Harvested Prey Predator Model: Infected Predator with Differential Predation Rate

Nishant Juneja1* and Kulbhushan Agnihotri2
1* Research Scholar, I.K Gujral Punjab Technical University, Kapurthala, Punjab, India.
2 Professor, S.B.S. State Technical Campus, Ferozepur, Punjab, India
(*Corresponding Author)

Abstract
In this present work, a model on predator-prey system is proposed and analyzed in which only the predator population is infected by a disease. In this model different predation rate for susceptible and infected predators are considered and also both the species are harvested at different rate. An SI model is taken for predator species as recovery from disease is not incorporated in this model. Conditions for the existence and stability of disease free prey predator system are obtained. Conditions for endemic disease in predator species are discussed. The epidemiological threshold quantities have been obtained using next generation approach for the model system. It has been observed that reduced predation rate of infected predator and controlled harvesting of predator species will helpful in controlling the spread of disease.

Keywords: Carrying capacity, Harvesting, Predator, Prey, Predation rate

I. INTRODUCTION
The population dynamics is one of the important branches of mathematical biology. Thomas Malthus was the first researcher in this field. His idea that the rate at which a population grows is directly proportional to its current size seems to be fairly obvious over 150 years later. Later on Pierre-Francois Verhulst, a Belgian Mathematician
generalizes the Malthusian model by allowing the fact that populations encounter internal competition and this competition retards the rate of growth.

The study of infectious diseases represents one of the oldest and richest areas in mathematical biology and it is also important and interesting to study ecological systems under the influence of various epidemiological factors. A good volume of literature is available on ecology and epidemiology that results in developing a new important branch of mathematical biology named eco-epidemiology [1], [2], [3]. Most of the eco-epidemiological studies are limited to study the situations where only prey species can get infection [4], [5], [6]. A few investigations consider the spread of disease in predator species [7], [8], [9]. It is also well known that harvesting plays an important role in the stability of prey-predator system [10]. A small study has been carried out to know the impact of harvesting on a disease infected system. Morgane Cheve, Ronon Congar, Papa A Diop studied the effect of harvesting of only infected predator species on the prey-predator system. But so far, a very little work has been carried out for eco-epidemiology model in which both the prey predator populations are harvested. Also few researches have been undertaken where both the prey and predator population are invaded by some disease.

In the present paper, a harvested predator-prey model is proposed in which the disease is spreading from infected predator species to susceptible predator species. An SI model is considered for predator species as it is assumed that susceptible predator once infected will never recover. Differential predation rate are taken for susceptible and infected predators due to less mobility of infected predator. The objective of the paper is to study the dynamics of the system when both the prey-predator species are subjected to harvesting. The effect of controlled harvesting on the population dynamics is also investigated from the stability point of view. Numerical simulations are carried out to illustrate the theoretical prediction.

2. FORMULATION OF THE MODEL
Consider a prey predator system in which the predator species is infected with some disease and consequently it is divided into two categories, susceptible predator and infected predator. The population of susceptible and infected predator at any time‘t’ is given as \( y \) and \( z \). Hence the total population at any time‘t’ is given as

\[ P = y + z \]

The prey population follows logistic growth with ‘r’ as intrinsic growth rate and ‘k’ as the carrying capacity of the environment. The disease is transmitted in predator species with law of mass action having \( \lambda \) as transmission parameter. The susceptible and infected predators have \( \mu \) and \( \mu' \) as their respective mortality rates with \( \mu' > \mu \).
The predator and prey species are also subjected to harvesting having harvesting effort $E$ and $E_1$, respectively. Let $\gamma$ and $\gamma'$ be the catch ability coefficients of susceptible and infected predators with $\gamma' > \gamma$ as clearly understood. The susceptible and infected predators meet the prey with $\alpha$ and $\alpha'$ as their respective capture coefficients ($\alpha' < \alpha$).

Based on the above assumptions, we write the following equations for our eco-epidemiological model.

$$
\begin{align*}
\frac{dx}{dt} &= rx \left(1 - \frac{x}{k}\right) - \alpha xy - \alpha' xz - \gamma_1 E_1 x \\
\frac{dy}{dt} &= \alpha \beta xy - \mu y - \gamma Ey - \lambda yz \\
\frac{dz}{dt} &= \alpha' \beta xz - \mu' z - \gamma' Ez + \lambda yz \\
(0 &\leq x(0) \leq k, \ 0 \leq y(0), \ 0 \leq z(0))
\end{align*}
$$

(1)

3. **EQUILIBRIUM POINTS**

The system can have following different equilibriums

(a) A trivial equilibrium point $A(0,0,0)$, where all the predator and prey populations extinct.

(b) A predator free equilibrium point $B(\hat{x},0,0)$, where $\hat{x}=k (1-\gamma_1 E_1/r)$ which exist

if $E_1 < \frac{r}{\gamma_1}$

(3.1)

(c) A disease free equilibrium (DFE) $C(x',y',0)$, where

$$
x' = \frac{k}{R_0}, \ y' = \frac{r}{\alpha} \left(1 - \frac{1}{L}\right) - \frac{\gamma_1}{\alpha}, \text{ which exist if } \ E_1 < \frac{1}{\gamma_1} \left(r - \frac{r}{L}\right)
$$

(3.2)

where $L = \frac{\alpha \beta k}{\mu + \gamma E}$

(d) An endemic semi positive equilibrium $D(\bar{x},0,\bar{z})$, where

$$
\bar{x} = \frac{k}{L'}, \bar{z} = \frac{r}{\alpha'} \left(1 - \frac{1}{L'}\right) - \frac{\gamma_1}{\alpha}, \text{ which exist if } \ E_1 < \frac{1}{\gamma_1} \left(r - \frac{r}{L'}\right)
$$

(3.3)

where $L' = \frac{\alpha' \beta k}{\mu' + \gamma'E}$

If $L' < 1$, the disease will die out and if $L' > 1$, the disease will invade the population.
Again it is observed that $L' < L$.
So the existence conditions for equilibrium points $C(x', y', 0)$ and $D(\bar{\pi}, 0, \bar{\gamma})$ can be combined as

$$E_i < \frac{1}{\gamma_i} \left( r - \frac{r}{L'} \right)$$

(3.4)

(e) An endemic positive equilibrium $E(x^*, y^*, z^*)$, where

$$x^* = K \left[ 1 - \frac{(\alpha'\mu' - \alpha'\mu + (\alpha'\gamma' - \alpha'\gamma)E + \gamma E)}{r} \right],$$

$$y^* = \left[ \frac{\alpha'\beta x^* - \mu' - \gamma' E}{\lambda} \right],$$

$$z^* = \frac{\alpha'\beta x^* - \mu - \gamma E}{\lambda}$$

which exists if

$$r - \gamma_i E_i > (\alpha'\mu' - \alpha'\mu + (\alpha'\gamma' - \alpha'\gamma)E)$$

and

$$\frac{\alpha'\beta x^* - \mu'}{\gamma'} < E < \frac{\alpha'\beta x^* - \mu}{\gamma}$$

(3.5)

i.e. a reasonable harvesting effort is required for the persistence of all the three populations.

4. UNIFORM BOUNDEDNESS

Lemma: - All the solutions of the system (1) will be in the region

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 : 0 \leq x + y + z \leq \frac{l}{m} \right\}$$

as $t \to \infty$ for all positive initial values $(x(0), y(0), z(0)) \in \mathbb{R}^3$ where $m = \min(\gamma_i E_i, \gamma E, \gamma' E)$ and $l = rx + \alpha'\beta xy + \alpha'\beta xz$.

Proof: - Let us consider the following function:

$$W(t) = x(t) + y(t) + z(t)$$

$$\frac{dW(t)}{dt} \leq l - mW, \text{ where } m = \min(\gamma_i E_i, \gamma E, \gamma' E)$$

By comparison theorem [9], $W(t) \leq \frac{l}{m}$ as $t \to \infty$

Hence the proof.
5. **STABILITY ANALYSIS**

The variational matrix for the system (1) is given by

\[
\begin{pmatrix}
-\frac{rx}{k} + r\left(1 - \frac{x}{k}\right) - \alpha y - \alpha' z - \gamma, E, x & -\alpha x & -\alpha' x \\
\alpha' \beta y & \alpha' \beta x - \mu - \gamma E - \lambda z & -\lambda y \\
\alpha' \beta z & \lambda z & \alpha' \beta x - \mu' - \gamma'E + \lambda y
\end{pmatrix}
\]

The following theorems are direct consequences of linear stability analysis of the system (1)

**Theorem-1:** The trivial equilibrium \( A(0,0,0) \) is locally asymptotically stable if

\[
E_i > \frac{r}{\gamma_i}
\]

*Proof:* The Eigen values for the equilibrium \( A(0,0,0) \) are given by

\[
\xi_1 = r - \gamma_i E, \quad \xi_2 = -\mu - \gamma E, \quad \xi_3 = -\mu' - \gamma'E
\]

Clearly \( \xi_2, \xi_3 < 0 \),

Now \( \xi_1 < 0 \)

Iff \( E_i > \frac{r}{\gamma_i} \) \hspace{1cm} (5.1)

This means that stability of equilibrium point \( A(0,0,0) \) will result in non existence of equilibrium point \( B(\hat{x},0,0) \). Again if the harvesting effort is too much large, then all the populations will extinct, which will not be good for our biodiversity.

**Theorem -2:** The predator free equilibrium \( B(\hat{x},0,0) \) if exist, is locally asymptotically stable for \( E > (\alpha \beta \hat{x} - \mu') / \gamma \).

where \( \hat{x} = k \left(1 - \frac{\gamma}{r} E_i\right) \)

*Proof:* The Eigen values for the equilibrium \( B(\hat{x},0,0) \) are given by

\[
\xi_1 = -\frac{r\hat{x}}{k}, \quad \xi_2 = \alpha' \beta \hat{x} - \mu - \gamma E, \quad \xi_3 = \alpha' \beta \hat{x} - \mu' - \gamma'E
\]

Clearly \( \xi_1 < 0 \),
Now \( \xi_1 < 0 \text{ iff } E > (\alpha \beta \hat{x} - \mu) / \gamma \) \hspace{1cm} (A)

\( \xi_3 < 0 \text{ iff } E > (\alpha' \beta \hat{x} - \mu') / \gamma' \) \hspace{1cm} (B)

Now \( \alpha' < \alpha, \gamma' > \gamma \) and \( \mu' > \mu \).

So the condition (A) and condition (B) can be combined as \( E > (\alpha \beta \hat{x} - \mu) / \gamma \) which means that the predator free equilibrium point will become stable if \( E > (\alpha \beta \hat{x} - \mu) / \gamma \). \hspace{1cm} (5.2)

So excess harvesting can lead to predator extinction which should be avoided for biodiversity.

**Theorem-3:-** The disease free equilibrium (DFE) \( C(x', y', 0) \), if exist is locally asymptotically stable if \( \frac{\lambda y'}{\mu' + \gamma' E - \alpha' \beta \hat{x}} < 1 \).

*Proof:* The Characteristic roots corresponding to the equilibrium \( C(x', y', 0) \) are given by the equation

\[
(\xi - \alpha' \beta x' + \mu' + \gamma' E - \lambda y') \left( \xi^2 + \frac{r x'}{k} \xi + \alpha' \beta x' y' \right) = 0
\]

Here \( \xi_1 = \alpha' \beta x' - \mu' - \gamma' E + \lambda y' \) and \( \xi_2, \xi_3 \) are roots of second factor having the entire coefficients positive. So \( \xi_2, \xi_3 \) are clearly negative.

For \( \xi_1 < 0 \), we have

\[
\frac{\lambda y'}{\mu' + \gamma' E - \alpha' \beta \hat{x}} < 1
\] \hspace{1cm} (5.3)

Here the lesser value of predation rate \( \alpha' \) of infected predator is helping in making disease free point more stable. Moreover if \( A(0,0,0) \) is stable then \( C(x', y', 0) \) will not even exist.

**Theorem-4:-** The endemic semi positive equilibrium \( D(\bar{x}, 0, \bar{z}) \) where entire population becomes infected by the disease if exist, is locally asymptotically stable if \( \frac{\lambda \bar{z}}{a \beta \bar{x} - \mu - \gamma E} > 1 \)

*Proof:* The Characteristic roots corresponding to the equilibrium \( D(\bar{x}, 0, \bar{z}) \) are given by the equation

\[
\left( \xi + \frac{r \bar{x}}{k} \right) \left[ \xi \left( \xi - \alpha \beta \bar{x} + \mu + \gamma E + \lambda \bar{z} \right) \right] + \alpha' \bar{x} \left( \alpha' \beta \bar{z} \left( \xi - \alpha \beta \bar{x} + \mu + \gamma E + \lambda \bar{z} \right) \right] = 0
\]
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Clearly this equation will have negative roots if \( a\beta \bar{x} - \mu - \gamma E - \lambda \bar{z} < 0 \)
i.e \( \frac{\lambda \bar{z}}{a\beta \bar{x} - \mu - \gamma E} > 1 \) \( (5.4) \)

The infected predator population \( \bar{z} \) decreases due to differential predation rate of infected predator \( \alpha' \) and due to harvesting of prey. So \( \bar{z} \) decreases which results in decrease the value of \( \frac{\lambda \bar{z}}{a\beta \bar{x} - \mu - \gamma E} \) and so the stability of \( D(\bar{x},0,\bar{z}) \) is increased to some extent which is favorable for biodiversity.

**Theorem-5:** The positive endemic equilibrium \( E(x^*,y^*,z^*) \) is locally asymptotically stable for all parametric values.

**Proof:** The Characteristic roots corresponding to the equilibrium \( E(x^*,y^*,z^*) \) are given by the equation

\[
\xi^3 + \frac{rx^*}{k} \xi^2 + \left( \lambda^2 y^* z^* + \alpha^2 \beta x^* y^* + \alpha'^2 \beta x^* z^* \right) \xi + \left( \frac{\lambda^2 r x^* y^* z^*}{k} + 2\lambda \alpha \alpha' \beta x^* y^* z^* \right) = 0
\]

Here \( \xi_1, \xi_2, \xi_3 \) are roots of above equation having all the coefficients positive. So \( \xi_1, \xi_2, \xi_3 \) are clearly negative.

Hence the equilibrium point \( E(x^*,y^*,z^*) \) is always locally stable.

**Theorem-6:** A locally asymptotically stable non zero equilibrium point \( E(x^*,y^*,z^*) \) is always a globally stable.

**Proof:** For global stability, define a function a positive definite function

\[
V = A_1 \left( x - x^* - x^* \log \frac{x}{x^*} \right) + B_1 \left( y - y^* - y^* \log \frac{y}{y^*} \right) + C_1 \left( z - z^* - z^* \log \frac{z}{z^*} \right)
\]

\[
\frac{dV}{dt} = A_1 \left( \frac{x - x^*}{x} \right) \frac{dx}{dt} + A_2 \left( \frac{y - y^*}{y} \right) \frac{dy}{dt} + A_3 \left( \frac{z - z^*}{z} \right) \frac{dz}{dt}
\]

\[
= A_1 \left( x - x^* \right) \left[ r \left( 1 - \frac{x}{K} \right) - \alpha y - \alpha' z - \gamma_1 E_1 \right] + A_2 \left( y - y^* \right) [\alpha \beta x - \mu - \gamma E - \lambda z]
\]

\[
+ A_3 \left( z - z^* \right) [\alpha' \beta x - \mu' - \gamma' E + \lambda y]
\]
\[ A_i \left(x-x^*\right) \left[ -\frac{r}{K} \left(x-x^*\right) - \alpha \left(y-y^*\right) - \alpha' \left(z-z^*\right) \right] + A_2 \left(y-y^*\right) \left[ \alpha \beta \left(x-x^*\right) - \lambda \left(z-z^*\right) \right] + A_3 \left(z-z^*\right) \left[ \alpha' \beta \left(x-x^*\right) + \lambda \left(y-y^*\right) \right] \]

\[ = -\frac{A_r}{K} \left(x-x^*\right)^2 - \alpha A_1 \left(x-x^*\right) \left(y-y^*\right) - \alpha' A_1 \left(x-x^*\right) \left(z-z^*\right) + A_2 \alpha \beta \left(x-x^*\right) \left(y-y^*\right) \]

\[ \lambda A_2 \left(y-y^*\right) \left(z-z^*\right) + \alpha' A_3 \left(x-x^*\right) \left(z-z^*\right) + \lambda A_3 \left(z-z^*\right) \left(y-y^*\right) \]

\[ = -\frac{A_r}{K} \left(x-x^*\right)^2 + \alpha \left(x-x^*\right) \left(y-y^*\right) \left(-A_1 + A_2 \beta\right) + \alpha' \left(x-x^*\right) \left(z-z^*\right) \left(-A_1 + A_3 \beta\right) \]

By taking \( A_1 = A_2 \beta, A_3 = A_3 \)

\[ \frac{dV}{dt} = -\frac{A_r}{K} \left(x-x^*\right)^2 < 0 \]

So the equilibrium point \( E \left(x^*, y^*, z^*\right) \) is globally asymptotically stable.

6. NUMERICAL SIMULATIONS

Numerical simulations have been carried out to investigate the dynamics of the proposed 3-D model (1). Computer simulations have been performed using MATLAB, for different set of parameters.

Consider the following set of parametric values:

\[ r = 0.6, \ k = 50, \ \alpha_1 = 0.015, \ \alpha_2 = 0.012, \ E = 0.11, E_1 = 1.9, \ \lambda = 0.6 \]

\[ \mu_1 = 0.25, \ \mu_2 = 0.3, \ \beta = 0.3, \ \gamma_1 = 0.35, \ \gamma_2 = 0.2, \ \gamma_3 = 0.6, \]

(6.1)

The system (1) has equilibrium point \( A \left(0,0,0\right) \) for the data set (6.1). It is locally asymptotically stable by Theorem (2), as the computed value of \( E_1 \) is so large that the equation (5.1) is satisfied. The solution trajectories in phase plane with different initial values for the equilibrium point \( A \left(0,0,0\right) \) is shown in Fig.1.
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Fig. 1: Phase diagram depicting the Behavior of the model equation (1) for the stability of equilibrium point $A(0,0,0)$.

Again consider the following set of parametric values:

$$r = 0.6, \quad k = 50, \quad \alpha_1 = 0.015, \quad \alpha_2 = 0.012, \quad E = 0.11, \quad E_1 = 0.6, \quad \lambda = 0.6 \quad (6.2)$$

$$\mu_1 = 0.25, \quad \mu_2 = 0.3, \quad \beta = 0.3, \quad \gamma_1 = 0.35, \quad \gamma_2 = 0.2, \quad \gamma_3 = 0.6,$$

We had taken all the parametric values same as in data set (6.1) except $E_1$, the harvesting effort for prey is taken so small that existence condition (3.1) is satisfied. System (1) has equilibrium point $B(32.5,0,0)$ for the data set (6.2). It is locally asymptotically stable by Theorem-2, as the computed value of $E$ is so large that the equation (5.2) is satisfied. So the excess harvesting of predator can lead to predator extinction. The solution trajectories in phase plane with different initial values for the equilibrium point $B(32.5,0,0)$ is shown in Fig. 2.
Fig. 2. Phase diagram depicting the Behavior of the model equation (2.1) for the stability of equilibrium point $B(32.5,0,0)$

Now consider the parametric set of values

$$r = 0.6, \; k = 50, \; \alpha_1 = 0.515, \; \alpha_2 = 0.012, \; E = 0.5, \; E_i = 0.6, \; \lambda = 0.6, \; \mu_1 = 0.25, \; \mu_2 = 0.3, \; \beta = 0.3, \; \gamma_1 = 0.35, \; \gamma_2 = 0.2, \; \gamma_3 = 0.6,$$  \hspace{1cm} (6.3)

System (1) has disease free equilibrium point $C(2.2654,0.7045,0)$ for the data set (6.3). This equilibrium point exists as the harvesting effort of prey species is so small that it satisfies the equation (3.2). Moreover the equilibrium point $C(2.2654,0.7045,0)$ is locally asymptotically stable by Theorem-3. Here the lesser value of predation rate of infected predator is helping in the local stability of the disease free equilibrium point. The solution trajectories in phase plane with different initial values for the equilibrium point $C(2.2654,0.7045,0)$ is shown in Fig. 3.
Fig. 3. Phase diagram depicting the Behavior of the model equation (1) for the stability of Disease free equilibrium point $C(2.2654, 0.7045, 0)$

Now consider the parametric set of values
\[
\begin{align*}
    r &= 0.6, \quad k = 50, \quad \alpha_1 = 1.15, \quad \alpha_2 = 1.012, \quad E = 0.1, \quad E_1 = 0.6, \quad \lambda = 0.6 \\
    \mu_1 &= 0.25, \quad \mu_2 = 0.3, \quad \beta = 0.3, \quad \gamma_1 = 0.35, \quad \gamma_2 = 0.2, \quad \gamma_3 = 0.6,
\end{align*}
\]  

(6.4)

System (1) has disease free equilibrium point $D(1.1858, 0, 0.3713)$ for the data set (6.4). This equilibrium point exists as the harvesting effort of prey species is so small that it satisfies the equation (3.4). Moreover the equilibrium point $D(1.1858, 0, 0.3713)$ is locally asymptotically stable by Theorem-4. The value of $\bar{z}$ decreases due to low predation rate of infected predator and also due to harvesting of prey. As a result the value of $\frac{\lambda \bar{z}}{\alpha \beta \bar{x} - \mu - \gamma E}$ decreases which helps in destabilization of equilibrium point $D(1.1858, 0, 0.3713)$ which is good for the biodiversity. The solution trajectories in phase plane with different initial values for the equilibrium point $D(1.1858, 0, 0.3713)$ is shown in Fig. 4.
Finally consider the following parametric set of values

\[ r = 0.6, \; k = 50, \; \alpha_1 = 0.115, \; \alpha_2 = 0.012, \; E = 0.11, E_i = 1, \; \lambda = 0.6 \]

\[ \mu_1 = 0.25, \; \mu_2 = 0.3, \; \beta = 0.3, \; \gamma_1 = 0.35, \; \gamma_2 = 0.2, \; \gamma_3 = 0.6, \] \tag{6.5} \]

System (1) has disease free equilibrium point \( E(15.4408, 0.5174, 0.4345) \) for the data set (6.5). This equilibrium point exists as reasonable harvesting efforts of prey and predator species are applied so as to satisfy the equation (3.5). Here all the three populations co-exist. The role of controlled harvesting is justified here. The solution trajectories in phase plane with different initial values for the equilibrium point \( D(1.1858, 0, 0.3713) \) is shown in Fig. 5.
Fig. 5. Phase diagram depicting the behavior of the model equation (1) for the stability of the interior equilibrium point $D(1.1858, 0, 0.3713)$

7. CONCLUSION
In this paper, a Prey-Predator model with disease in predator is proposed and analyzed. Both the prey as well as predator species are subjected to harvesting. Predation rate for infected predator is considered less as compared with the susceptible predator due to less mobility of infected ones. Local stability analysis of biologically feasible equilibrium points has been done. The conditions for the existence of equilibrium points and their stability have been established. It has been observed that the differential predation rate of the infected predator is helpful in making the disease free equilibrium point more stable, which means it will help in making the system disease free. The infected predator population $\mathcal{Z}$ decreases due to less differential predation rate of infected predator $\alpha'$. The harvesting of prey also reduce $\mathcal{Z}$. As $\mathcal{Z}$ decreases, the value of $\frac{\lambda\mathcal{Z}}{a\beta\lambda - \mu - \gamma E}$ also decreases and so the stability of equilibrium point $D(1.1858, 0, 0.3713)$ is decreased to some extent which is good for our biological environment.

The role of controlled harvesting is also justified. It is observed that if harvesting effort of prey i.e. $E_1$ is greater than $r \gamma_1$, then all the populations will extinct. The predator free equilibrium point will become stable if $E > (a\beta\lambda - \mu) / \gamma$. So excess
harvesting can lead to predator extinction. So a reasonable harvesting effort must be applied for the coexistence of all the species.

REFERENCES