Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 3869-3877
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http://www.ripublication.com/gjpam.htm

Symmetric identities for twisted Changhee polynomials associated with p-adic integral on $\mathbb{Z}_p$

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Abstract

In this paper, we investigate some new and interesting symmetric identities for the twisted Changhee polynomials associated with fermionic p-adic invariant integral on $\mathbb{Z}_p$.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: Changhee polynomials, twisted Changhee polynomials, fermionic p-adic integral.

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1. Introduction

For given odd prime number p, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integer, the field of p-adic rational numbers and the complication of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normally defined by $|\ p\ |_p = \frac{1}{p}$.

The *fermionic* p*-adic integral on* $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)du_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [1-4,6,15,17,18])}, \quad (1.1)$$

where $f(x)$ is a continuous functions on $\mathbb{Z}_p$.

From (1.1), we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [1-4,6,15,17,18])}, \quad (1.2)$$

where $f_1(x) = f(x+1)$.

It is not difficult to show that

$$\int_{\mathbb{Z}_p} f_n(x)du_{-1}(x) + \int_{\mathbb{Z}_p} f(x)du_{-1}(x) = 2\sum_{l=0}^{n-1}(-1)^l f(l), \text{ (see [16])} \quad (1.3)$$

where $f_n(x) = f(x+n)$ and $n \equiv 1$ (mod2).

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{ (see [1,2,5,6,8,9,10,11,16])}. \quad (1.4)$$

When $x = 0$, $E_n = E_n(0)$, $(n \geq 0)$ are called *Euler numbers*.

By (1.2), the Witt's formula for Euler polynomials is given by

$$\int_{\mathbb{Z}_p} e^{(x+y)t}du_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \quad (1.5)$$

By (1.5), we note that

$$\int_{\mathbb{Z}_p} (x+y)^n du_{-1}(y) = E_n(x), (n \geq 0), \text{ (see [1,2,6,10])}. \quad (1.6)$$

The *Stirling number of the first kind* is defined as follows:

$$(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, (n \geq 1), \text{ (see [13,14,15])}. \quad (1.7)$$

The *Stirling number of the second kind* is defined by the generating function to be

$$\frac{1}{n!}(e^t - 1)^m = \sum_{n=m}^{\infty} S_2(n, m)\frac{t^n}{n!}, \text{ (see [13,14,15]).} \tag{1.8}$$

The λ-*Changhee polynomials* are defined by the generating function to be

$$\frac{2}{(1+t)^\lambda + 1}(1+t)^{\lambda x} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}, \text{ (see [13]).} \tag{1.9}$$

When $x = 0$, $Ch_{n,\lambda} = Ch_{n,\lambda}(0)$ are called Changhee numbers.

From (1.2) and (1.9), we have

$$\int_{\mathbb{Z}_p} (1+t)^{\lambda(x+y)} du_{-1}(y) = \frac{2}{(1+t)^\lambda + 1}(1+t)^{\lambda x} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.10}$$

Thus, by (1.10), we easily get

$$Ch_{n,\lambda}(x) = \int_{\mathbb{Z}_p} (\lambda x + \lambda y)_n du_{-1}(y), n \geq 0. \tag{1.11}$$

From (1.11) and (1.7), we note that

$$\begin{aligned} Ch_{n,\lambda}(x) &= \sum_{l=0}^{n} S_1(n,l)\lambda^l \int_{\mathbb{Z}_p} (x+y)^l du_{-1}(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^l E_l(x), (n \geq 0). \end{aligned} \tag{1.12}$$

Recently, some authors have studies generalization of Changhee polynomials arising from p-adic integral on $\mathbb{Z}_p$, and investigated the properties of Changhee numbers and polynomials (see [3,4,12-15,17,18]). In this paper, we give some explicit and new symmetric identities for twisted Changhee polynomials arising from the fermionic intergal on $\mathbb{Z}_p$.

2. Symmetric identities for twisted Changhee polynomials

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \left\{\omega | \omega^{p^n} = 1\right\}$ is the cyclic group of order p^n.

For $\varepsilon \in T_p$ and $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the *twisted Changhee polynomials* are defined as

$$\left(\frac{2}{(1+\varepsilon t)^{\lambda}+1}\right)(1+\varepsilon t)^{\lambda x} = \sum_{n=0}^{\infty} Ch_{n,\lambda,\varepsilon}(x)\frac{t^n}{n!}, \text{ (see [15])}. \tag{2.1}$$

When $x = 0$, $Ch_{n,\lambda,\varepsilon} = Ch_{n,\lambda,\varepsilon}(0)$ are called the *twisted Changhee numbers*.

Let us assume that $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$ in this section. From (1.3), we have

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^{\lambda x+\lambda n} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (1+\varepsilon t)^{\lambda x} d\mu_{-1}(x) = 2\sum_{l=0}^{n-1}(-1)^l(1+\varepsilon t)^{\lambda l}. \tag{2.2}$$

Thus, by (2.2), we get

$$\begin{aligned} Ch_{n,\lambda,\varepsilon}(n) + Ch_{m,\lambda,\varepsilon} &= 2\sum_{l=0}^{n-1}(-1)^l \sum_{k=0}^{m} \lambda^k l^k S_1(m,k)\varepsilon^n \\ &= 2\sum_{k=0}^{m} \lambda^k S_1(m,k) C_k(n-1)\varepsilon^n, \end{aligned} \tag{2.3}$$

where $C_k(n-1) = \sum_{l=0}^{n-1}(-1)^l l^k$. From (2.3), we can derive the following integral equation:

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^{\lambda x+\lambda n} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (1+\varepsilon t)^{\lambda x} d\mu_{-1}(x) = \frac{2\int_{\mathbb{Z}_p}(1+\varepsilon t)^{\lambda x} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{\lambda n x} d\mu_{-1}(x)}. \tag{2.4}$$

It is easy to show that

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^{nx} d\mu_{-1}(x) = \frac{2}{(1+\varepsilon t)^n + 1}, \ (n \in \mathbb{N} \ with \ n \equiv 1 \ (\text{mod } 2)) \tag{2.5}$$

By using above equations, we obtain the following theorems.

Theorem 2.1. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(mod\ 2)$, $w_2 \equiv 1(mod\ 2)$. Then we have

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{m=0}^{n-i}\binom{n}{i} Ch_{i,w_1}(w_2 x) w_2^m S_1(n-i,m) C_m(w_1-1)\varepsilon^{n-i} \\ &= \sum_{i=0}^{n}\sum_{m=0}^{n-i}\binom{n}{i} Ch_{i,w_2}(w_1 x) w_1^m S_1(n-i,m) C_m(w_2-1)\varepsilon^{n-i}. \end{aligned}$$

where $n \in \mathbb{N} \cup \{0\}$.

Theorem 2.2. For $n \geq 0$, we have

$$Ch_{n,\lambda,\varepsilon}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}(x)_l \lambda^{k-l} S_1(k,l) Ch_{n-l,\lambda,\varepsilon}\varepsilon^{n-l}.$$

Theorem 2.3. For $n \geq 0$, we have

$$Ch_{n,\lambda,\varepsilon}(x) = \sum_{m=0}^{n} Ch_{m,\varepsilon}(x)\lambda^{n-m} S_1(n,m).$$

Theorem 2.4. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(mod\ 2)$, $w_2 \equiv 1(mod\ 2)$, we have

$$\sum_{l=0}^{w_1-1}(-1)^l Ch_{n,w_1,\varepsilon}(w_2 x + \frac{w_2}{w_1} l) = \sum_{l=0}^{w_2-1}(-1)^l Ch_{n,w_2,\varepsilon}(w_1 x + \frac{w_1}{w_2} l).$$

Theorem 2.5. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(mod\ 2)$, $w_2 \equiv 1(mod\ 2)$, we have

$$\sum_{l=0}^{w_1-1}\sum_{k=0}^{n}(-1)^l S_1(n,k) w_1^k E_k(w_2 x + \frac{w_2}{w_1} l) = \sum_{l=0}^{w_2-1}\sum_{k=0}^{n}(-1)^l S_1(n,k) w_2^k E_k(w_1 x + \frac{w_1}{w_2} l).$$

3. The proofs of theorems

Let $w_1, w_2 \in \mathbb{N}$ be odd numbers. From the fermionic p-adic integral on $\mathbb{Z}_p$, we get

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1 x_1 + w_2 x_2} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1 w_2 x} d\mu_{-1}(x)} = \frac{2(1+(1+\varepsilon t)^{w_1 w_2})}{((1+\varepsilon t)^{w_1}+1)((1+\varepsilon t)^{w_2}+1)}. \tag{3.1}$$

Now, we consider the following fermionic p-adic integral on $\mathbb{Z}_p$ associated with twisted Changhee polynomials

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1 x_1 + w_2 x_2 + w_1 w_2 x} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1 w_2 x} d\mu_{-1}(x)} \\ &= \frac{2(1+\varepsilon t)^{w_1 w_2 x}(1+(1+\varepsilon t)^{w_1 w_2})}{((1+\varepsilon t)^{w_1}+1)((1+\varepsilon t)^{w_2}+1)}.\end{aligned} \tag{3.2}$$

Now, we observe that

$$\begin{aligned}\frac{2\int_{\mathbb{Z}_p}(1+\varepsilon t)^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1x}d\mu_{-1}(x)} &= 2\sum_{l=0}^{w_1-1}(-1)^l(1+\varepsilon t)^l \\ &= 2\sum_{k=0}^{\infty}\left(\sum_{k=0}^{w_1-1}(-1)^l\sum_{m=0}^{k}l^m S_1(k,m)\right)\frac{\varepsilon^k t^k}{k!} \\ &= 2\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_1(k,m)\sum_{l=0}^{w_1-1}(-1)^l l^m\right)\frac{\varepsilon^k t^k}{k!} \\ &= 2\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_1(k,m)C_m(w_1-1)\right)\frac{\varepsilon^k t^k}{k!}.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} I &= \left(\frac{1}{2}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1(x_1+w_2x)}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1w_2x}d\mu_{-1}(x)}\right) \\ &= \frac{1}{2}\left(\sum_{i=0}^{\infty}Ch_{i,w_1,\varepsilon}(w_2x)\frac{t^i}{i!}\right)\left(2\sum_{l=0}^{w_1-1}(1+\varepsilon t)^{w_2l}(-1)^l\right) \\ &= \frac{1}{2}\left(\sum_{i=0}^{\infty}Ch_{i,w_1,\varepsilon}(w_2x)\frac{t^i}{i!}\right)\left(2\sum_{k=0}^{\infty}\sum_{m=0}^{k}w_2^m S_1(k,w)C_m(w_1-1)\right)\frac{\varepsilon^k t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\binom{n}{i}Ch_{i,w_1}(w_2x)\sum_{m=0}^{n-i}w_2^m S_1(n-i,m)C_m(w_1-1)\right\}\frac{\varepsilon^{n-i}t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\sum_{m=0}^{n-i}\binom{n}{i}Ch_{i,w_1}(w_2x)w_2^m S_1(n-i,m)C_m(w_1-1)\right\}\frac{\varepsilon^{n-i}t^n}{n!}.\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} I &= \left(\frac{1}{2}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_2(x_2+w_1x)}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1x_1}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1w_2x}d\mu_{-1}(x)}\right) \\ &= \frac{1}{2}\left(\sum_{i=0}^{\infty}Ch_{i,w_2,\varepsilon}(w_1x)\frac{t^i}{i!}\right)\left(2\sum_{l=0}^{w_2-1}(1+\varepsilon t)^{w_1l}(-1)^l\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\sum_{m=0}^{n-i}\binom{n}{i}Ch_{i,w_2,\varepsilon}(w_1x)w_1^m S_1(n-i,m)C_m(w_2-1)\right\}\frac{\varepsilon^{n-i}t^n}{n!}.\end{aligned} \tag{3.5}$$

By comparing the coefficients on the both sides of (3.4) and (3.5), we obtained the Theorem 2.1.

Now, we consider

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,\lambda,\varepsilon}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}(1+\varepsilon t)^{(x+y)\lambda}d\mu_{-1}(y) = \frac{2}{(1+\varepsilon t)^{\lambda}+1}(1+\varepsilon t)^{\lambda x} \\ &= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda,\varepsilon}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(\lambda x)_m\frac{\varepsilon^m t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda,\varepsilon}(\lambda x)_{n-l}\right)\frac{\varepsilon^{n-l}t^n}{n!}.\end{aligned} \tag{3.6}$$

Thus, by (3.6), we get

$$Ch_{n,\lambda,\varepsilon}(x) = \sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda,\varepsilon}(\lambda x)_{n-l}\varepsilon^{n-l}. \tag{3.7}$$

From (3.6) and (3.7), we proved the Theorem 2.2 and Theorem 2.3.

From (3.2), we get that

$$\begin{aligned}I &= \left(\frac{(1+\varepsilon t)^{w_1w_2x}}{2}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1x_1}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{Z_p}(1+\varepsilon t)^{w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1w_2x}d\mu_{-1}(x)}\right) \\ &= \left(\frac{(1+\varepsilon t)^{w_1w_2x}}{2}\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1x_1}d\mu_{-1}(x_1)\right)\left(2\sum_{l=0}^{w_1-1}(-1)^l(1+\varepsilon t)^{w_2l}\right) \\ &= \sum_{l=0}^{w_1-1}(-1)^l\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1x_1+w_1w_2x+w_2l}d\mu_{-1}(x_1) \\ &= \sum_{l=0}^{w_1-1}(-1)^l\int_{\mathbb{Z}_p}(1+\varepsilon t)^{w_1(x_1+w_2x+\frac{w_2}{w_1}l)}d\mu_{-1}(x_1) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{w_1-1}(-1)^lCh_{n,w_1,\varepsilon}(w_2x+\frac{w_2}{w_1}l)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.8}$$

On the other hand, by (3.2), we get

$$\begin{aligned} I &= \left(\frac{(1+\varepsilon t)^{w_1 w_2 x}}{2} \int_{\mathbb{Z}_p} (1+\varepsilon t)^{w_2 x_2} d\mu_{-1}(x_2) \right) \left(\frac{2\int_{Z_p} (1+\varepsilon t)^{w_1 x_1} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} (1+\varepsilon t)^{w_1 w_2 x} d\mu_{-1}(x)} \right) \\ &= \left(\frac{(1+\varepsilon t)^{w_1 w_2 x}}{2} \int_{\mathbb{Z}_p} (1+\varepsilon t)^{w_2 x_2} d\mu_{-1}(x_2) \right) \left(2 \sum_{l=0}^{w_2-1} (-1)^l (1+\varepsilon t)^{w_1 l} \right) \\ &= \sum_{l=0}^{w_2-1} (-1)^l \int_{\mathbb{Z}_p} (1+\varepsilon t)^{w_2 x_2 + w_1 w_2 x + w_1 l} d\mu_{-1}(x_2) \\ &= \sum_{l=0}^{w_2-1} (-1)^l \int_{\mathbb{Z}_p} (1+\varepsilon t)^{w_2 (x_2 + w_1 x + \frac{w_1}{w_2} l)} d\mu_{-1}(x_2) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{w_2-1} (-1)^l Ch_{n,w_2,\varepsilon}(w_1 x + \frac{w_1}{w_2} l) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.9}$$

Thus, by comparing the coefficients on the both sides of (3.8) and (3.9), we can obtain the Theorem 2.4.

From (1.12), we consider that

$$Ch_{n,w_1,\varepsilon}(w_2 x + \frac{w_2}{w_1} l) = \sum_{k=0}^{n} S_1(n,k) w_1^k E_k(w_2 x + \frac{w_2}{w_1} l), \ (n \geq 0), \tag{3.10}$$

and

$$Ch_{n,w_2,\varepsilon}(w_1 x + \frac{w_1}{w_2} l) = \sum_{k=0}^{n} S_1(n,k) w_2^k E_k(w_1 x + \frac{w_1}{w_2} l), \ (n \geq 0). \tag{3.11}$$

By (3.10) and (3.11), we proved the Theorem 2.5.

Remark 3.1. Note that several authors have studied for the symmetric identities related to special polynomials in the several areas (see [1,2,3,7,9,16,18]).

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