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Some Recurrence Relations of $\overline{H}$-Function

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Abstract

In the present paper, the authors have established three recurrence relations of $\overline{H}$-Function.

INTRODUCTION

The $\overline{H}$-function occurring in the paper will be defined and represented by Buschman and Srivastava [1] as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{1.1}$$

where
$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters,

$\alpha_j \geq 0 (j=1,...,P), \beta_j \geq 0 (j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j (j=1,...,N)$ and $B_j (j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (1.1) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} |A_j \alpha_j| - \sum_{j=M+1}^{Q} |\beta_j B_j| - \sum_{j=N+1}^{P} |\alpha_j| > 0 \tag{1.3}$$

and $|\arg(z)| < \frac{1}{2}\pi\,\Omega$ (1.4)

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by (Rathie [3],p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}[z] = 0\left(|z|^{\gamma}\right), \gamma = \min_{1\leq j\leq N}\left[\operatorname{Re}\left(b_j / \beta_j\right)\right], |z| \to 0 \tag{1.5}$$

If we take $A_j = 1(j=1,...,N)$, $B_j = 1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [2].

Recurrence Formulae

$$(a_1-a_2)\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1;1),(a_2,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1;1),(a_2-1,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$-\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1-1,\alpha_1;1),(a_2,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right] \quad n\geq 2 \tag{2.1}$$

Proof: We have to show that

$$(a_1-a_2)\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1;1),(a_2,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$-\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1;1),(a_2-1,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$+\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1-1,\alpha_1;1),(a_2,\alpha_1;1)(a_j;\alpha_j;A_j)_{3,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right] = 0$$

Let $\phi(s)=\left\{\dfrac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=3}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s\right\}$

Then L.H.S.

$$=(a_1-a_2)\frac{1}{2\pi i}\int_L \phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)ds$$

$$-\frac{1}{2\pi i}\int_L \phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+1+\alpha_1 s)ds$$

$$+\frac{1}{2\pi i}\int_L \phi(s)\Gamma(1-a_1+1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)ds$$

$$=\frac{1}{2\pi i}\int_L \phi(s)\left[\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)-\Gamma(1-a_1+\alpha_1 s)\right.$$

$$\left.\Gamma(2-a_2+\alpha_1 s)+\Gamma(1-a_2+\alpha_1 s)\Gamma(2-a_1+\alpha_1 s)\right]ds$$

$$=\frac{1}{2\pi i}\int_L \phi(s)\left[(a_1-a_2)\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)\right.$$

$$-\Gamma(1-a_1+\alpha_1 s).(1-a_2+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)$$

$$\left.+(1-a_1+\alpha_1 s)\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)\right]ds$$

$$=\frac{1}{2\pi i}\int_L \phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_2+\alpha_1 s)\left[0\right]ds=0$$

=R.H.S.

$$(b_1\alpha_1-a_1\beta_1+\beta_1)\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=\beta_1\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1-1,\alpha_1;1),(a_j;\alpha_j;A_j)_{2,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$+\alpha_1\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p}\\(1+b_1,\beta_1;1),(b_j,\beta_j)_{2,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

(2.2)

Where $m,n\geq 0$.

Proof:

$$(b_1\alpha_1 - a_1\beta_1 + \beta_1)\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$-\beta_1\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1-1,\alpha_1;1),(a_j;\alpha_j;A_j)_{2,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$-\alpha_1\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p}\\(1+b_1,\beta_1;1),(b_j,\beta_j)_{2,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]=0$$

Let $\phi(s)=\left\{\dfrac{\prod_{j=2}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s\right\}$

Then L.H.S.

$$=(b_1\alpha_1-a_1\beta_1)\frac{1}{2\pi i}\int_L\phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(b_1-\beta_1 s)ds$$

$$-\beta_1\frac{1}{2\pi i}\int_L\phi(s)\Gamma(b_1-\beta_1 s)\Gamma(1-a_1+1+\alpha_1 s)ds$$

$$-\alpha_1\frac{1}{2\pi i}\int_L\phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(1+b_1-\beta_1 s)ds$$

$$=\frac{1}{2\pi i}\int_L\phi(s)\left[(b_1\alpha_1-a_1\beta_1+\beta_1)\Gamma(1-a_1+\alpha_1 s)-\Gamma(b_1-\beta_1 s)\right.$$

$$-\beta_1\Gamma(1-a_1+\alpha_1 s)\Gamma(1-a_1+\alpha_1 s)\Gamma(b_1-\beta_1 s)$$

$$\left.-\alpha_1\Gamma(1-a_1+\alpha_1 s)(b_1-\beta_1 s)\Gamma(b_1-\beta_1 s)\right]ds$$

$$=\frac{1}{2\pi i}\int_L\phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(b_1-\beta_1 s)$$

$$\left[b_1\alpha_1-a_1\beta_1+\beta_1-\beta_1+a_1\beta_1-\beta_1\alpha_1 s-\alpha_1 b_1+\beta_1\alpha_1 s\right]ds$$

$$=\frac{1}{2\pi i}\int_L\phi(s)\Gamma(1-a_1+\alpha_1 s)\Gamma(b_1-\beta_1 s)[0]ds\;=0$$

=R.H.S.

$$(b_q\alpha_1-a_1\beta_q+\beta_q)\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$= \beta_q \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_1-1,\alpha_1;1),(a_j;\alpha_j;A_j)_{2,n},(a_j;\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q} \end{matrix} \right. \right]$$

$$-\alpha_1 \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q-1},(b_{q+1},\beta_q;B_q) \end{matrix} \right. \right] \tag{2.3}$$

Where $n \geq 1, 1 \leq m \leq q-1$.

Proof:

$$(b_q\alpha_1 - a_1\beta_q + \beta_q)\overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q} \end{matrix} \right. \right]$$

$$-\beta_q \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_1-1,\alpha_1;1),(a_j;\alpha_j;A_j)_{2,n},(a_j;\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q} \end{matrix} \right. \right]$$

$$+\alpha_1 \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,n},(a_j;\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q-1},(b_q+1,\beta_q;1) \end{matrix} \right. \right] = 0$$

Let $$\phi(s) = \left\{ \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q-1}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} x^s \right\}$$

Then L.H.S.

$$= (b_q\alpha_1 - a_1\beta_q + \beta_q)\frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(1-a_1+\alpha_1 s)}{\Gamma(1-b_q+\beta_q s)}ds$$

$$-\beta_q \frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(1-a_1+1+\alpha_1 s)}{\Gamma(1-b_q+\beta_q s)}ds + \alpha_1 \frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(1-a_1+\alpha_1 s)}{\Gamma(1-b_q-1+\beta_q s)}ds$$

$$= \frac{1}{2\pi i}\int_L \phi(s)\left[(b_q\alpha_1 - a_1\beta_q + \beta_q)\frac{\Gamma(1-a_1+\alpha_1 s)}{(-b_q+\beta_q s)\Gamma(-b_q+\beta_q s)} \right.$$

$$\left. -\beta_q \frac{(1-a_1+\alpha_1 s)\Gamma(1-a_1+\alpha_1 s)}{(-b_q+\beta_q s)\Gamma(-b_q+\beta_q s)} + \alpha_1 \frac{\Gamma(1-a_1+\alpha_1 s)}{\Gamma(-b_q+\beta_q s)} \right] ds$$

$$= \frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(1-a_1+\alpha_1 s)}{\Gamma(-b_q+\beta_q s)}\left[(b_q\alpha_1 - a_1\beta_q + \beta_q)\frac{1}{(-b_q+\beta_q s)} \right.$$

$$\left. -\beta_q \frac{(1-a_1+\alpha_1 s)}{(-b_q+\beta_q s)} + \alpha_1 \right] ds$$

$$=\frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(1-a_1+\alpha_1 s)}{(-b_q+\beta_q s)\Gamma(-b_q+\beta_q s)}\Big[b_q\alpha_1-a_1\beta_q+\beta_q$$

$$-\beta_q+\alpha_1\beta_q-\alpha_1\beta_q s-\alpha_1\beta_q+\alpha_1\beta_q s\Big]ds$$

=R.H.S.

For $A_j=1(j=1,...,N)$, $B_j=1(j=M+1,...,Q)$ in (2.1),(2.2)and (2.3), we get the results in terms of Fox's H-function [2].

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