Modular Colorings of Cycle Related Graphs

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Abstract

For \(k \geq 2\), a modular \(k\)-coloring of a graph \(G\) without isolated vertices is a coloring of the vertices of \(G\) with the elements in \(\mathbb{Z}_k\) having the property that for every pair of adjacent vertices of \(G\), the sums of the colors of their neighbours are different in \(\mathbb{Z}_k\). The minimum \(k\) for which \(G\) has a modular \(k\)-coloring is the modular chromatic number of \(G\). In this paper, we determine the modular chromatic number of Fan, Helm graph, Friendship graph and gear graph.

Keywords: Modular coloring, modular chromatic number, Fan, Helm graph, Friendship graph and gear graph.

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1. INTRODUCTION

For a vertex \(v\) of a graph \(G\), let \(N(v)\) denote the neighborhood of \(v\) (the set of adjacent vertices to vertex \(v\)). For a graph \(G\) without isolated vertices, let \(c: V(G) \to \mathbb{Z}_k\) (\(k \geq 2\)) be a vertex coloring \([1]\) of \(G\) where adjacent vertices may be colored the same. The color sum \(\sigma(v)\) of a vertex \(v\) of \(G\) is defined as the sum in \(\mathbb{Z}_k\) of the colors of the vertices in \(N(v)\), that is \(\sigma(v) = \sum_{u \in N(v)} c(u)\).

The coloring \(c\) is called a modular sum \(k\)-coloring or simply a modular \(k\)-coloring of \(G\), if \(\sigma(x) \neq \sigma(y)\) in \(\mathbb{Z}_k\) for all pairs \(x, y\) of adjacent vertices of \(G\). A coloring \(c\) is called
modular coloring if \( c \) is a modular k-coloring for some integer \( k \geq 2 \). The modular chromatic number \( mc(G) \) is the minimum \( k \) for which \( G \) has a modular k-coloring. This concept was introduced by Okamoto, Salehi and Zhang [4] [6].

Okamoto, Salehi and Zhang proved in [4] that: every nontrivial connected graph \( G \) has a modular k-coloring for some integer \( k \geq 2 \) and \( mc(G) \geq \chi(G) \), where \( \chi(G) \) denotes the chromatic number of \( G \); for the cycle \( C_n \) of length \( n \), \( mc(C_n) \) is 2 if \( n \equiv 0 \) (mod 4) and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph \( G \), \( mc(G) = \chi(G) \); for the wheel \( W_n = C_n + K_1 \), \( n \geq 3 \), \( mc(W_n) = \chi(W_n) \), where + denotes the join of two graphs and in [5] that for \( m \), \( n \geq 2 \), \( mc(P_m \times P_n) = 2 \). M.Paramaguru and R.Sampath Kumar proved in [3] that: every two vertex-disjoint nonempty bipartite graphs \( G \) and \( H \), \( mc(G + H) = 4 \); for \( m \geq 2 \) and \( n \geq 2 \), \( mc(P_m + P_n) = 4 \); for \( m \geq 2 \) and \( n \geq 2 \), \( mc(P_m + C_{2n}) = 4 \); for \( n \geq 2 \), \( r, s \geq 1 \), \( mc(P_m + K_{r,s}) = 4 \). In [2] we already obtained the Modular chromatic number of circular Halin graphs of level two.

In this article we consider the modular k-coloring of special classes of cycle related graphs.

We reproduce certain observations which would be used in this article.

**Observation 1.1** [5] If \( H \) is a complete subgraph of order \( k \) in a graph \( G \) then \( mc(G) \geq k \).

**Observation 1.2** [5] For every non trivial connected graph \( mc(G) \geq \chi(G) \).

### 2. MAIN RESULTS

Fan \( F_n = P_t + K_1 \), where the vertex of \( K_1 \) is called the apex vertex \( x \) and the vertices \( v_1, v_2, v_3, \ldots, v_t \) are the path vertices.

**Theorem 2.1**

For any \( n \geq 2 \), \( mc(F_n) = 3 \).

**Proof.**

**Case 1.** \( G = F_2 \).

\( F_2 \) is \( K_3 \) and hence by Observation 1.1 \( mc(G) \geq 3 \).

Now the coloring \( c:V(G) \rightarrow \mathbb{Z}_3 \) by \( c(v) = \begin{cases} 0 & \text{if } v = x, \\ 1 & \text{if } v = v_1, \\ 2 & \text{if } v = v_2. \end{cases} \)

achieves the inequality \( mc(G) \leq 3 \) which implies \( mc(G) = 3 \).

**Case 2.** \( G = F_3 \).
Define a coloring \( c : V(G) \to \mathbb{Z}_3 \) by \( c(v) = \begin{cases} 
0 & \text{if } v = x, \\
1 & \text{if } v = v_t \text{ for } t = 1, 2, 3. 
\end{cases} \)

This gives \( mc(G) \leq 3 \), which proves \( mc(G) = 3 \).

**Case 3.** \( n = 4i + j \) where \( j = 1, 2, 3 \) and \( i \equiv 0 \pmod{3} \)

Define a coloring \( c : V(F_n) \to \mathbb{Z}_3 \) by \( c(v) = \begin{cases} 
2 & \text{if } v = x, \\
1 & \text{if } v = v_t \text{ for } t \equiv 1 \pmod{4}, \\
0 & \text{otherwise}. 
\end{cases} \)

Then the colors of the path vertices can be listed as

1000 1000 1000 . . . 1000 1 or
1000 1000 1000 . . . 1000 10 or
1000 1000 1000 . . . 1000 100 as \( j = 1, 2, 3 \).

Therefore \( \sigma(x) = \) the color sum of \( N(x) = \left\lfloor \frac{n}{4} \right\rfloor \pmod{3} \).

Hence \( \sigma(x) = 1 \).

For \( v = v_1 \), \( c(N(v_1)) = \{2, 0\} \) for \( j = 1, 2, 3 \). Hence \( \sigma(v) = 2 \).

**Subcase 1.** For \( j = 1, 3 \) the colors of \( N(v_n) = \{0, 2\} \). Hence \( \sigma(v) = 2 \).

**Subcase 2.** For \( j = 2 \), the colors of \( N(v_n) = \{1, 2\} \). Hence \( \sigma(v) = 0 \).

For \( v = v_t, t \equiv 0 \pmod{2} \) and \( t \neq 4i + 2 \), \( c(N(v_t)) = \{1, 2, 0\} \). Hence \( \sigma(v) = 0 \).

For \( v = v_t, t \equiv 1 \pmod{2} \) where \( t \neq 1, t \neq 4i + 1 \) and \( t \neq 4i + 3 \), \( c(N(v_t)) = 0, 2, 0 \).

Hence \( \sigma(v) = 2 \).

Since \( \sigma(x) \neq \sigma(y) \) for all \( x, y \) of adjacent vertices, we have

Therefore \( mc(F_n) \leq 3 \), where \( j = 1, 2, 3 \) and \( i \equiv 0 \pmod{3} \).

**Case 4.** \( n = 4i + j \) where \( j = 1, 2, 3 \), and \( i \equiv 1 \pmod{3} \)

![Modular coloring of \( F_{13} \).](image)
Define a coloring \( c: V(F_n) \to \mathbb{Z}_3 \) by
\[
c(v) = \begin{cases} 
1 & \text{if } v = v_t \text{ for } t \equiv 1 \pmod{4}, \\
0 & \text{otherwise}
\end{cases}
\]
Then the colors of the path vertices can be listed as
\[
1000 1000 1000 \ldots 1000 1 \text{ or } 1000 1000 1000 \ldots 1000 10 \text{ or } 1000 1000 1000 \ldots 1000 100 \text{ as } j = 1, 2, 3.
\]
Therefore \( \sigma(x) = \) the color sum of \( N(x) = \left\lfloor \frac{n}{4} \right\rfloor \pmod{3} \).
Hence \( \sigma(x) = 2 \).
For \( v = v_1 \), \( c(N(v_1)) = 0, 0 \) for \( j = 1, 2, 3 \). Hence \( \sigma(v) = 0 \).
**Subcase 1.**
For \( j = 1, 3 \) the colors of \( N(v_n) = 0, 0 \). Hence \( \sigma(v) = 0 \).

**Subcase 2.**
For \( j = 2, v = v_n \), the colors of \( N(v_n) = \{1, 0\} \). Hence \( \sigma(v) = 1 \).
For \( v = v_t, t \equiv 0 \pmod{2} \) and \( t \neq 4i + 2 \), \( c(N(v_t)) = 1, 0, 0 \). Hence \( \sigma(v) = 1 \).
For \( v = v_t, t \equiv 1 \pmod{2} \) where \( t \neq 1, t \neq 4i + 1 \) and \( t \neq 4i + 3 \), \( c(N(v_t)) = 0, 0, 0 \).
Hence \( \sigma(v) = 0 \).

Since \( \sigma(x) \neq \sigma(y) \) for all \( x, y \) of adjacent vertices, we have
Therefore \( mc(F_n) \leq 3 \) where \( j = 1, 2, 3, \) for \( i \equiv 1 \pmod{3} \)
**Examples.**

> **Fig: 2.2.** Modular coloring of \( F_5 \)

> **Fig: 2.3.** Modular colorings of \( F_{17} \)

**Case 5.** \( n = 4i + j; j = 1, 2, 3, \) where \( i \equiv 2 \pmod{3} \)
Define a coloring \( c: V(F_n) \to \mathbb{Z}_3 \) by
\[
c(v) = \begin{cases} 
1 & \text{if } v = x; v = v_t \text{ for } t \equiv 1 \pmod{4}, \\
0 & \text{otherwise}
\end{cases}
\]
Then the colors of the path vertices can be listed as
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1000 1000 1000. . .1000 1 or
1000 1000 1000. . . 1000 10 or
1000 1000 1000. . .1000 100 as j = 1, 2, 3.
Therefore $\sigma(x) = \text{the color sum of } N(x) = \lceil \frac{n}{4} \rceil \left( \text{mod } 3 \right)$.

Hence $\sigma(x) = 0$.

For $v = v_1$, $c(N(v_1)) = \{1, 0\}$ for $j = 1, 2, 3$. Hence $\sigma(v) = 1$.

Subcase 1. For $j = 1, 3$ the colors of $N(v_n) = \{0, 1\}$. Hence $\sigma(v) = 1$.

Subcase 2. For $j = 2$, the colors of $N(v_n) = 1, 1$. Hence $\sigma(v) = 2$.

For $v = v_1$, $t \equiv 0 \pmod{2}$ and $t \neq 4i + 2$, $c(N(v_i)) = 1, 1, 0$. Hence $\sigma(v) = 2$.

For $v = v_t$, $t \equiv 1 \pmod{2}$ where $t \neq 1, t \neq 4i + 1$ and $t \neq 4i + 3$, $c(N(v_i)) = 0, 1, 0$.

Hence $\sigma(v) = 1$.

Since $\sigma(x) \neq \sigma(y)$ for all $x, y$ of adjacent vertices, we have

Hence $mc(F_9) \leq 3$ where $j = 1, 2, 3$, and $i \equiv 2 \pmod{3}$.

Example.

![Fig: 2.4. Modular colorings of F9.](image)

Case 6. $mc(F_{4n}) = 3$ where $n \geq 1$

Define a coloring $c: V(F_{4n}) \to \mathbb{Z}_3$ by $c(v) =$

\begin{align*}
2 & \text{ if } v = v_{3+4t} \text{ for } t = 0, 1, 2, \ldots, (n-1), \\
1 & \text{ if } v = v_{2+4t} \text{ for } t = 0, 1, 2, \ldots, (n-2), \\
0 & \text{ otherwise.}
\end{align*}

Then the colors of the path vertices can be listed as

0120 0120 0120 0120. . . 0120

Therefore $\sigma(x) = \text{the color sum of } N(x) = \lceil \frac{3n}{4} \rceil \left( \text{mod } 3 \right)$.

Hence $\sigma(x) = 0$. 
For \( v = v_1 \), \( c(N(v_1)) = \{1, 0\} \). Hence \( \sigma(v) = 1 \).

For \( v = v_t \), \( t \equiv 0 \) (mod 2) then \( c(N(v_t)) = 0, 2, 0 \). Hence \( \sigma(v) = 2 \).

For \( v = v_t \), \( t \equiv 1 \) (mod 2) then \( c(N(v_t)) = 0, 1, 0 \). Hence \( \sigma(v) = 1 \).

Since \( \sigma(x) \neq \sigma(y) \) for all \( x, y \) of adjacent vertices, we have

Hence \( \text{mc}(F_{4n}) \leq 3 \).

Example.

![Figure 2.5 Modular coloring of \( F_{12} \).](image_url)

The consolidation of all six cases completes the proof.

A helm graph is a graph obtained from the wheel \( W_n = C_n + K_1 \) by attaching a pendant cycle \( C_n \). A helm graph is denoted by \( H_n \) where \( n \) is the number of vertices of the cycle \( C_n \). Let \( x \) denote the central vertex, \( v_1, v_2, \ldots, v_n \) be the vertices on the cycle and \( w_1, w_2, w_3, \ldots, w_n \) be the corresponding pendant vertices.

**Theorem 2.2**

If \( n \) is even, then \( \text{mc}(H_n) = 3 \) for \( n \geq 3 \).

**Proof.**

**Case 1.** \( \text{mc}(H_n) = 3 \), \( n \geq 4 \) where \( n \) is even

Consider the coloring \( c: V(H_n) \to \mathbb{Z}_3 \) defined by

\[
c(v) = \begin{cases} 
0 & \text{if } v = x, \ v_1, \ v_2, \ldots, v_n, \\
1 & \text{if } v = w_t \text{ where } t = 1, 3, 5, \ldots, (n - 1), \\
2 & \text{elsewhere}.
\end{cases}
\]

\( \sigma(x) = \text{color sum of } N(x) = 0 \) since \( N(x) = \{ v_1, v_2, \ldots, v_n \} \) and \( c(v_i) = 0 \) for all \( i \).

Similarly, \( \sigma(w_i) = 0 \) since each \( w_i \) has a single neighbor colored 0.

Since \( N(v_i) = \{ v_{i-1}, v_{i+1}, x, w_i \} = 0, 0, 0, 1 \) or \( 0, 0, 2 \) as \( i = \) odd or even.

Therefore \( \sigma(v_i) = 1 \) if \( i \) is odd and 2 if \( i \) is even.

here \( \sigma(x) \neq \sigma(y) \) for any adjacent vertices \( x \) and \( y \).
Therefore $mc(H_n) \leq 3$ for $n \geq 4$. Equality follows from Observation 1.1.

Example:

![Figure 2.6. Modular coloring of $H_6$](image)

**Case 2.** $mc(H_n) = 4$, $n \geq 3$, where $n$ is odd.

Consider the coloring $c: V(H_n) \to \mathbb{Z}_4$ defined by

$$
c(v) = \begin{cases} 
0 & \text{for } v = x, v = v_t \text{ for all } t, \\
1 & \text{for } v = w_t \text{ where } t \text{ is odd, } t \neq n, \\
2 & \text{for } v = w_t \text{ where } t \text{ is even}, \\
3 & \text{for } v = w_n.
\end{cases}
$$

As in case 1, it can be easily verified that $\sigma(x) = 0$ and $\sigma(w_t) = 0$ for all $t = 1, 2, \ldots, n$.

It can be also verified that $\sigma(v_t) = 1$ if $t$ is odd and $t \neq n$.

$\sigma(v_1) = 1$ if $t$ is even and $\sigma(v_n) = 3$.

To sum up

$$
\sigma(v) = \begin{cases} 
0 & \text{for } v = x, w_t \text{ for all } t, \\
1 & \text{for } v = v_t \text{ where } t = 1, 3, 5, \ldots, (n-2), \\
2 & \text{for } v = v_t \text{ where } t = 2, 4, 6, \ldots, (n-1), \\
3 & \text{elsewhere}.
\end{cases}
$$

Here $\sigma(x) \neq \sigma(y)$ for all $x, y$ of adjacent vertices.

Then if $n$ is odd then $mc(H_n) \leq 4$. It follows that $mc(H_n) \geq 4$, by Observation 1.2.

Thus $mc(H_n) = 4$.  

Eg: $H_7$
3. MODULAR COLORING IN FRIENDSHIP GRAPH:

Friendship graph: A friendship graph is the one-point union of \( n \) copies of the cycle \( C_3 \) denoted by \( C_3^{(n)} \). It has \( 2n+1 \) vertices and \( 3n \) edges. Let \( x \) be the common vertex identified. Let \( v_t \) and \( v_t' \) be the vertices of the \( t^{th} \) cycle \( C_3 \) for \( t = 1, 2, \ldots, n \) adjacent to \( x \) in the clockwise direction.

**Theorem III**

For the friendship graph \( G = C_3^{(n)} \), \( mc(G) = 3 \).

**Proof.**

By Observation 1.1, we have \( mc(C_3^{(n)}) \geq 3 \).

Consider the coloring \( c: V(G) \rightarrow \mathbb{Z}_3 \) defined by

\[
c(v) = \begin{cases} 
0 & \text{if } v = x, \\
1 & \text{if } v = v_t \text{ for } t = 1, 2, \ldots, n, \\
2 & \text{if } v = v_t' \text{ for } t = 1, 2, \ldots, n. 
\end{cases}
\]

Therefore \( \sigma(x) = 0 \), since the color sum of \( N(x) = 3n \).

Also \( \sigma(v_t) = 2 \), for all \( t = 1, 2, \ldots, n \) and \( \sigma(v_t') = 1 \), for all \( t = 1, 2, \ldots, n \)

Then \( \sigma(v) = \begin{cases} 
0 & \text{if } v = x, \\
2 & \text{if } v = v_i \text{ where } i = 1, 3, 5, \ldots, (2n - 1), \\
1 & \text{elsewhere}. 
\end{cases} \)

Hence \( \sigma(x) \neq \sigma(y) \) for all \( x, y \) of adjacent vertices.

Then \( mc(C_3^{(n)}) = 3 \).

\( \blacksquare \)

Eg: \( C_3^{(6)} \).
4. MODULAR COLORING IN GEAR GRAPH:

Gear graph: A gear graph denoted by $G(n)$ is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of the wheel graph $W_n$. Then $G(n)$ has $2n+1$ vertices and $3n$ edges. Let $x$ be the apex vertex. Let $v_t$ be the vertices on the cycle adjacent to $x$ and $w_t$ be the vertices on the cycle between $v_t$ and $v_{t+1}$, on the cycle, subscripts are taken modulo $n$, for $t = 1, 2, 3, \ldots, 2n$ in the clockwise direction where $v_t, t = 1, 3, 5, \ldots, (2n-1)$.

**Theorem IV**

For a gear graph $G(n)$, $mc(G(n)) = 2$.

**Proof.**

Consider the coloring $c: V(G(n)) \rightarrow \mathbb{Z}_2$ defined by $c(v) = \begin{cases} 1 & \text{if } v = x, \\ 0 & \text{elsewhere.} \end{cases}$

Obiviously $\sigma(x) = 0$.

For $t = 1, 2, \ldots, n$, it can be easily verified that $\sigma(v_t) = 1$ and $\sigma(w_t) = 0$.

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices $x$ and $y$ in $G(n)$.

Then $mc(G(n)) \leq 2$. Equality holds by the Definition.

Eg: $G(5)$
REFERENCES


