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Two-parameter regularization method for determining the heat source

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Abstract

In this paper, we study the inverse problem of determining a spacewise dependent heat source in one-dimensional heat equation in a bounded domain where the additional data are given at some fixed time. The solution is given by a two-parameter regularization. We also propose a model function approach to the Morozov principle for choosing regularization parameter. Numerical examples show that the regularization method is effective and stable.

AMS Subject Classification: 35R30, 35K05

Keywords: Inverse problem, Heat source, Model function, Morozov principle, Regularization

1. INTRODUCTION

In this paper, we consider an initial-boundary value problem for a second-order parabolic equation in the form

$$\left.\begin{array}{llll} u_t(x,t) - u_{xx}(x,t) & = f(x), & (x,t) & \in Q = (0,1) \times (0,1], \\ u(x,0) & = 0, & x & \in [0,1], \\ u(0,t) = u(1,t) & = 0, & t & \in (0,1], \end{array}\right\} \tag{1.1}$$

where the right hand side function $f(x)$ is unknown. The additional data discussed in this study are observations at a final moment $t = 1$ given by

$$u(x,1) = g(x), \qquad x \in [0,1], \tag{1.2}$$

where g is a known function which satisfies the homogeneous Dirichlet boundary condition.

The identification of parameter in parabolic equation is an ill-posed problem that has been receiving considerable attention from many researchers in a variety of fields. The mathematical model $(1.1)-(1.2)$ arises in various physical and engineering settings. In physics, people often need to determine the heat source from some information of measurement temperature. According to the Hadamard requirements (existence, uniqueness and stability of the solution), the inverse problem is ill-posed mathematically [11,14]. For stable reconstruction, we have some regularization techniques [7]. Engl et al. [8] established the uniqueness of inverse source problem of parabolic and hyperbolic equations and analyzed the convergence rate of the regularized solution.

In [1], the author considered the determination of $f(x)$ by the spectral theory from the overspecified boundary data. In [20], optimization methods are applied to the inverse problem and the numerical solution of the inverse problem is obtained by the Landweber iteration method.

Hasanov [9] applied conjugate gradient method to identify the unknown spacewise and time dependent heat sources of the variable coefficient heat conduction equation. Castro et al. [2] applied the reproducing kernel method for backward heat conduction type problem. In [6], Egger investigated the regularizing properties of semi-iterative regularization methods in Hilbert scales for linear ill-posed problems and perturbed data. A number of techniques including iterative regularization methods [12,13], simplified Tikhonov regularization method [19] and generalized Tikhonov regularization [17] have been proposed for solving the inverse source problem.

Cheng et al. [3] employed a new strategy for a priori choice of regularizing parameter in Tikhonov's regularization. In [15], Kunisch and Zou proposed a two parameter algorithm to choose some reasonable regularization parameters in single parameter regularization method.

In recent years, there has been a growing interest in the multi-parameter Tikhonov regularization method that uses multiple constraints as a means of improving the quality of inversion. In [16], Lu and Pereverzev obtained a model function approximation of the multi-parameter discrepancy principle leading to efficient iteration algorithms for choosing regularization parameters. Wang [18] determined the regularization parameters by the damped Morozov principle.

The problem $(1.1)-(1.2)$ can be formulated as an operator equation as follows

$$(Kf)(x) = g(x) \tag{1.3}$$

where K is a linear compact operator.

Motivated by the above facts, in this paper, we give an approximate solution of $f(x)$ by a two parameter Tikhonov regularization method which minimizes the functional

$$J_{\alpha,\beta}(f) = \|Kf - g_\delta\|^2 + \alpha\|f\|^2 + \beta\|Bf\|^2 \tag{1.4}$$

where B is an unbounded self-adjoint strictly positive operator.

The main novelty of this paper lies in the following aspects: In Section 2, we simply recall some preliminaries. In Section 3, we propose Two-parameter regularization method and obtain a conditional stability result. Also we give convergence estimate under a posteriori regularization parameter choice rule. In Section 4, we present algorithms for solving the regularization solution and determine the regularization parameters by using model function approach to the damped Morozov principle. In addition, numerical example and their simulation is exploited to demonstrate the usefulness and effectiveness of the method.

2. PRELIMINARIES

By the method of separation of variables, the solution of (1.1) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2}(f, X_n)X_n, \tag{2.1}$$

where $\{X_n = \sqrt{2}\sin\, n\pi x, (n = 1,2,\cdots)\}$ is an orthogonal basis in $L^2(0,1)$ and

$$f_n = (f, X_n) = \sqrt{2}\int_0^1 f\,(x)\sin\,(n\pi x)dx. \tag{2.2}$$

From (1.2) and (2.1), we define the operator $K: f \to g$ by

$$\begin{aligned} g(x) = Kf(x) &= \sum_{n=1}^{\infty}(g, X_n)X_n \\ &= \sum_{n=1}^{\infty} \frac{1 - e^{-n^2\pi^2}}{n^2\pi^2}(f, X_n)X_n. \end{aligned} \tag{2.3}$$

It is easy to see that K is a linear compact operator and the singular values $\{\sigma_n\}_{n=1}^{n=\infty}$ of K satisfy

$$\sigma_n = \frac{1 - e^{-n^2\pi^2}}{n^2\pi^2} \tag{2.4}$$

and

$$(g, X_n) = \frac{1 - e^{-n^2\pi^2}}{n^2\pi^2}(f, X_n)(X_n, X_n), \tag{2.5}$$

that is, $(f, X_n) = \sigma_n^{-1}(g, X_n)$. Therefore

$$\begin{aligned} f(x) = K^{-1}g(x) &= \sum_{n=1}^{\infty} \frac{1}{\sigma_n}(g, X_n)X_n \\ &= \sum_{n=1}^{\infty} \frac{n^2\pi^2}{1 - e^{-n^2\pi^2}}(g, X_n)X_n. \end{aligned} \tag{2.6}$$

Since the data $g(x)$ is based on (physical) observation, there must be measurement errors and we assume the measured data function $g_\delta(x) \in L^2(0,1)$ which satisfies

$$\|g - g_\delta\|_{L^2(0,1)} \leq \delta \tag{2.7}$$

where the constant $\delta > 0$ represents noise level.

Let a Hilbert scale $\{X^r\}_{r\in\mathbb{R}}$ be a family of Hilbert spaces X^r with the inner products $\langle u, v\rangle_r := \langle B^r u, B^r v\rangle$ where B is an unbounded self-adjoint strictly positive operator in a dense domain of Hilbert space X. More precisely, X^r is defined as the completion of the intersection of domains of all operators $\{B^r\}_{r\in\mathbb{R}}$, endowed with the norm $\|u\|_r := \langle u, u\rangle_r^{1/2}, \|t\|_0 = \|\cdot\|$.

Lemma 1. For $n \geq 1$, there holds

$$\frac{1}{1 - e^{-n^2\pi^2}} \leq 2. \tag{2.8}$$

3. TWO PARAMETER TIKHONOV REGULARIZATION METHOD

The multi-parameter regularization method has been studied in different papers. In [4] , the authors used multi-parameter regularization for atmospheric remote sensing. A multi-parameter regularization method was applied to estimate parameters of a jump diffusion process in [5]. In [10], the authors discussed the regularization algorithm with two parameters to reconstruct the heat conduction coefficient in parabolic equation.

By [16] , one can easily prove that the minimizer $f_{\alpha,\beta}^{\delta}(x)$ of (1.4) satisfies the following normal equation:

$$(K^*K + \alpha I + \beta B^2)f_{\alpha,\beta}^{\delta}(x) = K^*g_\delta(x), \tag{3.1}$$

Because K is a linear self-adjoint compact operator, that is, $K^* = K$, we have the equivalent form

$$f^{\delta}_{\alpha,\beta}(x) = [K^2 + \alpha I + \beta B^2]^{-1} K^* g_{\delta}(x). \tag{3.2}$$

We define the Hilbert scale operator B by

$$B^r f := \sum_{n=1}^{\infty} (n\pi)^r \langle f, X_n \rangle X_n, \tag{3.3}$$

which yields $B^2 f = -f''$. With this choice, we have $X = L^2[0,1]$ and $X^2 = H^2[0,1] \cap H^1_0[0,1]$. Next we define a function of a compact self-adjoint operator K by the spectral mapping theorem in the following way:

If $f(x)$ is a real-valued continuous function on the spectrum $\sigma(K)$, we define $f(K)$ by

$$f(K)x = \sum_{n} f(\lambda_n)(x, \omega_n)\omega_n, \tag{3.4}$$

where K is compact self-adjoint, $\lambda_n \in \sigma(K)$ and ω_n are the corresponding orthogonal eigenvectors.

The operator K^2 and B^2 have common orthogonal eigenvectors, so we obtain

$$f^{\delta}_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} \frac{\frac{1 - e^{-n^2\pi^2}}{n^2\pi^2}}{\alpha + \beta n^2\pi^2 + \left(\frac{1 - e^{-n^2\pi^2}}{n^2\pi^2}\right)^2} (g_{\delta}, X_n) X_n. \tag{3.5}$$

Furthermore the operator B satisfies the following hypothesis:

Hypothesis I There exist positive constants N and p such that

$$\hat{f} \in W_{p,N} = \{f \in X \,|\, \|f\|_p \le N\} \tag{3.6}$$

which implies that $\hat{f} = B^{-p} v$ with $v \in X$ and $\|v\| \le N$.

From (2.6), we can obtain the following conditional stability.

Theorem 1. If $\|f(x)\|_p \le N$, then $\|f(x)\| \le 2^{\frac{p}{p+2}} \|g\|^{\frac{p}{p+2}} N^{\frac{2}{p+2}}$.

Proof. From (2.6) and Holder inequality, we have

$$\|f(x)\|^2 = \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} |g_n| \right)^2$$

$$= \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} \right)^2 |g_n|^{\frac{4}{p+2}} |g_n|^{\frac{2p}{p+2}}$$

$$\leq \left(\sum_{n=1}^{\infty} \left(\left(\frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} \right)^2 |g_n|^{\frac{4}{p+2}} \right)^{\frac{p+2}{2}} \right)^{\frac{2}{p+2}}$$

$$\times \left(\sum_{n=1}^{\infty} \left(|g_n|^{\frac{2p}{p+2}} \right)^{\frac{p+2}{p}} \right)^{\frac{p}{p+2}}$$

$$= \left(\sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} \right)^{p+2} |g_n|^2 \right)^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}}$$

$$\leq \left(\sum_{n=1}^{\infty} 2^p \, (n\pi)^{2p} \left(\frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} |g_n| \right)^2 \right)^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}}$$

$$= 2^{\frac{2p}{p+2}} \left(\sum_{n=1}^{\infty} ((n\pi)^p |f_n|)^2 \right)^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}}$$

$$= 2^{\frac{2p}{p+2}} \|f(x)\|_p^{\frac{4}{p+2}} \|g\|^{\frac{2p}{p+2}}$$

The proof is completed.

□

We consider an extension of the classical Morozov discrepancy principle for two parameter is

$$\left\| K f_{\alpha,\beta}^{\delta}(x) - g_\delta(x) \right\| = c\delta, c \geq 1. \tag{3.7}$$

By the following lemma, we have to verify the link condition characterizing the smoothing properties of the operator K relative to the operator B^{-1}.

Lemma 3.2. For $n \geq 1$, there holds

$$\|Kf\| \geq \frac{1}{2} \|B^{-2} f\|. \tag{3.8}$$

Proof.

$$\|Kf(x)\| = \left\|\sum_{n=1}^{\infty} \frac{1-e^{-n^2\pi^2}}{n^2\pi^2}(f, X_n)X_n\right\|.$$

By Lemma 2.1 we have,

$$\|Kf(x)\| \geq \frac{1}{2}\left\|\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}(f, X_n)X_n\right\|.$$

$$\geq \frac{1}{2}\|B^{-2}f\|$$

This completes the proof of the lemma. □

Under the above lemma and hypotheses I, we obtain the following estimate that has been proven in [16].

Lemma 3.3. [16] Let $K\hat{f} = g$ and $f_{\alpha,\beta}^{\delta}$ be the regularization solution of $Kf = g_\delta$. Then under the Hypotheses I, one has

$$\left\|f_{\alpha,\beta}^{\delta} - \hat{f}\right\| \leq 2N^{2/(2+p)}((c+1)\delta)^{p/(p+2)} = O(\delta^{p/(p+2)}), \tag{3.9}$$

for $p \in [1,2]$ and for every (α, β) satisfying the standard Morozov discrepancy principle.

4. NUMERICAL IMPLEMENTATION

After obtaining the theoretical results, we propose the numerical schemes for the inverse problem. The regularization parameter plays a major role in the numerical simulation. In fact, the effectiveness of a regularization method depends strongly on the choice of the regularization parameter.

From (3.5), we do an approximate truncation for the series by choosing the sum of the first N terms and use trapezoidal rule to approach the integral. After considering an equidistant grid $0 = x_0 < \cdots < x_M = 1,\ x_i = \frac{i}{M}, i = 0, \cdots, M)$, we get

$$f_{\alpha,\beta}^{\delta}(x_i) = 2\sum_{j=0}^{M}\sum_{n=1}^{N} \frac{\frac{n^2\pi^2}{1-e^{-n^2\pi^2}} g_\delta(x_j)\, sin\,(n\pi x_i)\, sin\,(n\pi x_j) h}{\alpha\left(\frac{n^2\pi^2}{1-e^{-n^2\pi^2}}\right)^2 + \beta n^2\pi^2\left(\frac{n^2\pi^2}{1-e^{-n^2\pi^2}}\right)^2 + 1} \tag{4.1}$$

where $h = \frac{1}{M}$ and $j = 0, \cdots, M$

Let $V_j = g(x_j)$ and $Z_i = f(x_i)$, $i,\ j = 0,\cdots,M$. We apply the noise data generated in the form

$$V^\delta = V + \hat{\delta} \times randn(size(V)). \tag{4.2}$$

where $\hat{\delta}$ is a noisy level. The function $randn(\cdot)$ generates arrays of random numbers whose elements are normally distributed. The error level δ is computed as

$$\|V - V^\delta\|_2^2 = \frac{1}{M+1}\sum_{j=0}^{M}(V_j - V_j^\delta)^2 = \delta^2. \tag{4.3}$$

We compute Z^δ by V^δ and the relative error $e_r(Z)$ by

$$e_r(Z) = \frac{\|Z^\delta(\cdot) - Z(\cdot)\|_2}{\|Z(\cdot)\|_2} \tag{4.4}$$

where $\|\cdot\|_2$ is given by (4.3).

In the following, we deal with the a-posteriori choice of the regularization parameter set (α, β) based on a damped Morozov discrepancy principle for the multi-parameter Tikhonov regularization method.

The damped Morozov discrepancy principle is given by

$$\|Kf_{\alpha,\beta}^\delta - g_\delta\|^2 + \alpha^\gamma\|f_{\alpha,\beta}^\delta\|^2 + \beta^k\|Bf_{\alpha,\beta}^\delta\|^2 = c\delta^2, \tag{4.5}$$

where $c \geq 1$ is a constant, $\gamma > 1$ and $k > 1$ are called the damped coefficients.

For fixed α and β, denote by $F(\alpha,\beta)$, the minimum of the regularization functional (1.4),

$$F(\alpha,\beta) := \|Kf_{\alpha,\beta}^\delta - g_\delta\|^2 + \alpha\|f_{\alpha,\beta}^\delta\|^2 + \beta\|Bf_{\alpha,\beta}^\delta\|^2. \tag{4.6}$$

From now on, we call $F(\alpha,\beta)$ the optimal function of the regularization functional $J(\alpha,\beta;f)$.

Lemma 4.1. The first partial derivatives of the optimal function $F(\alpha,\beta)$ are given by $\partial_\alpha F(\alpha,\beta) = \frac{\partial F(\alpha,\beta)}{\partial\alpha} = \|f^\delta\|^2, \partial_\beta F(\alpha,\beta) = \frac{\partial F(\alpha,\beta)}{\partial\beta} = \|Bf^\delta\|^2$.

In terms of $F(\alpha,\beta)$, the discrepancy principle (4.5) can be written as

$$\begin{aligned} G(\alpha,\beta) &= F(\alpha,\beta) - (\alpha - \alpha^\gamma)\partial_\alpha F(\alpha,\beta) - (\beta - \beta^k)\partial_\beta F(\alpha,\beta) \\ &= c\delta^2. \end{aligned} \tag{4.7}$$

We call $G(\alpha,\beta)$ as the discrepancy function. To solve the equation (4.7), we use model function method. The idea of the model function method is to construct a locally approximate function of $F(\alpha,\beta)$ with some constants to be updated at each iteration. This approximate function has an explicit expression, denoted by $m(\alpha,\beta)$, which we call the model function of $F(\alpha,\beta)$. Then the exact discrepancy equation (4.7) is approximated by the simple equation

$$m(\alpha,\beta)-(\alpha-\alpha^{\gamma})\frac{\partial m(\alpha,\beta)}{\partial\alpha}-(\beta-\beta^{k})\frac{\partial m(\alpha,\beta)}{\partial\beta}-c\delta^{2}=0. \tag{4.8}$$

In fact, $m(\alpha,\beta)$ approximates the optimal function $F(\alpha,\beta)$ locally with the constants C,D,T to be determined by requiring that

$$m(\alpha,\beta)|_{\alpha=x,\beta=y}=F(\alpha,\beta)|_{\alpha=x,\beta=y}, \tag{4.9}$$

$$\frac{\partial m(\alpha,\beta)}{\partial\alpha}|_{\alpha=x,\beta=y}=\frac{\partial F(\alpha,\beta)}{\partial\alpha}|_{\alpha=x,\beta=y}, \tag{4.10}$$

$$\frac{\partial m(\alpha,\beta)}{\partial\beta}|_{\alpha=x,\beta=y}=\frac{\partial F(\alpha,\beta)}{\partial\beta}|_{\alpha=x,\beta=y}, \tag{4.11}$$

where x,y are approximate values of regularization parameters obtained at each iteration. This means that C,D,T depend on x and y. From the techniques presented in [18], we use the linear model function as follows.

$$m(\alpha,\beta)=T+C\alpha+D\beta, \tag{4.12}$$

where $C=C(x,y),D=D(x,y),T=T(x,y)$. It can be easily verified that these linear model functions satisfy differential equation (4.8) that approximates the exact discrepancy equation (4.7). From (4.9), we have

$$C=\|f(x,y)\|^{2},D=\|Bf(x,y)\|^{2},T=\left\|Kf(x,y)-g^{\delta}\right\|^{2}. \tag{4.13}$$

For the linear model function (4.12), we get a locally approximate Morozov discrepancy equation(4.8) of the form

$$G(\alpha,\beta)=m(\alpha,\beta)-(\alpha-\alpha^{\gamma})\frac{\partial m(\alpha,\beta)}{\partial\alpha}-(\beta-\beta^{k})\frac{\partial m(\alpha,\beta)}{\partial\beta}-c\delta^{2}=0.$$

From this, we have

$$G(\alpha,\beta)=\left\|Kf_{\alpha,\beta}^{\delta}-g^{\delta}\right\|^{2}+\alpha^{\gamma}\left\|f_{\alpha,\beta}^{\delta}\right\|^{2}+\beta^{k}\left\|Bf_{\alpha,\beta}^{\delta}\right\|^{2}-c\delta^{2}. \tag{4.14}$$

For the local convergence of the linear model function (4.12), we introduce the following relaxation form of the approximation discrepancy equation

$$\hat{G}(\alpha,\beta_k)=G(\alpha,\beta_k)+\lambda_{k,1}[G(\alpha,\beta_k)-G(\alpha_k,\beta_k)]=0, \tag{4.15}$$

where the relaxation factor λ_k is determined by requiring that $\hat{G}(0,\beta_k)=-\hat{\sigma}|G(\alpha,\beta_k)|,\hat{\sigma}\in(0,1)$; that is,

$$\lambda_k = \frac{G(0,\beta_k) + \hat{\sigma}|G(0,\beta_k)|}{G(\alpha_k,\beta_k) - G(0,\beta_k)}. \tag{4.16}$$

In the following, we state the algorithm by using the relaxation form of the approximate discrepancy equation:

Given $\alpha_0 > 0, \beta_0 > 0, \epsilon_1 > 0, \epsilon_2 > 0$, set $k = 0$.

Step 1: Solve $f_{\alpha,\beta}^{\delta}$ by (3.5) for (α_k, β_k).

Step 2: Compute $G(\alpha_k, \beta_k)$ and $G(0, \beta_k)$ where $C = C(\alpha_k, \beta_k), D = D(\alpha_k, \beta_k)$, $T = T(\alpha_k, \beta_k)$.

Step 3: If $G(\alpha_0, \beta_0)G(\alpha_k, \beta_k) \leq 0$, then go to step 8; otherwise, solve for α_{k+1} the relaxation discrepancy equation

$$\hat{G}(\alpha, \beta_k) = G(\alpha, \beta_k) + \lambda_{k,1}[G(\alpha, \beta_k) - G(\alpha_k, \beta_k)] = 0, \tag{4.17}$$

$$\text{where } \lambda_{k,1} = \frac{G(0,\beta_k) + \hat{\sigma}|G(0,\beta_k)|}{G(\alpha_k,\beta_k) - G(0,\beta_k)}.$$

Step 4: Solve $f_{\alpha,\beta}^{\delta}$ by (3.5) for (α_{k+1}, β_k).

Step 5: Compute $G(\alpha_{k+1}, \beta_k)$ and $G(\alpha_{k+1}, 0)$ where $C = C(\alpha_{k+1}, \beta_k), D = D(\alpha_{k+1}, \beta_k)$, $T = T(\alpha_{k+1}, \beta_k)$.

Step 6: If $G(\alpha_0, \beta_0)G(\alpha_{k+1}, \beta_k) \leq 0$, then $\alpha_k = \alpha_{k+1}$ and go to step 8; otherwise, solve for β_{k+1} the relaxation discrepancy equation

$$\hat{G}(\alpha_{k+1}, \beta) = G(\alpha_{k+1}, \beta) + \lambda_{k,2}[G(\alpha_{k+1}, \beta) - G(\alpha_{k+1}, \beta_k)] = 0, \tag{4.18}$$

$$\text{where } \lambda_{k,2} = \frac{G(\alpha_{k+1},0) + \hat{\sigma}|G(\alpha_{k+1},0)|}{G(\alpha_{k+1},\beta_k) - G(\alpha_{k+1},0)}.$$

Step 7: If $\frac{|\alpha_{k+1}-\alpha_k|}{\alpha_{k+1}} < \epsilon_1$ and $\frac{|\beta_{k+1}-\beta_k|}{\beta_{k+1}} < \epsilon_2$, then $\alpha_k = \alpha_{k+1}, \beta_k = \beta_{k+1}$ and go to step 8; otherwise, set $k = k + 1$ and go to step 1.

Step 8: Stop and return α_k, β_k and $f_{\alpha_k,\beta_k}^{\delta}$ values.

Firstly we get the measurable data $g(x)$ from solving the equation (1.1) by using the implicit finite difference method when the exact solution $f(x)$ is given. Secondly $g_\delta(x)$ is obtained by (4.2) and the regularization solution $f_{\alpha,\beta}^{\delta}$ is obtained. In our implementations, we use the equal space and time step sizes as $\frac{1}{100}$.

We have performed numerical experiment to test the stability of our algorithm for different noise levels and initial data. The stopping criterion for the iteration is chosen as $\epsilon_1 = \epsilon_2 = 10^{-2}$. In all experiments, some chosen parameters are $M = 100$ and $\gamma = k = 1.4$.

Example 4.1. Let us consider $f(x) = \pi^2 \sin \pi x, x \in (0,1)$. The exact solution of the forward problem for this $f(x)$ is $u(x,t) = (1 - e^{\pi^2 t}) \sin \pi x$ of the problem (1.1).

The source term $f(x)$ is to be recovered from the noise observation data g_δ with $c = 1, N = 5$. In the experiment, we compare the performance of the two-parameter regularization solution $f_{\alpha,\beta}^{\delta}$ and the standard single-parameter regularization solutions f_α^δ and f_β^δ given by

$$f_\alpha^\delta = f^\delta(\alpha, 0) = (K^*K + \alpha I)^{-1} K^* g_\delta, \tag{4.19}$$

$$f_\beta^\delta = f^\delta(0, \beta) = (K^*K + \beta B^*B)^{-1} K^* g_\delta \tag{4.20}$$

Table 1. The relative error in source $f_{\alpha,\beta}^{\delta}$ for different noise levels with $\alpha_0 = \beta_0 = 0.05$.

$\hat{\delta}$	α	β	$\|g - g^\delta\|_2$	$er(f_{\alpha,\beta}^{\delta})$	iter
0.01	$1.3149 * 10^{-6}$	$1.7302 * 10^{-7}$	0.01206	0.01790	10
0.005	$1.3410 * 10^{-6}$	$3.2025 * 10^{-7}$	0.00516	0.00718	10
0.001	$3.2635 * 10^{-7}$	$2.3785 * 10^{-8}$	0.00105	0.00472	11

In Table 1, we present some numerical results of the Example 4.1 solved by the two-parameter regularization method with different noise levels $\hat{\delta}$. The regularization parameters α and β obtained by model function approach to the damped Morozov principle are given in the second and third columns. The fourth and fifth columns of the table give the errors in the observation data g and relative error in the computed source term respectively. The last column shows the number of iterations.

Table 2. The relative error in source f_α^δ for different noise levels with $\alpha_0 = 0.05$.

$\hat{\delta}$	α	$\|g - g^\delta\|_2$	$er(f_\alpha^\delta)$	iter
0.01	$3.99932 * 10^{-6}$	0.00910596	0.0226715	7
0.005	$1.01574 * 10^{-6}$	0.00525713	0.0174744	8
0.001	$3.1552 * 10^{-7}$	0.000911926	0.00523301	8

Table 3. The relative error in source f_β^δ for different noise levels with $\beta_0 = 0.05$.

$\hat{\delta}$	β	$\|g - g^\delta\|_2$	$er(f_\beta^\delta)$	iter
0.01	$1.69477 * 10^{-}$	0.0103861	0.0208723	10
0.005	$2.13227 * 10^{-}$	0.00501385	0.00991339	10
0.001	$5.31096 * 10^{-}$	0.00108376	0.00597924	11

In Table 2-3, we present some numerical results of the Example 4.1 solved by the single-parameter regularization method with different noise levels $\hat{\delta}$ repectively.

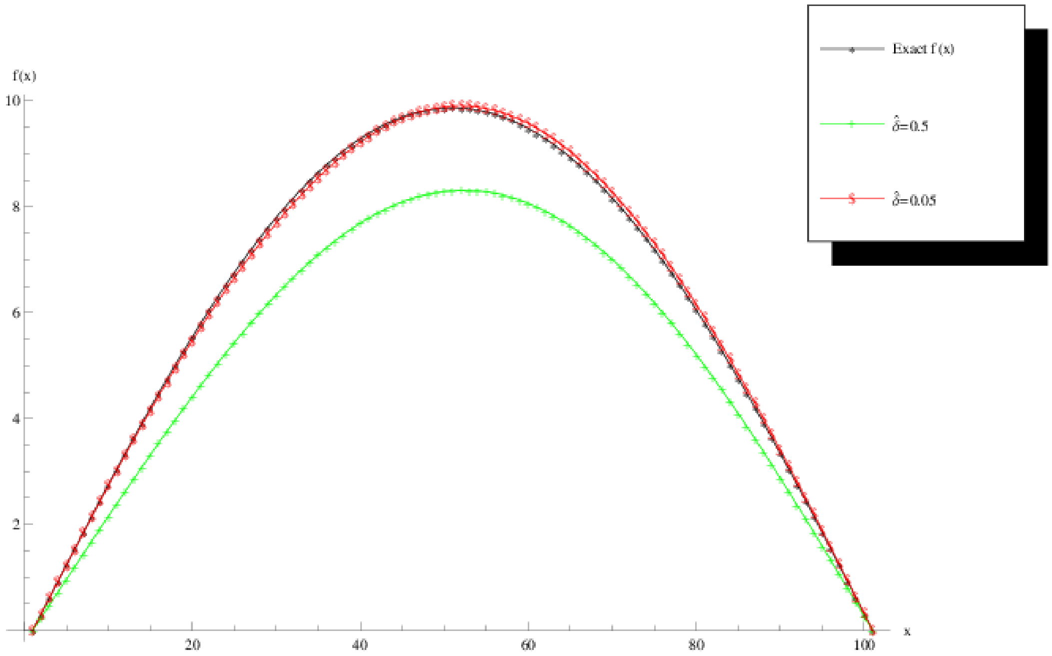

Figure 1. Exact and computed source term for different $\hat{\delta}$.

Figure 1 shows the plot of the approximation of the unknown source function $f(x)$ of the Example 4.1 by the two-parameter regularization method for different noise levels $\hat{\delta}$. From this, we see that the efficiency of reconstruction of source term depends on the noise level.

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