f_i(y) = \infty$ uniformly with respect to $k \in [0, T]$ for $i = 1, \cdots, n$, then there is a $\Delta > 0$, such that for $R > \Delta$, $T_\lambda : K \setminus \Omega_R \to K$ is completely continuous.

(iii) If $T_\lambda : K \setminus \{0\} \to \chi$, then for $y \in K$ we have

$$\begin{aligned}\|T_\lambda y\| &\geq M_1 \sum_{l=1}^{T} G(k,k) \sum_{i=1}^{n} f_i(y(l)) \sum_{i=1}^{n} a_i(l) \\ &\geq M_1 \tilde{m} \sum_{i=1}^{n} f_i(y(l)) \\ &\geq \frac{M_1 \tilde{m}}{M_2 \tilde{M}} \|T_\lambda y\| = \sigma \|T_\lambda y\|. \end{aligned} \tag{2.16}$$

$$\begin{aligned}\|T_\lambda y\| &\leq M_2 \sum_{l=1}^{T} G(k,k) \sum_{i=1}^{n} f_i(y(l)) \sum_{i=1}^{n} a_i(l) \\ &\leq M_2 \tilde{M} \sum_{i=1}^{n} f_i(y(l)) \end{aligned} \tag{2.17}$$

Proof. The proof is similarly to Lemma 4 [11]. (i) We split $a_i f_i(y(k)) + e_i(k)$ into two terms $\frac{1}{2} a_i f_i(y(k))$ and $\frac{1}{2} a_i f_i(y(k)) + e_i(k)$. Then the first is always positive and used to carry out the estimates of the operator. We will make the second term $\frac{1}{2} a_i f_i(y(k)) + e_i(k)$ positive by choosing appropriate domains of f_i. Noting that $a_i(k)$ is continuous and positive on [0,T], and $\lim_{\|y\| \to 0} f_i(y) = \infty$, for $i = 1, \cdots, n$. Therefore, we can choose $\delta > 0$, such that

$$f_i(y(k)) \geq 2 \frac{\max_{k \in [1,T]} \{|e_i(k)| + 1\}}{\min_{k \in [1,T]} \{a_i(k)\}}, \quad k \in [1,T], \quad y \in \mathbb{R}_+^n, \quad 0 < \| y \| \leq \delta.$$

Now for $r \in (0, \delta)$ and $y \in \bar{\Omega} \setminus \{0\}$, we have

$$\begin{aligned} a_i f_i(y(k)) + e_i(k) &\geq \frac{1}{2} a_i f_i(y(k)) + e_i(k) \\ &\geq a_i \frac{\max_{k \in [1,T]} \{|e_i(k)| + 1\}}{\min_{k \in [1,T]} \{a_i(k)\}} + e_i(k) > 0, \end{aligned}$$

and from Lemma 2.4, we obtain

$$\begin{aligned} 0 \;&\leq\; T_\lambda^i(y)(k) = \lambda \sum_{l=0}^{T} G(k,l) \Big(a_i(l) f_i(y_1(l), \cdots, y_n(l)) + e_i(l) \Big) \\ &\leq\; \lambda M_2 \tilde{M} \sum_{l=0}^{T} \Big(f_i(y_1(l), \cdots, y_n(l)) + e_i(l) \Big), \quad k \in [0, T+2]. \end{aligned}$$

Thus

$$|T_\lambda^i(y)|_0 \leq \lambda M_2 \tilde{M} \sum_{l=0}^{T} \Big(f_i(y_1(l), \cdots, y_n(l)) + e_i(l) \Big), \quad i = 1, 2, \cdots, n. \tag{2.18}$$

For $k \in [1, T]$, we have from Lemma 2.3 and (2.18) that

$$\begin{aligned} T_\lambda^i(y)(k) &= \lambda \sum_{l=0}^{T} G(k,l) \Big(a_i(l) f_i(y_1(l), \cdots, y_n(l)) + e_i(l) \Big) \\ &\geq \lambda M_1 \tilde{m} \sum_{l=0}^{T} \Big(f_i(y_1(l), \cdots, y_n(l)) + e_i(l) \Big) \\ &\geq \frac{M_1 \tilde{m}}{M_2 \tilde{M}} \\ &\geq \sigma |T_\lambda^i(y)|_0, \quad i = 1, 2, \cdots, n. \end{aligned}$$

So, for $k \in [1, T]$,

$$\sum_{i=1}^{n} T_\lambda^i(y)(k) \geq \sigma \sum_{i=1}^{n} |T_\lambda^i(y)|_0 = \sigma \| T_\lambda y \|.$$

Hence

$$\min_{k \in [1,T]} \sum_{i=1}^{n} T_\lambda^i(y)(k) \geq \sigma \| T_\lambda y \|,$$

Thus, $T : \bar{\Omega} \setminus \{0\}) \to K$. According to Arzela - Ascoli theorem and hypothesis (A1), (A2), we know that $T_\lambda : \bar{\Omega}_r \setminus \{0\} \to K$ is completely continuous.

Let $y_m(k),\ y_0(k) \in K \setminus \{0\}$ with $y_m(k) \to y_0(k)$ as $m \to \infty$. From (1.1) and since $f_i(k, \xi)$ is continuous in ξ, as $m \to \infty$, we have

$$\begin{aligned} |T_\lambda^i y_m(k) - T_\lambda^i y_0(k)| &\leq \frac{\sigma_i^{-1}}{\sigma_i^{-1} - 1} \left(\sum_{0}^{T} \lambda \Big(|a_i(l)|\, |f_i(y_m(l)) - f_i(y_0(l))| + |e_m(l) - e_0(l)| \Big) \right) \\ &\to 0. \end{aligned}$$

Hence $|T_\lambda^i y_m - T_\lambda^i y_0| \to 0$. It follows that the operator T_λ is continuous.
(ii) If $\lim_{\|y\| \to \infty} f_i(y(k)) = \infty$, there is an $\hat{R} > 0$, such that

$$f_i(y(k)) \geq 2 \frac{\max_{k \in [1,T]} \{|e_i(k)| + 1\}}{\min_{k \in [1,T]} \{a_i(k)\}}, \quad k \in [1, T], \quad y \in \mathbb{R}_+^n, \ \| y \| \geq \hat{R}.$$

Let $\Delta = \hat{R}(T+1)$. Then for $\hat{R} > \Delta$, $y \in K \setminus \Omega_R$, we have that $\|y\| = \sum_{i=1}^{n} |y|_0 \geq \sum_{i=1}^{n} y_i(k)) \geq \sigma \|y\| \geq \hat{R}$, and therefore

$$a_i(k) f_i(y(k)) + e_i(k) \geq \frac{1}{2} a_i(k) f_i(y(k)) + e_i(k) > 0, \quad k \in [1, T].$$

Similar to (i), we have that $T_\lambda : K \setminus \Omega_R \to K$ is completely continuous. ■

Lemma 2.6. Assume that (A1), (A2) hold. Let $y = (y_1, \cdots, y_n) \in K \setminus \{0\}$ and $\eta > 0$. If there exist $\lim_{\|y\| \to 0} \frac{f_i(y)}{\|y\|}$ we have

$$f_i(y_i(k), \cdots, y_n(k)) \geq \eta \sum_{i=1}^{n} y_i(k), \ k \in [1, T] \tag{2.19}$$

then

$$\|T(y)\| \geq \lambda \eta m \|y\|.$$

Proof. From definition of K and (2.19), we have

$$\begin{aligned}
\|T_\lambda(y)\| &= \sum_{i=1}^{n} |T_\lambda^i(y)|_0 \geq |T_i(y)|_0 \\
&= \lambda \max_{k \in [0, T+1]} \sum_{l=0}^{T} G(k, l) \Big(a_i(l) f_i(y_1(l), \cdots, y_n(l)) + (e_i(l) \Big) \\
&\geq \lambda \sum_{l=1}^{T} G(k, l) \Big(a_i(l) f_i(y_1(l), \cdots, y_n(l)) \Big) \\
&\geq \lambda \eta M \sum_{i=1}^{n} y_i(l) \\
&\geq \lambda \eta \|y\| M \geq \lambda \eta m \|y\|.
\end{aligned}$$

The proof is complete. ■

Let $\hat{f} : [1, \infty) \to \mathbb{R}_+$ be the function given by

$$\hat{f}(\theta) = \max \left\{ f(y) : y \in \mathbb{R}_+^n, \text{and } 1 \leq \|y\| \leq \theta \right\}.$$

It is easy to see that $\hat{f}(\theta)$ is nondecreasing function on $[1, \infty)$. The following lemma is the same as Lemma 3.6 in [16].

Lemma 2.7. Assume (A2) holds. If $\lim_{y\to\infty}\frac{f(y)}{y}$ exists (which can be infinity) then $\lim_{\theta\to\infty}\frac{\hat{f}(\theta)}{\theta}$ exists and $\lim_{\theta\to\infty}\frac{\hat{f}(\theta)}{\theta} = \lim_{y\to\infty}\frac{f(y)}{y}$.

Lemma 2.8. Assume that (A1), (A2) hold. Let $r > 0$. If there exist $\epsilon > 0$, such that

$$\tilde{f}_i(r) \le \epsilon r,\ i = 1, 2, \cdots, n, \tag{2.20}$$

then

$$\|T(y)\| \le \left(\lambda\epsilon C + \frac{1}{2}\right)\|y\|, \quad \text{for } y \in \partial K_r,$$

where

$$C = \sum_{i=1}^{n}\lambda\sum_{l=0}^{T}G(k,l)a_i(l) \text{ and } r = 2\lambda\sum_{l=0}^{T}G(k,l)e_i(l).$$

Proof. Suppose $y \in \partial K_r$, i.e, $y \in K \setminus \{0\}$ and $\|y\| = r$, then it follows from (2.20) that

$$\begin{aligned}
T_\lambda^i(y)(k) &= \lambda\sum_{l=0}^{T}G(k,l)\Big(a_i(l)f_i(y_1(l),\cdots,y_n(l)) + e_i(l)\Big)\\
&\le \lambda\sum_{l=0}^{T}G(k,l)\Big(a_i(l)\tilde{f}_i(r) + e_i(l)\Big)\\
&\le \lambda\sum_{l=0}^{T}G(k,l)\Big(a_i(l)\epsilon r + e_i(l)\Big)\\
&\le \lambda\sum_{l=0}^{T}G(k,l)\Big(a_i(l)\epsilon r + e_i(l)\Big), \quad k \in [0, T+2], \quad i = 1, 2, \cdots, n.
\end{aligned}$$

So,

$$|T_\lambda^i|_0 \le \lambda\sum_{l=0}^{T}G(k,l)\Big(a_i(l)\epsilon r + e_i(l)\Big),\ \ i = 1, 2, \cdots, n.$$

Therefore,

$$\|T_\lambda(y)\| = \sum_{i=1}^{n}|T_\lambda^i(y)|_0 \le \lambda\sum_{i=1}^{n}\left(\max_{k\in[0,T+1]}\sum_{l=0}^{T}G(k,l)\Big(a_i(l)\epsilon r + e_i(l)\Big)\right) = \|y\|\left(\lambda\epsilon C + \frac{1}{2}\right).$$

The proof is complete. ■

3. Main result

Now we are in the position to establish our main result.

Theorem 3.1. Let (A1), (A2) hold. Assume the $\lim_{\|y\|\to 0} f_i(y) = \infty, i = 1, \cdots, n$ uniformly with respect to $k \in [1, T]$. Then,

(i) there exists a $\lambda_1 > 0$, such that (1.1) - (1.2) has a positive solution for $0 < \lambda < \lambda_1$.

(ii) if $\lim_{\|y\|\to\infty} \frac{f_i(y)}{\|y\|} = 0$ then, for all $\lambda > 0$, (1.1) - (1.2) has a positive solution.

(iii) if $\lim_{\|y\|\to\infty} \frac{f_i(y)}{\|y\|} = \infty$ uniformly with respect to $k \in [1, T]$, then, for sufficiently small $\lambda > 0$, (1.1) - (1.2) has two positive solutions.

Proof. Part (i): Since $lim_{\|y\|\to 0} f_i(y) = \infty$ uniformly with respect to the $k \in [1, T]$ for $i = 1, \cdots, n$, by Lemma 2.5 (i) there is a $\delta > 0$, such that for $r \in (0, \delta)$, $T_\lambda : \bar{\Omega}_r \setminus \{0\} \to K$ is defined and $a_i(s) f_i(s) + e_i(s)$ is nonnegative.

Now fix a number $r_1 < \delta$, if we choose

$$\lambda_1 = \frac{r_1}{M_2 \tilde{M}} \sum_{i=1}^{n} f_i(y(l))$$

for $\lambda < \lambda_1$, (2.17) implies

$$\|T_\lambda y\| < \|y\|, \ y \in \partial\Omega_{r1}. \tag{3.1}$$

On the other hand, for $\lambda < \lambda_1$, in view of $\lim_{\|y\|\to 0} f_i(y) = \infty$, there is a positive number $r_2 < r_1$ such that

$$f_i(y) \geq \eta\|y\|, \ \ i = 1, \cdots, n,$$

for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$ and $0 < \|y\| \leq r_2$, where $\eta > 0$ is chosen so that

$$\lambda m\eta > 1.$$

Thus, if $y = (y_1, \cdots, y_n) \in \partial\Omega_{r2}$ then

$$f_i(y_i(k)) \geq \eta \sum_{i=1}^{n} y_i(k), \ \ k \in [1, T].$$

Then by Lemma 2.6 implies that

$$\|T_\lambda y| \geq \lambda m\eta\|y\| > \|y\| \ \text{ for } \ \ y \in \partial\Omega_{r2}. \tag{3.2}$$

It follows from (3.1), (3.2) and Theorem (1.1) that T_λ has a fixed point in $\bar{\Omega}_{r_1} \setminus \Omega r_2$. Consequently, (1.1) - (1.2) has a positive solution for $0 < \lambda < \lambda_1$. ■

Part (ii): From assumption, there is $r_1 > 0$ such that

$$f_i(y) \geq \eta \|y\|, \ \ i = 1, \cdots, n,$$

for $y = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$ and $0 < \|y\| \leq r_1$, where $\eta > 0$ is chosen so that

$$\lambda m \eta > 1.$$

If $y = (y_1, \cdots, y_n) \in \partial\Omega_{r_1}$ then

$$f_i(y_i(k)) \geq \eta \sum_{i=1}^{n} y_i(k), \ k \in [1, T].$$

Then Lemma (2.8) implies

$$\|T_\lambda(y)\| \geq \lambda m \eta \|y\| \geq \|y\|. \tag{3.3}$$

for $y \in \partial\Omega_{r_1}$.

We now determine $\partial\Omega_{r_2}$. Since $\lim_{\|y\|\to\infty} \sigma\|y\| = 0$, it follows from Lemma (2.7) that $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = 0$, $i = 1, \cdots, n$. Therefore there exists $r_2 = \max\{2r_1, (T+1)H_1\}$ and $\|y\| = \sum_{i=1}^{n} |y|_0 \geq \sum_{i=1}^{n} y_i(k)) \geq \frac{\|y\|}{T+1} \geq H_1$, which implies

$$\tilde{f}_i(r) \leq \epsilon r_2, \ i = 1, 2, \cdots, n,$$

where the constant $\epsilon > 0$ satisfies

$$\lambda\epsilon C + \frac{1}{2} < 1$$

and C is defined in Lemma 2.8. Thus, we have by Lemma 2.8

$$\|T_\lambda(y)\| \leq \left(\lambda\epsilon C + \frac{1}{2}\right)\|y\| < \|y\|, \text{ for } \ y \in \partial\Omega_{r_2}. \tag{3.4}$$

It follows from (3.3), (3.4) and Theorem 1.1 that T_λ has a fixed point in $\bar{\Omega}_{r_2} \setminus \Omega r_1$. Consequently, (1.1) - (1.2) has a positive solution for $0 < \lambda < \lambda_1$.

Part (iii): Since $lim_{\|y\|\to 0} f_i(y) = \infty$ uniformly with respect to the $k \in [1, T]$ for $i = 1, \cdots, n,$, by Lemma2.5 (i) implies (1.1), (1.2) has a positive solution $\bar{\Omega}_{r_2} \setminus \Omega_{r_1}$ for $\lambda \in (0, \lambda_1)$.

On the other hand, since $\lim_{\|y\|\to\infty} f_i(y) = \infty$ by Lemma 2.5(ii), there is $\Delta > 0$, such that for $R > \Delta$, $T_\lambda : K \setminus \{0\} \to K$ is defined and $a_i(s) f_i(s) + e_i(s)$ is nonnegative. For a fixed number $R_3 > \max\{\Delta, r_1\}$. If we choose

$$\lambda_0 = \frac{R_3}{M_1 \tilde{(}m) \sum_{i=1}^{n} f_i(y(l))}$$

for $\lambda < \lambda_0$, (2.16) implies

$$\|T_\lambda y\| < \|y\|, \ \ y \in \partial\Omega_{R_3}. \tag{3.5}$$

Since $\lim_{\|y\|\to\infty} \frac{f_i(y)}{\|y\|} = \infty$, there is a positive number $\hat{H}$ such that

$$f_i(y)) \geq \eta\|y\| \ \ k \in [1, T]$$

for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$ and $\|y\| \geq \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda m\eta > 1.$$

Let $R_4 = \max\left\{2R_3, \frac{\hat{H}}{\sigma}\right\}$. If $y = (y_1, \cdots, y_n) \in \partial\Omega_{R_4}$, then

$$\|y\| = \sum_{i=0}^{n} |y|_0 \geq \min_{k\in[1,T]} \sum_{i=1}^{n} y_i(k) \geq \sigma\|y\| = \sigma R_4 \geq \hat{H},$$

which implies that

$$f_i(y_i(k)) \geq \eta \sum_{i=1}^{n} y_i(k), \quad for \quad k \in [1, T].$$

It follows from Lemma 2.6 that

$$\|T_\lambda y\| \geq \lambda m\eta\|y\| > \|y\| \ \ \text{for} \ \ y \in \partial\Omega_{R_4}. \tag{3.6}$$

It follows from (3.5), (3.6), Theorem 1.1 that T_λ has two fixed point y_1 in $\bar{\Omega}_{r_2} \setminus \Omega_{r_1}$ and y_2 in $\bar{\Omega}_{R_4} \setminus \Omega_{R_3}$ such that

$$r_2 < \|y_1\| < r_1 < R_3 < \|y_2\| < R_4.$$

Thus, we can say that (1.1) - (1.2) has two positive solutions for $0 < \lambda < \min\{\lambda_1, \lambda_0\}$. ■

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