Fixed Point Theorems for Approximating a Common Fixed Point for a Family of Nonself, Strictly Pseudo Contractive and Inward Mappings in Real Hilbert Spaces

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Abstract
In this paper, we introduce iterative methods of Mann type for approximating fixed point and common fixed point of a finite family of nonself, $k$-strictly pseudo contractive and inward mappings in real Hilbert spaces and we prove the weak and strong convergence theorems of the iterative methods.

Keywords and phrases: Common fixed point; $k$-strictly pseudo contractive mapping; nonself mapping; Mann’s iterative method, uniformly convex; uniformly smooth Banach space; weak and strong convergence.

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1. INTRODUCTION
In this paper, our aim is to study a common fixed for the finite family of strictly pseudo contractive mappings in Banach spaces, in particular, in real Hilbert spaces. The class of pseudo contractive mappings will play a great role in many of the existence for
solutions for nonlinear problems and hence the class of strictly pseudo contractive mappings as a subclass has a significant role.

Let \( E \) be a Banach space with its dual \( E^* \) and \( K \) be non empty, closed and convex subset of \( E \). Then a mapping \( T : K \rightarrow E \) is said to be strongly pseudo contractive if there exists a positive constant \( k \) and \( j(x - y) \in J(x - y) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \leq k \|x - y\|^2
\]
and a mapping \( T : K \rightarrow E \) is called pseudo contractive if there exists
\( j(x - y) \in J(x - y) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \quad \text{for all } x, y \in K.
\]

Where \( J : E \rightarrow 2^{E^*} \) is normalized duality mapping given by
\[
Jx = \left\{ f \in E^* : \langle f, x \rangle = \|x\|^2, \|f\|^2 \right\}
\]
where, \( \langle \ldots \rangle \) denotes the generalized duality pairing which is analogous to an inner product in Hilbert space. It can be seen that \( J \) is single valued if \( E \) is smooth. \( T \) is called \( \alpha \)-strictly pseudo contractive if there exists \( \alpha \in (0,1) \) and \( j(x - y) \in J(x - y) \) such that,
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \alpha \|I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in K
\]
\[
T \text{ is called Lipschitzian if and only if } \|Tx - Ty\| \leq L\|x - y\| \quad \text{for } x, y \in K \quad \text{some } L \geq 0
\]
and if \( L < 1 \), \( T \) is contraction mapping, whereas, if \( L \leq 1 \), the mapping is called nonexpansive. \( T \) is called contractive if
\[
\|Tx - Ty\| < \|x - y\| \quad \text{for all } x \neq y \text{ and } x, y \in K
\]
\[
T \text{ is called quasi-nonexpansive if } \|Tx - P\| \leq \|x - P\|, \forall x \in K \text{ and } P \text{ is a fixed point of } T
\]
\[
(1.1) \quad \text{and } (1.2) \text{ are equivalent to}
\]
\[
\|x - y\| \leq \|(1 + t)(x - y) - kt(Tx - Ty)\| \quad \text{for all } x, y \in E \text{ and } t > 0
\]
and
\[
\|x - y\| \leq \|(1 + t)(x - y) - t(Tx - Ty)\| \quad \text{for all } x, y \in K \text{ and } t > 0
\]
respectively.
We see that every contraction mapping is nonexpansive, every nonexpansive mapping is pseudo contractive and strongly pseudo contractive as well but the converse is not true.

In fact, if $T$ is non expansive then
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|Tx - Ty\| \|jx - jy\| \leq \|x - y\| \|x - y\| = \|x - y\|^2 \tag{1.9}\]

Thus, $T$ is pseudo contractive mapping.

In Hilbert spaces, $H$ (1.1) and (1.2) are reduced to the followings;
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \text{ for all } x, y \in K, \text{ for some } k \in (0,1) \tag{1.10}
\]
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \text{ } x, y \in K \tag{1.11}
\]
respectively.

The class of pseudo contractive mappings is more general than the class of nonexpansive mappings and is strong linked with that of accretive mappings; That is, a mapping $A: K \rightarrow E$ is said to be accretive if for all $x, y \in K$ and $t > 0$
\[
\|x - y\| \leq \|(x - y + t(Ax - Ay))\| \text{ holds.} \tag{1.12}
\]
(1.12) is equivalent to there exists $j(x - y) \in J(x - y)$ such that
\[
\langle Ax - Ay, j(x - y) \rangle \geq 0 \text{ for all } x, y \in K. \tag{1.13}
\]

We observe that $A$ is accretive if and only if $T = I - A$ is pseudo contractive mapping and the zero of $A$ is the fixed point of $T$ and vice versa. As a result, finding fixed point or common fixed point of pseudo contractive mapping (mappings) hence its subclass k-strictly pseudo contractive mappings is essential. We denote the set of fixed points of the mapping $T: K \rightarrow E$ by $F(T) = \{x \in K : T(x) = x\}$ where $K$ is a non empty closed and convex subset of a real Banach space $E$.

We shall have the following definitions:

**Definition 1.1** A Banach space $E$ is said to be Uniformly convex if for every $0 < \varepsilon < 2$ , there is a $\delta > 0$ such that for all $x, y \in S = \{x \in E : \|x\| = 1\}$, if $\|x - y\| > \varepsilon(x \neq y)$, then
\[
\frac{\|x + y\|}{2} \leq 1 - \delta.
\]
If \( \dim E \geq 2 \), the modulus of convexity of \( E \) is a function defined by
\[
\delta_E(t) = \inf \left\{ 1 - \frac{\|x + y\|}{2}, \|x\| = 1 & \|x - y\| = t, x, y \in E \right\}, 0 \leq t \leq 2,
\]
and \( E \) is said to be uniformly convex if \( \delta_E(t) > 0 \) for all \( 0 < t \leq 2 \).

**Definition 1.2** A uniformly smooth space, \( E \) is a normed space in which for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for \( x, y \in E, \|x\| = 1, \|y\| \leq \delta, \|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\| \).

If \( \dim E \geq 2 \), then the modulus Smoothness of \( E \) is defined by
\[
\rho_E(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1, \|x\| = 1, \|y\| = t \right\},
\]
and \( E \) is uniformly smooth if and only if \( \lim_{t \to 0} \frac{\rho_E(t)}{t} = 0 \).

Let \( p, q > 1 \) be real numbers. Then \( E \) is said to be \( p \)-uniformly convex (respectively \( q \)-uniformly smooth) if there is a constant \( c > 0 \) such that \( \delta_E(t) \geq ct^p \) (respectively \( \rho_E(t) \leq ct^q \)).

Hilbert spaces, the Lebesgue \( L_p \) and the sequences \( l_p \) are examples of uniformly convex and uniformly smooth Banach spaces for \( p \in (1, \infty) \) (see, for example, in [14]).

**Definition 1.3.** Let \( E \) be a Banach space, let \( S \) be defined by \( S = \{x \in E : \|x\| = 1\} \). Then \( E \) is said to have Gateaux differentiable norm if
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
events for all \( x, y \in S \). \( E \) is said to be smooth if its norm is Gateaux differentiable.

If for each \( y \in S \) the limit in (1.16) exists uniformly for each \( x \in S \), then the norm of \( E \) is said to be uniformly Gateaux differentiable. The norm of \( E \) is Frechet differentiable if for each \( x \in S \) the limit in (1.16) exists uniformly for each \( y \in S \). The norm of \( E \) is uniformly Frechet differentiable if the limit (1.16) exists uniformly for each \( x, y \in S \).

Thus, it is clear that every Frechet differentiable norm is Gateaux differentiable, however, the converse is not in general true. This can be seen (See, for example, in [2]).

Mann’s iterative process was first introduced by Mann in [10], since then considerably research works have been made to approximate fixed point of nonexpansive and k-
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strictly pseudo contractive mappings, via Mann’s iterative method [10] (see, for example [15]). Mann’s iterative method generates a sequence \( \{x_n\} \) according to the recursive formula, 

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n, \quad \alpha_n \in (0, 1), n \geq 0, x_0 \in K
\]  

(1.17)

If \( T \) is a nonexpansive self mapping on a nonempty, closed convex subset \( K \) of a real uniformly convex Banach space \( E \) with a Frechet differentiable norm with fixed point and if the sequence \( \{\alpha_n\} \subset (0,1) \) satisfies \( \sum_1^\infty \alpha_n (1-\alpha_n) = \infty \), then \( \{x_n\} \) converges to a fixed point of \( T \) weakly (See, for example [15]).

Since the convergence is weak, it is expected the result to be slow, however, Nakajo and Takahashi in [13] proposed a modified Mann type algorithm for nonexpansive and self mapping on a nonempty, closed and convex subset of a real Hilbert space \( H \) which is known as CQ algorithm and can be restated as;

\[
\begin{align*}
x_0 & \in K, \\
y_n & = \alpha_n x_n + (1 - \alpha_n)T x_n, \\
K_n & = \left\{ z \in K : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(k-\alpha_n)\|x_n - T x_n\|^2 \right\}, \\
Q_n & = \left\{ z \in K : \langle x_n - z, x_0 - z \rangle \geq 0 \right\}, \\
x_{n+1} & = P_{K_n \cap Q_n} x_0.
\end{align*}
\]

(1.18)

Here \( P_K \) the metric projection of the Hilbert space \( H \) onto a nonempty, closed convex subset \( K \) of \( H \).

Nakajo and Takahashi in [13] proved that the sequence \( \{x_n\} \) generated by the algorithm (1.17) converges strongly to a fixed point of \( T \) provided that \( \{\alpha_n\} \subset (0,1) \) satisfies \( \limsup_\infty \alpha_n < 1 \).

Marino and Xu in [20] extended the CQ algorithm (1.17) for \( k \)-strictly pseudo contractive mapping in Hilbert spaces. They proved the following theorem;

**Theorem 1.1 MXu** [20] Let \( K \) be a closed and convex subset of a Hilbert space \( H \). Let \( T : K \to K \) be a \( k \)-strict pseudo contractive mapping for some \( 0 \leq k < 1 \) and assume that \( F(T) \) is nonempty. Let \( \{x_n\} \) be the sequence generated by Mann’s algorithm (1.17).
Assume that the sequence \( \{\alpha_n\} \) is chosen such that \( k < \alpha_n < 1 \) for all \( n \) and \( \sum (\alpha_n - k)(1 - \alpha_n) = \infty \). Then the sequence \( \{x_n\} \) converges weakly to a fixed point of \( T \).

We notice that, if \( T \) is nonexpansive, then \( \kappa = 0 \) and Theorem 1.1 reduces to Reich’s theorem in Hilbert space setting (see, for example [15]). Since the convergence is weak, in order to get strong convergence Marino and Xu modified Mann’s algorithm to CQ algorithm for strict pseudo contractive mappings and they proved strong convergence in the following theorem;

**Theorem 1.2** MXu[12] Let \( K \) be a closed and convex subset of a Hilbert space \( H \). Let \( T: K \to K \) be a \( \kappa \)-strict pseudo contractive mapping for some \( 0 < k < 1 \) and assume that \( F(T) \) is nonempty. Let \( \{x_n\} \) be the sequence generated by the following CQ algorithm;

\[
\begin{align*}
    y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
    K_n &= \left\{ z \in K : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n}\|x_n - Tx_n\|^2 \right\}, \\
    Q_n &= \left\{ z \in K : \langle x_n - z, x_n - z \rangle \geq 0 \right\}, \\
    x_{n+1} &= P_{C \cap Q_n} x_0.
\end{align*}
\]

Assume that the sequence \( \{\alpha_n\} \) is chosen such that \( \alpha_n < 1 \) for all \( n \). Then the sequence \( \{x_n\} \) converges strongly to \( P_{F(T)}x_0 \).

However, the above results are for self mappings. On the other hand, in many cases nonself or family of nonself, \( k \)-strictly pseudo contractive mappings arises when the domain is a proper subset of the given space, thus finding a fixed point or a common fixed point (if it exists) is essential and is our purpose in this paper to modify Mann’s algorithm to approximate a fixed point and a common fixed point of the class of nonself and \( k \)-strictly pseudo contractive mappings in a real Hilbert space which is more general class than the class of nonexpansive mappings.

Attempts retraction have been made to approximate fixed point of nonself mappings by using projection for sunny nonexpansive of the real Banach spaces \( E \) on to its closed and convex subset \( K \) of \( E \).

**Definition 1.4** Let \( E \) be a real Banach space. Then
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a) a subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to K \) such that \( P(x) = x \) for all \( x \in K \).

b) a map \( P : E \to K \) is said to be a retraction if \( P^2 = P \). It follows that if a map \( P \) is a retraction, then \( P(y) = y \) for all \( y \) in the range of \( P \).

c) a map \( P : E \to K \) is said to be sunny if \( P(Px + t(x - Px)) = Px \) for all \( x \in E \) and \( t \geq 0 \).

d) a subset \( K \) of \( E \) is said to be a sunny nonexpansive retract of \( E \) if there exists a sunny nonexpansive retraction of \( E \) onto \( K \) and it is said to be a nonexpansive retract of \( E \) if there exists a nonexpansive retraction of \( E \) onto \( K \).

For example, in Hilbert space \( H \), the metric projection \( P_K \) is a sunny nonexpansive retraction from \( H \) onto any closed convex subset \( K \) of \( H \).

As a result, a number of research efforts have been made to find iterative methods for approximating a fixed point or a common fixed point (when it exists) for nonexpansive, pseudo contractive mappings and \( k \)-strictly pseudo contractive mappings as well via projection for sunny nonexpansive retraction. In particular, attempts to modify the Mann iteration method [14] for nonexpansive mappings and strict pseudo contractions so that strong convergence is guaranteed have recently been made. (See, for example, in [12, 14, 18-19] and their references). To mention a few;

Recently, Zhou in [20] modified the Mann’s iterative process [14] for nonself and \( k \)-strictly pseudo contractive mapping to have strong convergence in Hilbert spaces setting. He proved the following theorem.

**Theorem 1.3** Zhou [20] Let \( K \) be a closed and convex subset of a real Hilbert space \( H \). Let \( T : K \to H \) be a \( k \)-strictly pseudo contractive and nonself mapping such that \( F(T) \) is non empty. For \( u \in K \) and sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in (0,1). Let the sequence \( \{x_n\} \) be defined by

\[
\begin{align*}
x_1 &= x \in K, \\
y_n &= P_K(\alpha_n x_n + (1 - \alpha_n)Tx_n), \\
x_{n+1} &= \beta_n u + (1 - \beta_n)Ty_n, n \geq 1,
\end{align*}
\]

Where \( P_K \) is the projection of \( H \) onto the closed and convex subset \( K \), then if \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy the following conditions;
\[ \lim_{n \to \infty} \beta_n = 0 \text{ and } k \leq \beta_n \leq \varepsilon < 1 \text{ for all } n \geq 1; \quad \sum_{n=1}^{\infty} \beta_n = \infty \]

\[ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty \text{ or } \lim_{n \to \infty} \frac{\beta_n}{\beta_{n+1}} = 1, \text{ then the sequence } \{x_n\} \]

Converges strongly to a fixed point \( x \) of \( T \), and \( x = P_{F(T)} u \).

Motivated by the result of Zhou in [20], Hao in [7] extended to uniformly convex Banach spaces. He proved strong convergence theorems for approximating a common fixed point of a finite family of \( k \)-strict pseudo-contractive mappings in Banach spaces via metric projection, however, the computation of metric projection is costly, even in Hilbert spaces it requires another approximation method.

Thus, it is our purpose in this paper to introduce an iterative method for approximating fixed point or a common fixed point of nonself, \( k \)-strictly pseudo contractive and inward mapping or family of mappings without the computation of metric projection.

On the other hand, in 2015, Colao and Marino in [3] introduced Krasnoselskii-Mann’s iterative method for approximating a fixed point of nonself, nonexpansive mapping with additional assumptions; the mapping to be inward and strictly convexity of the domain in lowering the requirement of metric projection.

**Definition 1.5** A mapping \( T : K \to H \) is said to be inward (or to satisfy the inward condition) if for any \( x \in K \), \( T x \in IK(x) = \{x + c(u - x) : c \geq 1 \& u \in K\} \) and \( T \) is said to satisfy weakly inward condition if \( T x \in \overline{IK(x)} \) (the closure of \( IK(x) \)).

**Definition 1.6** A subset \( K \) of \( E \) is said to be strictly convex if for any \( x, y \in \partial K, x \neq y, 0 < t < 1, tx + (1-t)y \in \text{int}(K), \) that is, no line segment joining any two points of \( K \) totally lies on the boundary of \( K \).

They also proved weak and strong convergence for the following theorem.

**Theorem 1.4 CM** [3] Let \( K \) be a convex, closed and nonempty subset of a Hilbert space \( H \) and \( T : K \to H \) be a nonself mapping and let for any given \( x \in K \), \( h(x) = \inf \{\lambda \geq 0 : \lambda x + (1 - \lambda)x \in K\} \). Then the algorithm defined by
is well defined and assume that; K is strictly convex set and T is nonexpansive, nonself and inward mapping with F(T) is non empty, then the sequence \(\{x_n\}\) converges weakly to \(p \in F = F(T)\) and moreover if \(\sum_{n=1}^{\infty} (1 - \alpha_n) < \infty\), then the convergence is strong.

More recently, Takele and Reddy in [5&6] extended the theorem of CM to approximate a common fixed point of family of nonself, nonexpansive and inward mappings in real Hilbert spaces and real uniformly convex Banach spaces respectively. They also proved weak and strong convergence theorems. To be precise we put their main theorem in [5] as follow.

**Theorem 1.5 TR [5]** Let \(T_1, T_2, \ldots, T_N : K \to H\) be family of, nonself, nonexpansive and inward mappings on a non empty, closed and strictly convex subset K of a real Hilbert space H with \(F = \bigcap_{k=1}^{N} F(T_k)\) is non empty, \(T_k = T_k(\text{Mod} N), x_0 \in K\) and if for each we define \(h_k : K \to \mathbb{R}\) by \(h_k(x) = \inf \{\lambda \geq 0 : \lambda x + (1 - \lambda)T_kx \in K\}\), \(\alpha \in (0,1)\) be fixed. Then the sequence \(\{x_n\}\) given by

\[
\begin{align*}
x_0 \in K \\
\alpha_0 = \max \left\{ \frac{1}{2}, h(x_0) \right\}, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \\
\alpha_{n+1} = \max \{\alpha_n, h(x_{n+1})\}
\end{align*}
\]

is well-defined and if \(\{\alpha_n\} \subset [\varepsilon, 1 - \varepsilon] \subset (0,1)\) for some \(\varepsilon \in (0,1)\), then \(\{x_n\}\) converges weakly to some element p of \(F = \bigcap_{k=1}^{N} F(T_k)\). Moreover, if \(\sum_{n=1}^{\infty} (1 - \alpha_n) < \infty\) and \((F,K)\) satisfies S-condition, then the convergence is strong.

Our concern is that is it possible to approximate fixed point of the family of nonself, k-strictly pseudo contractive mappings which is more general than the family of nonexpansive mappings? Our objective in this paper is to approximate a fixed point and a common fixed point (if it exists) for the family of nonself, k-strictly pseudo
contractive and inward mappings which is more general class than the class of nonexpansive mappings in real Hilbert spaces which is a positive answer to our concern.

2. PRELIMINARY CONCEPTS

Let $K$ be a non-empty, closed and convex subset of a real Banach space $E$. Let $T : K \rightarrow E$ be nonself mapping; in the sequel, we shall need the following assumptions;

Lemma 2.1. (See, for example, Proposition 2.1 in [9]) Suppose $K$ is a closed and convex subset of a Hilbert space $H$. Let $T : K \rightarrow K$ be a mapping on $K$. Then

(a) if $T$ is a $\kappa$-strict-pseudo contractive, then $T$ satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{k + 1}{1 - k} \|x - y\|, \text{ for all } x, y \in K;$$

(b) if $T$ is a $\kappa$-strict pseudo contractive, then the mapping $I - T$ is demi closed (at 0). That is, if $\{x_n\}$ is a sequence in $K$ such that $x_n \rightharpoonup x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly, then $Tx = x$;

(c) if $T$ is a $\kappa$-quasi-strict pseudo contraction, then the fixed point set $F(T)$ of $T$ is closed.

Remark: The proposition can be easily extended to for nonself mapping $T : K \rightarrow H$ with the same conclusion.

Definition 2.1. A sequence $\{x_n\}$ in $K$ is said to be Fejer monotone with respect to a subset $F$ of $K$ if $\forall x \in F, \|x_{n+1} - x\| \leq \|x_n - x\| \forall n$.

Lemma 2.2. (See, for example, (MARINO and Xu [12], lemma 1.1) Let $H$ be a Hilbert space. Then, for all $\lambda \in [0,1]$, $\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$, for all $x, y \in H$.

Definition 2.2 Let $F, K$ be two closed and convex nonempty sets in a Hilbert space $H$ and $F \subset K$. For any sequence $\{x_n\} \subset K$, if $\{x_n\}$ converges strongly to an element $x \in K \setminus F$, $x_n \neq x$, implies that $\{x_n\}$ is not Fejer-monotone with respect to the set $F \subset K$, we called that, the pair $(F, K)$ satisfies $S$-condition.
Lemma 2.3 (see in [5,3]) Suppose \( T_1, T_2, \ldots, T_N : K \rightarrow E \) be nonself mappings. If for each \( k \in \{1, 2, \ldots, N\} \) we define \( h_k : K \rightarrow \mathbb{R} \) by \( h_k(x) = \inf \{ \lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in K \} \), then

\[
\begin{align*}
a) & \quad \forall x \in K, h_k(x) \in [0, 1] \text{ and } h_k(x) = 0 \text{ if and only if } T_k(x) \in K; \\
b) & \quad \forall x \in K, \alpha_k \in [h_k(x), 1], \text{ then } \alpha_k x + (1-\alpha_k)T_k(x) \in K; \\
c) & \quad \text{If } T_k \text{ is inward mapping, then } , \forall x \in K, h_k(x) < 1; \\
d) & \quad \text{If } T_k x \not\in K, \text{ then } h_k(x) + (1-h_k(x))T_k x \in \partial K.
\end{align*}
\]

3. MAIN RESULT

Suppose \( T_1, T_2, \ldots, T_N : K \rightarrow H \) be family of nonself and k-strictly pseudo contractive mappings on a non empty, closed and convex subset, \( K \) of a real Hilbert space \( H \), our objective is to introduce an iterative method for common fixed point of the family and determine conditions for convergence of the iterative method.

In lowering the requirement of metric projection calculation, which is expensive in many cases, we impose the condition that the mappings to be inward, and as similar to Lemma 2.1 in [9] we prove the following lemma which will be useful to prove the main results.

We will prove our main theorem, which is given below;

Lemma 3.1 (Extension of lemma 2.1 for nonself mapping). Let \( K \) be a closed and convex subset of a Hilbert space \( H \). Let \( T : K \rightarrow H \) be a mapping on \( K \). Then

if \( T \) is a \( \kappa \)-strict-pseudo contractive, then \( T \) satisfies the Lipschitz condition,

\[
\|Tx - Ty\| \leq \frac{k + 1}{1 - k} \|x - y\|, \text{ for all } x, y \in K.
\]

Proof Let for every \( K y, x \in K \), since \( T \) is a \( \kappa \)-strict-pseudo contractive mapping for some \( 0 \leq k < 1 \) we have the following,

\[
\begin{align*}
\|Tx - Ty\|^2 & \leq \|x - y\|^2 + k \langle x - y, - (Tx - Ty) \rangle \\
& = (1 + k) \|x - y\|^2 + k \|Tx - Ty\|^2 - 2k \langle x - y, Tx - Ty \rangle \\
& \leq (1 + k) \|x - y\|^2 + k \|Tx - Ty\|^2 + 2k \|x - y\| \|Tx - Ty\|
\end{align*}
\]
Thus, \((1 - k)\|Tx - Ty\|^2 - 2k\|x - y\|\|Tx - Ty\| - (1 + k)\|x - y\|^2 \leq 0\).

Solving the quadratic inequality for \(\|Tx - Ty\|\) by completing the square method as

\[
\|Tx - Ty\|^2 - \frac{2k}{(1 - k)}\|x - y\|\|Tx - Ty\| + \frac{k^2}{(1 - k)^2}\|x - y\|^2 \leq \left(\frac{1 + k}{(1 - k)}\|x - y\|\right)^2 + \frac{k^2}{(1 - k)^2}\|x - y\|^2
\]

\[
= \frac{1}{(1 - k)^2}\|x - y\|^2.
\]

Which implies that \(\|Tx - Ty\| \leq \frac{k + 1}{(1 - k)}\|x - y\|\) \(\) for all \(x, y \in K\).

\textbf{Theorem 3.2} Let \(K\) be non-empty, closed and convex subset of a real Hilbert space \(H\). Let \(T : K \rightarrow H\) be \(k\)-pseudo contractive for some \(0 \leq k < 1\), nonself and inward mapping with non-empty fixed pint set \(F(T)\). Let \(h : K \rightarrow \mathbb{R}\) be defined for all \(x \in K\) by \(h(x) = \inf \{\lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in K\}\). Let the sequence \(\{x_n\}\) be generated by the algorithm

\[
\begin{cases}
  x_1 \in K, \alpha > k, \\
  \alpha_n = \max\{\alpha, h(x_n)\}, \\
  x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n, \\
  \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}.
\end{cases}
\]

Then \(\{x_n\}\) is well-defined and if \(\sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty\), then \(\{x_n\}\) converges weakly to some \(x \in F(T)\).

\textbf{Proof:} By lemma 2.3 \(\{x_n\}\) is well defined and in \(K\). And also \(\{\alpha_n\}\) is nondecreasing and \(k < \alpha_n < 1\), hence by theorem 1.1 in [12], \(\{x_n\}\) converges weakly to some element \(x \in F(T)\).

\textbf{Theorem 3.3} Let \(K\) be non-empty, closed and strictly convex subset of a real Hilbert space \(H\). Let \(T : K \rightarrow H\) be \(k\)-pseudo contractive for some \(0 \leq k < 1\), nonself and inward mapping with non-empty fixed pint set \(F(T)\). Let \(h : K \rightarrow \mathbb{R}\) be defined for all
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Let $x \in K$ by $h(x) = \inf \{ \lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in K \}$. Let the sequence $\{x_n\}$ be generated by the algorithm

$$
\begin{align*}
x_1 &\in K, \alpha > k, \\
\alpha_i &= \max \{ \alpha, h(x_1) \}, \\
x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)Tx_n,
\end{align*}
$$

Then $\{x_n\}$ is well-defined and if $\sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) < \infty$, then $\{x_n\}$ converges strongly to some $x \in F(T)$.

Proof Let $p \in F(T)$. Then from lemma 2.1, lemma 2.2 and $T$ is $k$-strictly pseudo contractive, it can be shown that $\|x_n - p\|$ is decreasing and bounded below, hence converges as follow:

$$
\begin{align*}
\|x_{n+1} - p\|^2 &= \|x_n p + (1 - \alpha_n)Tx_n - p\|^2 \\
&= \|\alpha_n (x_n - p) + (1 - \alpha_n)(Tx_n - p)\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n (1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)(\|x_n - p\|^2 + k\|x_n - Tx_n\|^2) - \alpha_n (1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - p\|^2 - (1 - \alpha_n)(\alpha_n - k)\|x_n - Tx_n\|^2 \leq \|x_n - p\|^2,
\end{align*}
$$

Thus $\|x_n - p\|$ is decreasing and bounded below, hence converges.

Thus $\lim_{n \to \infty} \|x_n - p\|$ exists, let $\lim_{n \to \infty} \|x_n - p\| = r$ for some $r \geq 0$, then $\lim_{n \to \infty} \|x_{n+1} - p\| = r$.

Hence, cancelation of terms in the right hand-side and convergence of $\|x_n - p\|$ we have

$$
\sum_{n=1}^{\infty} (1 - \alpha_n)(\alpha_n - k)\|x_n - Tx_n\|^2 \leq \sum_{n=1}^{\infty} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) < \infty. \tag{3.2}
$$

Again from the lemma 2.1 we have

$$
\|x_n\| = \|x_n - p + p\| \leq \|x_n - p\| + \|p\| \leq M_1,
$$

and

$$
\|Tx_n\| = \|Tx_n - Tp + p\| \leq \frac{k + 1}{1 - k} \|x_n - p\| + \|p\| \leq M_2 \tag{3.3}
$$
hold for some $M_1, M_2 \geq 0$.

Thus, the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded, hence $\{x_n - Tx_n\}$ is bounded. (3.4)

Since $\sum_{n=1}^{\infty} (\alpha_n - k)(1-\alpha_n) < \infty$, we have $\lim_{n \to \infty} (1-\alpha_n)(\alpha_n - k) = 0$, since $\alpha_n > k$ and $\{\alpha_n\}$ is nondecreasing, we must have $\lim_{n \to \infty} (1-\alpha_n) = 0$ equivalently $\lim \alpha_n = 1$. (3.5)

Also, we have $\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - Tx_n\|$, thus $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. (3.6)

Thus $\{x_n\}$ is Cauchy sequence in $H$, which converges in $H$.

Since by lemma 2.3 $\{x_n\}$ is in $K$ and $K$ is closed, convex subset of the Hilbert space $H$.

Thus $x_n \to x \in K$.

Hence by lemma 2.3(a) $h(x) < 1$, thus by definition of $h$ we see that for any $\beta \in [h(x), 1)$, $\beta x + (1-\beta)Tx \in K$. (3.7)

Since $\lim \alpha_n = \lim \max \{\alpha_{n-1}, h(x_n)\} = 1$, there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[h(x_{n_k})]$ is increasing and $\lim_{k \to \infty} h(x_{n_k}) = 1$, hence for any $\beta < 1$,

$$\beta x_{n_k} + (1-\beta)Tx_{n_k} \notin K$$ (3.8)

Now let $\beta_1, \beta_2 \in (h(x), 1)$, $\beta_1 \neq \beta_2$ and $\beta_1 x + (1-\beta_1)Tx = z_1 \in K$, $\beta_2 x + (1-\beta_2)Tx = z_2 \in K$.

In particular, if $\beta \in [\beta_1, \beta_2]$, hence $\beta \in (h(x), 1)$ and $z = \beta x + (1-\beta)Tx \in K$ (3.9)

Since $x_n \to x \in K$ and $T$ is Lipschitzian, hence continuous and by lemma 2.3 (d)

$$\beta x_{n_k} + (1-\beta)Tx_{n_k} \to z = \beta x + (1-\beta)Tx \in \partial K$$ (3.10)

Similarly $z_1, z_2 \in \partial K$, since $\beta$ is arbitrary, $[z_1, z_2] \subset \partial K$.

Since $K$ is strictly convex $z_1 = z_2$, hence $x = Tx$.

Therefore, $x_n \to x \in F(T)$ in norm, this complete the proof of the theorem 3.4.

**Remark** Now we give example for k- strictly pseudo contractive, nonself and inward mapping that illustrate our result.
Let $K = \left[ \frac{-1}{2}, 1 \right] \subset H = \mathbb{R}$. Let $T : K \to \mathbb{R}$ by $Tx = -2x$. Thus

$$
\|Tx - Ty\|^2 = \|2x - 2y\|^2 = 4\|x - y\|^2 \leq \|x - y\|^2 + 3\|x - y\|^2 \\
= \|x - y\|^2 + \frac{1}{3}\|3x - 3y\|^2 \\
= \|x - y\|^2 + \frac{1}{3}\|(x - y) - (-2x + 2y)\|^2
$$

(3.12)

This shows that $T$ is $k$-strictly pseudo contractive for $k = \frac{1}{3} \in (0,1)$.

We see that $T$ is nonself and inward mapping, indeed,

$$
Tx = -2x = x + 2\left( -\frac{x}{2} - x \right) = x + c(u - x), \text{ where } c = 2 \geq 1 \text{ and } u = -\frac{x}{2} \in \left[ \frac{-1}{2}, \frac{1}{4} \right] \subset K.
$$

Thus $T$ is $k$-strictly pseudo contractive, nonself and inward mapping with fixed point 0.

**Theorem 3.4** Let $K$ be non-empty, closed and convex subset of a real Hilbert space $H$. Let $A : K \to H$ be non self mapping such that $I - A$ is $k$-pseudo contractive for some $0 \leq k < 1$, and inward mapping with non-empty fixed pint set. Let $h : K \to \mathbb{R}$ be defined for all $x \in K$ by $h(x) = \inf \{ \lambda \geq 0 : x + (\lambda - 1)Ax \in K \}$. Let the sequence $\{x_n\}$ be generated by the algorithm

$$
\begin{align*}
&x_1 \in K, \alpha > k, \\
&\alpha_1 = \max \{\alpha, h(x_1)\},
&x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I - A)x_n,
&\alpha_{n+1} = \max \{\alpha_n, h(x_{n+1})\}.
\end{align*}
$$

Then $\{x_n\}$ is well-defined and if $\sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ converges weakly to some $N(A)$. Moreover, if $K$ is strictly convex and $\sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) < \infty$, then the convergence is strong.
The proof of theorem 3.4 is the same as the proof of theorem 3.2 and 3.3 for $T = I - A$.

**Lemma 3.5** Let $T : K \to H$ be $k$-strictly pseudo contractive for some $k \in (0,1)$ and $\alpha \in (k,1)$, then $T_{\alpha} : K \to H$ defined by $T_{\alpha}x = \alpha x + (1 - \alpha)T(x)$ is non expansive and $F(T_{\alpha}) = F(T)$.

Proof: let $x, y \in K$ and $\alpha \in (k,1)$, then by lemma 2.2 in [12] and $T$ is $k$-strictly pseudo contractive we will have the following:

$$
\|T_{\alpha}x - T_{\alpha}y\|^2 = \|\alpha x + (1 - \alpha)Tx - (\alpha y + (1 - \alpha)Ty)\|^2 \\
= \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|^2 \\
\leq \alpha\|x - y\|^2 + (1 - \alpha)\|Tx - Ty\|^2 - \alpha(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
\leq \alpha\|x - y\|^2 + (1 - \alpha)\|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2 \\
- \alpha(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
= \|x - y\|^2 + (1 - \alpha)(k - \alpha)\|(x - y) - (Tx - Ty)\|^2 \leq \|x - y\|^2
$$

(3.13)

Since $k < \alpha$, we have $(1 - \alpha)(k - \alpha)\|(x - y) - (Tx - Ty)\|^2 < 0$.

Thus $\|T_{\alpha}x - T_{\alpha}y\| \leq \|x - y\|$, hence $T_{\alpha}$ is nonexpansive and $x \in F(T_{\alpha})$ if and only if $x \in F(T)$.

which completes the proof of the lemma.

**Theorem 3.6** Let $T_1, T_2, ..., T_N : K \to H$ be family of nonself, $k$-strictly pseudo contractive and inward mappings on a non-empty, closed strictly convex subset $K$ of a real Hilbert space $H$. Let $\alpha \in (k,1)$ $T_{\alpha} = T_{\alpha}(\text{Mod}(N))$ and for $i = 1, 2, ..., N,$

$$
h_i(x) = \inf \{ \lambda \geq 0 : \lambda x + (1 - \lambda)T_{\alpha}x \in K \}.
$$

Let $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Then the sequence $\{x_n\}$ given by
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\[
\begin{aligned}
x_1 & \in K, \beta > k \\
\beta_1 & = \max \{\beta, h_1(x_1)\}, \\
x_{n+1} & = \beta_n x_n + (1 - \beta_n) r_n x_n \\
\beta_{n+1} & = \max \{\beta_n, h_{n+1}(x_{n+1})\}
\end{aligned}
\]

is well-defined and if \( \{\beta_n\} \subset [\varepsilon, 1 - \varepsilon] \subset (0, 1) \) for some \( \varepsilon \in (0, 1) \) \( \{x_n\} \) converges weakly to some element \( p \) of \( F = \bigcap_{k=1}^{N} F(T_k) \). Moreover, if \( \sum_{n=1}^{\infty} (1 - \beta_n) < \infty \) and \( (F, K) \) satisfies S-condition, then the convergence is strong.

Proof suppose that for each \( i, T_i \) is inward mapping.

Then for each \( x \in K, T_{\alpha_i}(x) = x + c(u - x), c \geq 1 \) \& \( u \in K \), thus

\[
T_{\alpha_i}(x) = \alpha x + (1 - \alpha) T_{\alpha_i} x \\
= \alpha x + (1 - \alpha) \left[ x + c(u - x) \right] \\
= x + c(1 - \alpha)(u - x) \\
= x + d(u - x), d = c(1 - \alpha) \geq c \geq 1, u \in K
\]

Thus \( T_{\alpha_i} \) is inward mapping, hence \( \{T_{\alpha_i}\}_{i=1}^{N} \) is family of nonself, nonexpansive and inward mappings. Thus by the theorem of 1.5, lemma 3.1 and lemma 3.6 we complete the proof.

For example, let \( K = \left[-\frac{1}{2}, 1\right] \subset H = \mathbb{R} \), and let \( T_1, T_2 : K \to \mathbb{R} \) be defined by

\[
T_1(x) = -2x, \quad T_2(x) = \begin{cases} 
-x, & -\frac{1}{2} \leq x \leq 0 \\
\frac{1}{2}, & 0 < x \leq 1
\end{cases}
\]

Thus \( T_1 \) and \( T_2 \) are nonself, \( \frac{1}{3} \)-strictly pseudo contractive and inward mappings with common fixed point 0.

**Theorem 3.7** Suppose \( A_1, A_2, \ldots, A_N : K \to H \) be family of, nonself on a non-empty, closed and convex subset \( K \) of a Hilbert space \( H \), such that for each \( i = 1, 2, \ldots, N \), \( I - A_i \) is \( k \)-strictly pseudo contractive and inward mapping. Let \( \alpha \in (k, 1) \) for each \( x \in K \), \( A_{\alpha_i}x = (1 - \alpha) A_i x \) and \( A_{\alpha_i} = A_{\alpha_i} \pmod{N} \),

\[
h_i(x) = \inf \{ \lambda \geq 0 : x + (\lambda - 1) A_{\alpha_i} x \in K \}
\]

Let \( F = \bigcap_{i=1}^{N} N(A_i) \) is nonempty. Then the sequence \( \{x_n\} \) given by
\[
\begin{align*}
\begin{cases}
x_1 \in K, \beta > k \\
\beta_n = \max \{\beta, h_i(x_1)\}, \\
x_{n+1} = x_n + (\beta_n - 1)A_n x_n \\
\beta_{n+1} = \max \{\beta_n, h_n(x_{n+1})\}
\end{cases}
\end{align*}
\]

\(\varepsilon \in (0,1) \) \(\{x_n\}\) converges weakly to some element \( p \) of \( F = \bigcap_{i=1}^{N} N(A_i) \). Moreover, if
\[
\sum_{n=1}^{\infty} (1 - \beta_n) < \infty \quad \text{and} \quad (F,K) \text{ satisfies S-condition, then the convergence is strong.}
\]

Proof: Let \( T_{a_i} = (I - A_{a_i}) \). Then from theorem 3.5, 3.7 and \( N(A) = F(T) \) we complete the proof of the theorem.

**Corollary 3.8** Suppose \( T: K \to H \) is nonself, nonexpansive and inward mapping on a non-empty closed convex subset K of a Hilbert space H.

Let \( x_1 \in K, h(x) = \inf \{\lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in K\} \). Then the sequence \( \{x_n\} \) given by
\[
\begin{align*}
\begin{cases}
x_1 \in K \\
\alpha_1 = \max \{\alpha, h(x_1)\}, \alpha > 0 \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \\
\alpha_{n+1} = \max \{\alpha_n, h(x_{n+1})\}
\end{cases}
\end{align*}
\]

Which agrees with Krasnoselskii-Mann type algorithm for \( \theta \)-strictly Pseudo contractive mapping.

**Corollary 3.9.** Let \( T_1, T_2, \ldots, T_N : K \to H \) be family of, nonself, nonexpansive and inward mappings on a non-empty, closed and strictly convex subset K of a real Hilbert space H with \( F = \bigcap_{i=1}^{N} F(T_i) \) is non empty. \( T_{a_n} = T_{a(n \text{mod} N)} \), \( x_1 \in K \) and if for each we define \( h_i : K \to \mathbb{R} \) by \( h_i(x) = \inf \{\lambda \geq 0 : \lambda x + (1 - \lambda)T_i x \in C\} \), \( \alpha \in (0,1) \) be fixed. Then the sequence \( \{x_n\} \) given by
\[
\begin{align*}
\begin{cases}
x_1 \in C \\
\alpha_1 = \max \{\alpha, h_1(x_1)\} \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \\
\alpha_{n+1} = \max \{\alpha_n, h_{n+1}(x_{n+1})\}
\end{cases}
\end{align*}
\]
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is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\{x_n\}$ converges weakly to some element $p$ of $F = \bigcap_{i=1}^{N} F(T_i)$. Moreover, if $\sum_{n=1}^{\infty} (1-\alpha_n) < \infty$ and $(F,K)$ satisfies S-condition, then the convergence is strong.

Proof. For $k = 0$, each $T_i$, for $i = 1,2,...,N$ is 0-pseudo contractive mapping, hence the corollary is the same as the theorem of TR [5].

**Remark:** If each $T_1, T_2,..., T_N : C \to H$ are self-mappings, then, for all $x, h_i(x) = 0, \forall i$.

Thus $\alpha_n = \alpha$, the above iterative methods reduces to Mann’s iterative method and converges weakly, since $\alpha > 0$, and $\alpha \to 0$, and $\sum \alpha(1-\alpha) = \infty$.

**Concluding remark:** Our theorems generalize the theorem of CM to the class of k-strictly pseudo contractive mappings which is more general than the class of nonexpansive mappings. Our theorems also extend the theorem of MXu [12] to nonself mapping and for the family of nonself mappings as well. We also lower the requirement for metric projection computation, which is costly and requires another approximation technique.

Finally, we raise the following open question:

**Open question 1.** Is it possible to extend our results to more general spaces such as; uniformly convex Banach spaces, uniformly smooth Banach spaces? If so under what conditions?

**Open question 2.** Is it possible to extend our results to the class of pseudo contractive mappings which is more general than the class of strictly pseudo contractive mappings? If so under what conditions?

**Open question 3.** Is it possible to extend our results to infinite family of k-strictly pseudo contractive mappings? If so under what conditions?

**Authors’ contributions**

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

**Competing interests**

The authors declare that they have no competing interests.
REFERENCES:


