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Modular Colorings of Cycle Related Graphs

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Abstract

For $k \geq 2$, a modular k-coloring of a graph G without isolated vertices is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ having the property that for every pair of adjacent vertices of G, the sums of the colors of their neighbours are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k-coloring is the modular chromatic number of G. In this paper, we determine the modular chromatic number of Fan, Helm graph, Friendship graph and gear graph.

Keywords: Modular coloring, modular chromatic number, Fan, Helm graph, Friendship graph and gear graph.

MSC AMS Classification 2010: 05C15

1. INTRODUCTION

For a vertex v of a graph G, let N(v) denote the neighborhood of v (the set of adjacent vertices to vertex v). For a graph G without isolated vertices, let c: $V(G) \rightarrow \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring [1] of G where adjacent vertices may be colored the same. The color sum $\sigma(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is $\sigma(v) = \sum_{u \in N(v)} c(u)$.

The coloring c is called a modular sum k-coloring or simply a modular k-coloring of G, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. A coloring c is called modular coloring if c is a modular k-coloring for some integar $k \geq 2$. The modular chromatic number mc(G) is the minimum k for which G has a modular k-coloring.This concept was introduced by Okamoto, Salehi and Zhang [4] [6].

Okamoto, Salehi and Zhang proved in [4] that: every nontrivial connected graph G has a modular k-coloring for some integar $k \geq 2$ and $mc(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G; for the cycle C_n of length n, $mc(C_n)$ is 2 if $n \equiv 0$ (mod 4) and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph G, $mc(G) = \chi(G)$; for the wheel $W_n = C_n + K_1$, $n \geq 3$, $mc(W_n) = \chi(W_n)$, where + denotes the join of two graphs and in [5] that for m, $n \geq 2$, $mc(P_m \times P_n) = 2$. M.Paramaguru and R.Sampath Kumar proved in [3] that: every two vertex-disjoint nonempty bipartite graphs G and H, mc(G + H) = 4; for $m \geq 2$ and $n \geq 2$, $mc(P_m + P_n) = 4$; for $m \geq 2$ and $n \geq 2$, $mc(P_m + C_{2n}) = 4$; for $n \geq 2$, r, $s \geq 1$, $mc(P_m + K_{r,s}) = 4$. In [2] we already obtained the Modular chromatic number of circular Halin graphs of level two.

In this article we consider the modular k-coloring of special classes of cycle related graphs.

We reproduce certain observations which would be used in this article.

Observation 1.1 [5] If H is a complete subgraph of order k in a graph G then $mc(G) \geq k$.

Observation 1.2 [5] For every non trivial connected graph $mc(G) \geq \chi(G)$.

2. MAIN RESULTS

Fan $F_t = P_t + K_1$, where the vertex of K_1 is called the apex vertex x and the vertices $v_1, v_2, v_3, \ldots, v_t$ are the path vertices

Theorem 2.1

For any $n \geq 2$, $mc(F_n) = 3$.

Proof.

Case 1. $G = F_2$.

F_2 is K_3 and hence by Observation 1.1 $mc(G) \geq 3$.

Now the coloring $c: V(G) \rightarrow \mathbb{Z}_3$ by $c(v) = \begin{cases} 0 & \text{if } v = x, \\ 1 & \text{if } v = v_1, \\ 2 & \text{if } v = v_2. \end{cases}$

achieves the inequality $mc(G) \leq 3$ which implies $mc(G) = 3$.

Case 2. $G = F_3$.

Define a coloring c: V(G) → $\mathbb{Z}_3$ by c(v) = $\begin{cases} 0 & \text{if } v = x, \\ 1 & \text{if } v = v_t \text{ for } t = 1,2,3. \end{cases}$

This gives mc(G) ≤ 3, which proves mc(G) = 3.

Case 3. n = 4i + j where j = 1, 2, 3 and i ≡ 0 (mod 3)

Define a coloring c: V(F_n) → $\mathbb{Z}_3$ by c(v) = $\begin{cases} 2 & \text{if } v = x, \\ 1 & \text{if } v = v_t \text{ for } t \equiv 1 \ (\text{mod } 4), \\ 0 & \text{otherwise.} \end{cases}$

Then the colors of the path vertices can be listed as

1000 1000 1000 . . . .1000 1 or

1000 1000 1000. . . . 1000 10 or

1000 1000 1000. . . .1000 100 as j = 1, 2, 3.

Therefore σ(x) = the color sum of N(x) = $\lceil \frac{n}{4} \rceil$ (mod 3).

Hence σ(x) = 1.

For v = v_1, c(N(v_1)) = {2, 0} for j = 1, 2, 3. Hence σ(v) = 2.

Subcase1. For j = 1, 3 the colors of N(v_n) = { 0, 2}. Hence σ(v) = 2.

Subcase 2. For j = 2, the colors of N(v_n) = { 1, 2}. Hence σ(v) = 0.

For v= v_t, t ≡ 0 (mod 2) and t ≠ 4i + 2, c(N(v_t)) = {1, 2, 0}. Hence σ (v) = 0.

For v= v_t, t ≡ 1 (mod 2) where t ≠ 1, t ≠ 4i + 1 and t ≠ 4i + 3, c(N(v_t)) = 0, 2, 0.

Hence σ (v) = 2.

Since $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices, we have

Therefore mc(F_n) ≤ 3, where j = 1, 2, 3 and i ≡ 0 (mod 3).

Example.

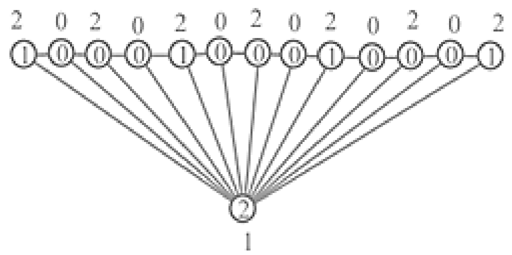


Fig: 2.1 Modular coloring of F_{13}.

Case 4. n=4i + j where j = 1, 2, 3, and i ≡ 1 (mod 3)

Define a coloring c: $V(F_n) \to \mathbb{Z}_3$ by $c(v) = \begin{cases} 1 & \text{if } v = v_t \text{ for } t \equiv 1 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$

Then the colors of the path vertices can be listed as

1000 1000 1000. . . .1000 1 or

1000 1000 1000. . . . 1000 10 or

1000 1000 1000. . . .1000 100 as j = 1, 2, 3.

Therefore σ(x) = the color sum of N(x) = $\lceil \frac{n}{4} \rceil \pmod 3$.

Hence $\sigma(x) = 2$.

For $v = v_1$, $c(N(v_1)) = 0, 0$ for j = 1, 2, 3. Hence $\sigma(v) = 0$.

Subcase1. For j = 1, 3 the colors of $N(v_n) = 0, 0$. Hence $\sigma(v) = 0$.

Subcase 2. For j = 2, $v = v_n$, the colors of $N(v_n) = \{1, 0\}$. Hence $\sigma(v) = 1$.

For $v = v_t$, $t \equiv 0 \pmod 2$ and $t \neq 4i + 2$, $c(N(v_t)) = 1, 0, 0$. Hence $\sigma(v) = 1$.

For $v = v_t$, $t \equiv 1 \pmod 2$ where $t \neq 1$, $t \neq 4i + 1$ and $t \neq 4i + 3$, $c(N(v_t)) = 0, 0, 0$.

Hence $\sigma(v) = 0$.

Since $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices, we have

Therefore $mc(F_n) \leq 3$ where j = 1, 2, 3, for $i \equiv 1 \pmod 3$

Examples.

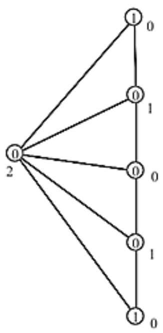


Fig: 2.2. Modular coloring of F_5

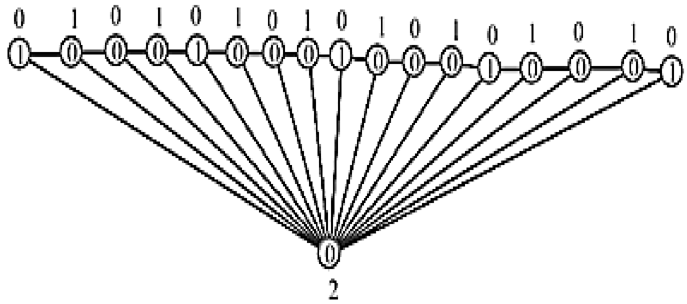


Fig:2.3. Modular colorings of F_{17}

Case 5. n = 4i + j; j = 1, 2, 3, where $i \equiv 2 \pmod 3$

Define a coloring c: $V(F_n) \to \mathbb{Z}_3$ by $c(v) = \begin{cases} 1 & \text{if } v = x;\ v = v_t \text{ for } t \equiv 1 \pmod 4, \\ 0 & \text{otherwise}. \end{cases}$

Then the colors of the path vertices can be listed as

1000 1000 1000. . . .1000 1 or

1000 1000 1000. . . . 1000 10 or

1000 1000 1000. . . .1000 100 as j = 1, 2, 3.

Therefore σ(x) = the color sum of N(x) = $\lceil \frac{n}{4} \rceil$ (mod 3).

Hence σ (x) = 0.

For v = v_1, c(N(v_1)) = {1, 0} for j = 1, 2, 3. Hence σ(v) = 1.

Subcase1. For j = 1, 3 the colors of N(v_n) = { 0, 1}. Hence σ(v) = 1.

Subcase 2. For j = 2, the colors of N(v_n) = 1, 1. Hence σ(v) = 2.

For v= v_t, t ≡ 0 (mod 2) and $t \neq 4i + 2$, c(N(v_t)) = 1, 1, 0. Hence σ(v) = 2.

For v= v_t, t ≡ 1 (mod 2) where $t \neq 1, t \neq 4i + 1$ and $t \neq 4i + 3$, c(N(v_t)) = 0, 1, 0.

Hence σ(v) = 1.

Since $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices, we have

Hence mc(F_n) ≤ 3 where j = 1, 2, 3, and $i \equiv 2$ (mod 3) .

Example.

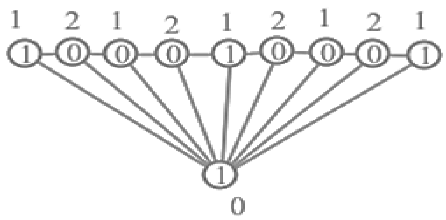


Fig: 2.4. Modular colorings of F_9.

Case 6. mc(F_{4n}) = 3 where n ≥ 1

Define a coloring c: V(F_{4n}) → $\mathbb{Z}_3$ by c(v) =

$$\begin{cases} 2 & \text{if } v = v_{3+4t} \text{ for } t = 0, 1, 2, \ldots, (n-1), \\ 1 & \text{if } v = v_{2+4t} \text{ for } t = 0, 1, 2, \ldots, (n-2), \\ 0 & \text{otherwise.} \end{cases}$$

Then the colors of the path vertices can be listed as

0120 0120 0120 0120. . . . 0120

Therefore σ(x) = the color sum of N(x) = $[\frac{3n}{4}]$(mod 3).

Hence σ(x) = 0.

For v = v_1, c(N(v_1)) = {1, 0}. Hence σ(v) = 1.

For v= v_t, t ≡ 0 (mod 2) then c(N(v_t)) = 0, 2, 0. Hence σ(v) = 2.

For v= v_t, t ≡ 1 (mod 2) then c(N(v_t)) = 0, 1, 0. Hence σ(v) = 1.

Since $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices, we have

Hence mc(F_{4n}) ≤ 3.

Example.

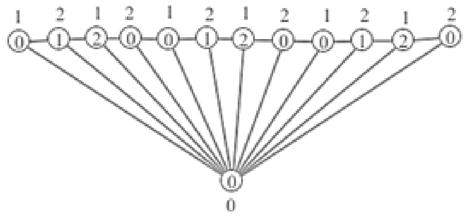


Fig: 2.5 Modular coloring of F_{12}.

The consolidation of all six cases completes the proof. ■

A helm graph is a graph obtained from the wheel $W_n = C_n + K_1$ by attaching a pendant cycle C_n. A helm graph is denoted by H_n where n is the number of vertices of the cycle C_n. Let x denote the central vertex, $v_1, v_2, \ldots, v_n$ be the vertices on the cycle and $w_1, w_2, w_3, \ldots, w_n$ be the corresponding pendant vertices.

Theorem 2.2

If n is even, then mc(H_n) = 3 for n ≥ 3.

Proof.

Case 1. mc(H_n) = 3, n ≥ 4 where n is even

Consider the coloring c: V(H_n) → $\mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 0 \text{ if } v = x,\ v_1,\ v_2, \ldots . v_n, \\ 1 \text{ if } v = w_t \text{ where } t = 1, 3, 5, \ldots ., (n-1), \\ 2 \quad \text{elsewhere} . \end{cases}$$

σ(x) = color sum of N(x) = 0 sine N(x) = { $v_1, v_2, \ldots v_n$} and c(v_i) = 0 for all i.

Similarly, σ(w_i) = 0 since each w_i has a single neighbor colored 0.

Since N(v_i) = {v_{i-1}, v_{i+1}, x, w_i} = 0, 0, 0, 1 or 0, 0, 0, 2 as i = odd or even.

Therefore σ(v_i) = 1 if i is odd and 2 if i is even.

here σ(x) ≠ σ(y) for any adjacent vertices x and y.

Therefore $mc(H_n) \leq 3$ for $n \geq 4$. Equality follows from Observation 1.1.

Example:

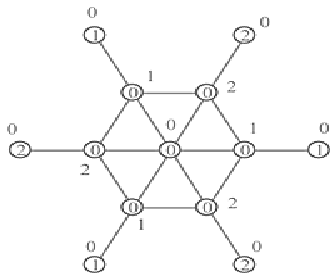


Fig: 2.6. Modular coloring of H_6

Case 2. $mc(H_n) = 4$, $n \geq 3$, where n is odd.

Consider the coloring $c: V(H_n) \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 0 \text{ for } v = x, v = v_t \text{ for all } t, \\ 1 \text{ for } v = w_t \text{ where } t \text{ is odd}, t \neq n, \\ 2 \text{ for } v = w_t \text{ where } t \text{ is even}, \\ 3 \text{ for } v = w_n. \end{cases}$$

As in case1, it can be easily verified that $\sigma(x) = 0$ and $\sigma(w_t) = 0$ for all $t = 1, 2, \ldots, n$.

It can be also verified that $\sigma(v_t) = 1$ if t is odd and $t \neq n$.

$\sigma(v_t) = 1$ if t is even and $\sigma(v_n) = 3$.

To sum up

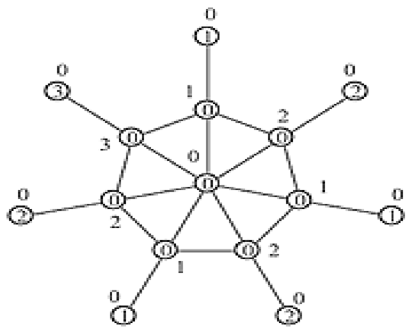
$$\sigma(v) = \begin{cases} 0 \text{ for } v = x, w_t \text{ for all } t, \\ 1 \text{ for } v = v_t \text{ where } t = 1, 3, 5, \ldots, (n-2), \\ 2 \text{ for } v = v_t \text{ where } t = 2, 4, 6, \ldots, (n-1), \\ 3 \quad \text{elsewhere}. \end{cases}$$

here $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices.

Then if n is odd then $mc(H_n) \leq 4$. It follows that $mc(H_n) \geq 4$, by Observation 1.2.

Thus $mc(H_n) = 4$. ■

Eg:H_7



3. MODULAR COLORING IN FRIENDSHIP GRAPH:

Friendship graph: A friendship graph is the one-point union of n copies of the cycle C_3 denoted by $C_3^{(n)}$. It has 2n+1 vertices and 3n edges. Let x be the common vertex identified. Let v_t and v_t' be the vertices of the t^{th} cycle C_3 for t = 1, 2, . . . , n adjacent to x in the clockwise direction.

Theorem III

For the friendship graph $G = C_3^{(n)}$, mc(G) = 3.

Proof.

By Observation 1.1, we have $mc(C_3^{(n)}) \geq 3$.

Consider the coloring c: $V(G) \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 0 \text{ if } v = x, \\ 1 \text{ if } v = v_t \text{ for al,l } t = 1, 2, \dots, n. \\ 2 \text{ if } v = v_t' \text{ for } t = 1, 2, \dots, n. \end{cases}$$

Therefore $\sigma(x) = 0$, since the color sum of N(x) = 3n.

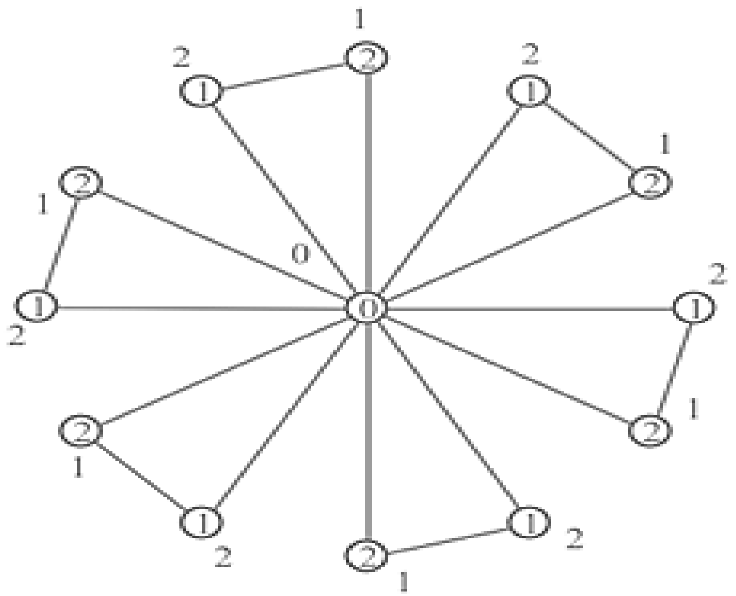
Also $\sigma(v_t) = 2$, for all t = 1, 2, . . . , n. and $\sigma(v_t') = 1$, for all t = 1, 2, . . . , n

$$\text{Then } \sigma(v) = \begin{cases} 0 \quad \text{if } v = x, \\ 2 \text{ if } v = v_i \text{ where } i = 1, 3, 5, \dots, (2n-1), \\ 1 \quad \text{elsewhere.} \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all x, y of adjacent vertices.

Then $mc(C_3^{(n)}) = 3$. ■

Eg: $C_3^{(6)}$.



4. MODULAR COLORING IN GEAR GRAPH:

Gear graph: A gear graph denoted by G(n) is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of the wheel graph W_n. Then G(n) has 2n+1 vertices and 3n edges. Let x be the apex vertex. Let v_t be the vertices on the cycle adjacent to x and w_t be the vertices on the cycle between v_t and v_{t+1} ,on the cycle, subscripts are taken modulo n, for t = 1, 2, 3, . . . , 2n in the clockwise direction where v_t, t = 1, 3, 5, . . . ,(2n-1).

Theorem IV

For a gear graph G(n), mc(G(n)) = 2.

Proof.

Consider the coloring c: V(G(n)) → $\mathbb{Z}_2$ defined by c(v) = $\begin{cases} 1 & \text{if } v = x, \\ 0 & \text{elsewhere.} \end{cases}$

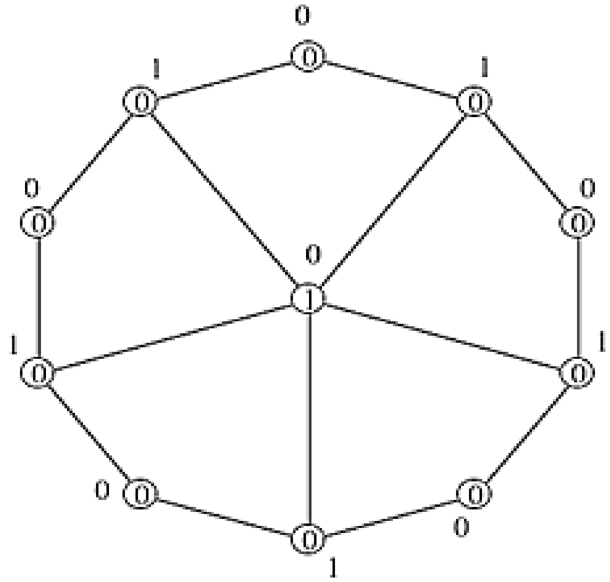
Obiviously $\sigma(x) = 0$.

For t = 1, 2, . . . , n, it can be easily verified that $\sigma(v_t) = 1$ and $\sigma(w_t) = 0$.

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in G(n).

Then mc(G(n)) $\leq$ 2. Equality holds by the Definition. ■

Eg:G(5)



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