Identities of symmetry for Carlitz’s twisted
(h, q)-tangent polynomials associated with
p-adic integral on \( \mathbb{Z}_p \)

C. S. Ryoo
Department of Mathematics,
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper, we discover symmetric properties for Carlitz’s twisted \((h, q)\)-tangent polynomials.


Keywords: Tangent numbers and polynomials, twisted \((h, q)\)-tangent polynomials, symmetric identities, Carlitz’s twisted \((h, q)\)-tangent number and polynomials.

1. Introduction

Many mathematicians have worked some identities of symmetry for \(q\)-extension of Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials (see [2, 3, 4, 6, 7, 8, 9, 10, 11]). Recently, Y. He derived several identities of symmetry for Carlitz’s \(q\)-Bernoulli numbers and polynomials in complex field (see [2]). D. Kim et al. [3] investigated some identities of symmetry for Carlitz’s \(q\)-Euler numbers and polynomials in complex field. J. Y. Kang and C. S. Ryoo obtained some identities of symmetry for \(q\)-Genocchi polynomials (see [1]). In [5], we obtained some identities of symmetry for Carlitz’s twisted \(q\)-Euler polynomials associated with \(p\)-adic integral on \( \mathbb{Z}_p \). Our aim in this paper is to discover special symmetric properties for Carlitz’s twisted \((h, q)\)-tangent polynomials. Throughout this paper we use the following notations. By \( \mathbb{Z}_p \) we denote the ring of \( p \)-adic rational integers, \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers, \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \), \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{Z} \) denotes the ring of rational integers, \( \mathbb{Q} \) denotes the field of rational numbers, \( \mathbb{C} \) denotes the set of complex numbers, and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( \nu_p \) be the normalized exponential valuation
of \( \mathbb{C}_p \) with \(|p|_p = p^{-v_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assumes that \(|q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \(|q - 1|_p < p^{-\frac{1}{p-1}} \) so that \( q^x = \exp(x \log q) \) for \(|x|_p \leq 1 \). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q} \quad (\text{cf. } [1, 2, 3, 4]).
\]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \(|x|_p \leq 1 \) in the present \( p \)-adic case. Let

\[ g \in UD(\mathbb{Z}_p) = \{ g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \}. \]

For \( g \in UD(\mathbb{Z}_p) \), the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} g(x)(-1)^x, \quad \text{see } [4]. \tag{1.1}
\]

First, we introduce the Carlitz's type twisted \((h, q)\)-tangent numbers \( T_{n,q,\zeta}^{(h)} \) and polynomials \( T_{n,q,\zeta}^{(h)}(x) \) and investigate their properties(see [5]). Let \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( h \in \mathbb{Z} \), and

\[
T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},
\]

where \( C_{p^m} = \{ \zeta | \zeta^{p^m} = 1 \} \) is the cyclic group of order \( p^m \). For \( \zeta \in T_p \), we denote by \( \phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \zeta^x \). For \( q \in \mathbb{C}_p \) with \(|1 - q|_p < 1 \) and \( \zeta \in T_p \), the Carlitz’s type twisted \((h, q)\)-tangent polynomials \( T_{n,q,\zeta}^{(h)}(x) \) are defined by

\[
T_{n,q,\zeta}^{(h)}(x) = \int_{\mathbb{Z}_p} w^y q^{hy}[2y + x]_q d\mu_{-1}(y). \tag{1.2}
\]

When \( x = 0 \), \( T_{n,q,\zeta}^{(h)}(0) = T_{n,q,\zeta}^{(h)} \) is called the \( n \)th Carlitz’s twisted \((h, q)\)-tangent numbers.

### 2. Symmetric identities for Carlitz’s twisted \((h, q)\)-tangent numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz’s twisted \((h, q)\)-tangent numbers \( T_{n,q,\zeta}^{(h)} \) and polynomials \( T_{n,q,\zeta}^{(h)}(x) \). Since \([x + 2y]_q = [x]_q + \)
Carlitz’s twisted \((h, q)\)-tangent polynomials associated with \(p\)-adic integral

\[ q^x \lfloor 2y \rfloor_q, \] we see that

\[
T_{n, q, \zeta}^{(h)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q}^{n-l} q^{xl} T_{l, q, \zeta}^{(h)}
\]

\[ = \left( q^x T_{q, \zeta}^{(h)} + [x]_{q} \right)^n \]

\[ = 2 \sum_{m=0}^{\infty} (-1)^m \zeta^m q^{hm} [x + 2m]_{q}^n, \] (2.1)

with the usual convention of replacing \((T_{q, \zeta}^{(h)})^n\) by \(T_{n, q, \zeta}^{(h)}\). Let \(w_1\) and \(w_2\) be odd numbers. Then we have

\[
\int_{\mathbb{Z}_p} \zeta^{w_1y} q^{w_1y} e^{\left[w_2x + \frac{2w_2}{w_1} j + 2y\right]} [w_1]_q^t d\mu_{-1}(y)
\]

\[ = \lim_{N \to \infty} \sum_{y=0}^{w_2 p^N - 1} \zeta^{w_1y} q^{w_1y} e^{[w_1 w_2 x + 2w_2 j + 2w_1 y]_q^t} (-1)^y
\]

\[ = \lim_{N \to \infty} \sum_{i=0}^{w_2 - 1} \sum_{y=0}^{p^N - 1} \zeta^{w_1(i+w_2 y)} q^{w_1 h(i+w_2 y)} e^{[w_1 w_2 x + 2w_2 j + 2w_1(i+w_2 y)]_q^t} (-1)^{i+w_2 y}
\] (2.2)

From (2.2), we can derive the following equation (2.3):

\[
\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2j} q^{w_2j} \int_{\mathbb{Z}_p} \zeta^{w_1y} q^{w_1y} e^{\left[w_2x + \frac{2w_2}{w_1} j + 2y\right]} [w_1]_q^t d\mu_{-1}(y)
\]

\[ = \lim_{N \to \infty} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} (-1)^{i+j} \zeta^{w_2j} \zeta^{w_1i} \zeta^{w_1 w_2 y} q^{w_2 h j} q^{w_1 h i} q^{w_1 w_2 y}
\times e^{[w_1 w_2 x + 2w_2 j + 2w_1 i + 2w_1 w_2 y]_q^t} (-1)^y
\] (2.3)

By the same method as (2.3), we obtain

\[
\sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1j} q^{w_1j} \int_{\mathbb{Z}_p} \zeta^{w_2y} q^{w_2y} e^{\left[w_1x + \frac{2w_1}{w_2} j + 2y\right]} [w_2]_q^t d\mu_{-1}(y)
\]

\[ = \lim_{N \to \infty} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} (-1)^{i+j} \zeta^{w_1j} \zeta^{w_2i} \zeta^{w_1 w_2 y} q^{w_1 h j} q^{w_2 h i} q^{w_1 w_2 y}
\times e^{[w_1 w_2 x + 2w_1 j + 2w_2 i + 2w_1 w_2 y]_q^t} (-1)^y
\] (2.4)
Therefore, by (2.3) and (2.4), we have the following theorem.

**Theorem 2.1.** For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\sum_{j=0}^{w_1-1} (-1)^j \zeta w_2^j q^{hw_2 j} \int_{\mathbb{Z}_p} \xi^{w_1 y} q^{w_1 hy} e^{\left[ w_2 x + \frac{2 w_2}{w_1} j + 2y \right]} d\mu_{-1}(y) = \sum_{j=0}^{w_2-1} (-1)^j \zeta w_1^j q^{hw_1 j} \int_{\mathbb{Z}_p} \xi^{w_2 y} q^{w_2 hy} e^{\left[ w_1 x + \frac{2 w_1}{w_2} j + 2y \right]} d\mu_{-1}(y).$$

(2.5)

By substituting Taylor series of $e^{\xi y}$ into (2.5) and after calculations, we obtain the following corollary.

**Corollary 2.2.** For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta w_2^j q^{hw_2 j} \int_{\mathbb{Z}_p} \xi^{w_1 y} q^{w_1 hy} e^{\left[ w_2 x + \frac{2 w_2}{w_1} j + 2y \right]} d\mu_{-1}(y) = [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta w_1^j q^{hw_1 j} \int_{\mathbb{Z}_p} \xi^{w_2 y} q^{w_2 hy} e^{\left[ w_1 x + \frac{2 w_1}{w_2} j + 2y \right]} d\mu_{-1}(y).$$

(2.6)

By (1.2) and Corollary 2.2, we have the following theorem.

**Theorem 2.3.** For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta w_2^j q^{hw_2 j} T_{n,q^{w_1},\zeta w_1}^{(h)} \left( w_2 x + \frac{2 w_2}{w_1} j \right) = [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta w_1^j q^{hw_1 j} T_{n,q^{w_2},\zeta w_2}^{(h)} \left( w_1 x + \frac{2 w_1}{w_2} j \right).$$

(2.7)

By (2.6), we can derive the following equation (2.7):
By (2.7) and Theorem 2.3, we have

\[
[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{h w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 h y} \left[ w_2 x + \frac{2w_2}{w_1} j + 2y \right]_q^n d\mu_{-1}(y)
\]

which equals

\[
= \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{h w_2 j} \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} [2j]_q^i q^{w_2(n-i)j} T_{n-i, q^{w_1}, \zeta^{w_1}}^{(h)} (w_2 x)
\]

where

\[
S_{n,i}^{(h)}(w_1, \zeta, q) = \sum_{j=0}^{w_1-1} (-1)^j \zeta^j q^{(n-i+h)j} [2j]_q^i,
\]

is called as the sums of twisted even \(q\)-integer powers. By the same method as (2.8), we get

\[
[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_2 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} q^{w_2 h y} \left[ w_1 x + \frac{2w_1}{w_2} j + 2y \right]_q^n d\mu_{-1}(y)
\]

which equals

\[
= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i, q^{w_2}, \zeta^{w_2}}^{(h)} (w_1 x) S_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}).
\]

By (2.8) and (2.9), we have the following theorem.

**Theorem 2.4.** For \(w_1, w_2 \in \mathbb{N}\) with \(w_1 \equiv 1 \pmod{2}\), \(w_2 \equiv 1 \pmod{2}\), we have

\[
\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} S_{n,i}^{(h)}(w_1, \zeta^{w_1}, q^{w_2}) T_{n-i, q^{w_1}, \zeta^{w_1}}^{(h)} (w_2 x)
\]

which equals

\[
= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} S_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}) T_{n-i, q^{w_2}, \zeta^{w_2}}^{(h)} (w_1 x).
\]

Observe that if \(h = 1\), then (2.10) reduces to Theorem 2.4 in [8]. If we take \(x = 0\) in Theorem 2.4, we also derive the interesting identity for Carlitz’s twisted \((h, q)\)-tangent numbers as follows:
Corollary 2.5. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have
\[
\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} S_{n,i}^{(h)}(w_1, \zeta^{w_2}, q^{w_2}) T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)} = \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} S_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}) T_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}.
\]

Acknowledgement

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

References