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Conformal anti-invariant Submersions from Sasakian manifolds

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Abstract

In this paper we define conformal anti-invariant submersions from almost contact metric manifolds onto Riemannian manifolds. We obtain some results on conformal anti-invariant submersions from Sasakian manifolds onto Riemannian manifolds. We also give the necessary and sufficient conditions for a conformal anti-invariant submersions to be harmonic and totally geodesic. Moreover, we obtain decomposition theorems by using the existence of conformal anti-invariant submersions from Sasakian manifolds onto Riemannian manifolds. Finally, we give some examples of conformal anti-invariant submersions such that characteristic vector field ξ is vertical.

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1. Introduction

The theory of smooth maps between Riemannian manifolds has been extensively studied in Riemannian geometry. Such maps are useful for comparing geometric structures between two manifolds.

In 1966, O'Neill [19] initiated the study of Riemannian submersion between Riemannian manifolds. It was found beneficial if one should study such submersions between manifolds with differentiable structures. When Watson was studying almost Hermitian submersions between almost Hermitian manifolds he found that the base manifold and each fibre have the same kind of structure as the total space, in most cases [25]. We note that almost Hermitian submersions have been extended to the almost contact metric submersions [7], locally conformal Kahler submersions [16] etc.

We have so many submersions. Some of them are: semi-Riemannian submersion and Lorentzian submersion [8], semi-invariant submersion [23], slant submersion ([6], [22]), contact-complex submersion [12], anti-invariant Riemannian submersions from Cosymplectic manifold [18] etc. As we know that Riemannian submersions are related with Physics and have their applications in the Yang-Mills theory [24], Kaluza-Klein theory ([5], [13]), supergravity and superstring theories ([14], [17]). In 2010, Sahin defined anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [21] etc.

As a generalization of Riemannian submersions, horizontally conformal submersion are introduced as follows [2]. Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimension *m* and *n* respectively. A smooth map $f : (M, g_M) \to (N, g_N)$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(U, V) = g_N(f_*U, f_*V), \tag{1.1}$$

for every $U, V \in (\ker f_*)^{\perp}$. It is evident that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Let f is a smooth map between given Riemannian manifolds and $x \in M$. Then, f is called horizontally weakly conformal map at x if either (i) $f_{*x} = 0$ or (ii) f_{*x} maps the horizontal space $\mathcal{H} = (\ker f_*)^{\perp}$ conformally onto $T_{f(x)}N$, *i.e.*, f_{*x} is surjective and f_* satisfies the equation (1.1) for U, V vectors tangent to $\mathcal{H}_x$. If a point x is of type (i), then it is called critical point of f. A point x of type (ii) is called regular. The number $\wedge(x)$ is called the square dilation which is necessarily non-negative. Its square root $\lambda(x) = \sqrt{\wedge(x)}$ is known as the dilation. A horizontally weakly conformal map f to be horizontally homothetic if the gradient of their dilation λ is vertical, i.e., $\mathcal{H}(grad\lambda) = 0$ at regular points. If a horizontally weakly conformal map f has no critical points, then f is called horizontally conformal submersion [2]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. The horizontal conformal maps were introduced independently by Fuglede in 1978 [9] and by Ishihara 1979 [15]. From the above argument, one can determine that the notion of horizontal conformal maps is a generalization of the concept of Riemannian submersions.

Next, we memorize the following description in [10]. Let $f : (M, g_M) \to (N, g_N)$ be a submersion. A vector field X on M is said to be projectable if there exists a vector field $\widehat{X}$ on N, such that $f_*(X_x) = \widehat{X}_{f(x)}$ for each $x \in M$. In this situation X and $\widehat{X}$ are called $f-$related. We call a horizontal vector field Y on (M, g_M) a basic vector fields if it is projectable. We know that if $\widehat{Y}$ is a vector field on N, then there exists a unique basic vector field Y on M, such that Y and $\widehat{Y}$ are $f-$related. The vector field Y is called the horizontal lift of $\widehat{Y}$.

We denote the kernel space of f_* by ker f_* and consider the orthogonal complementary space $\mathcal{H} = (\ker f_*)^{\perp}$ to ker f_*. Then the tangent bundle of M has the following decomposition

$$TM = (\ker f_*) \oplus (\ker f_*)^{\perp}. \tag{1.2}$$

We also denote the *range* of f_* by $rangef_*$ and consider the orthogonal complementary space $(rangef_*)^{\perp}$ to $rangef_*$ in the tangent bundle TN of N. Thus the tangent bundle TN of N has the following decomposition

$$TN = (rangef_*) \oplus (rangef_*)^{\perp}. \tag{1.3}$$

We know that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties. The concept of Conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifold has been studied in [1].

In this paper, we study conformal anti-invariant submersions from Sasakian manifolds onto Riemannian manifolds. The paper is organized as follows. In section 2, we collect main notions and formulae which we need in this paper.

In section 3, we introduce conformal anti-invariant submersions from almost contact metric manifolds onto Riemannian manifolds, investigate the geometry of leaves of the horizontal distribution and the vertical distribution. In section 4, we study the necessary and sufficient conditions for a conformal anti-invariant submersion to be harmonic and totally geodesic. In section 5, we prove that there are certain product structures on the total space of a conformal anti-invariant submersion from Sasakian manifold on Riemannian manifold such that ξ is vertical vector field. Finally in section 6, we give some examples of conformal anti-invariant submersion such that the characteristic vector field ξ is vertical.

2. Preliminaries

An $n-$dimensional Riemannian manifold M is said to be an almost contact metric manifold, if there exist on M, a $(1, 1)$ tensor field ϕ, a vector field ξ, a $1-$form η and Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

for any vector fields X, Y on M.

An almost contact metric manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the fundamental $2-$form of M. On the other hand the almost contact metric structure of M is said to be normal if $[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi$, for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ given by

$$[\phi, \phi]\ (X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{2.5}$$

A normal contact metric manifold is called a Sasakian manifold [4]. It can be proved that a Sasakian manifold is $K-$contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

for any vector fields X, Y on M.

Moreover, for a Sasakian manifold the following equations satisfy:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$\nabla_X \xi = -\phi X. \tag{2.9}$$

Definition 2.1. **([2])** Let (M, g_M) and (N, g_N) are two Riemannian manifold with Riemannian metrics g_M and g_N, respectively. If $f : (M, g_M) \to (N, g_N)$ be a differentiable map between given Riemannian manifolds, then f is called horizontally weakly conformal or semi-conformal at q if either

(i) $df_q = 0$, or

(ii) df_q maps the horizontal space $\mathcal{H}_q = (\ker(df_q))^{\perp}$ conformally onto $T_{f(q)}N$ *i.e.*, df_q is surjective and there exists a number $\Lambda(q) \neq 0$ such that

$$g_N(f_*U, f_*V) = \Lambda(q) g_M(U, V) \qquad (U, V \in \mathcal{H}_q), \tag{2.10}$$

where $q \in M$.

Watson introduced the fundamental tensors of a submersion in [19]. It is well known that the fundamental tensor play parallel role to that of the second fundamental form of an immersion. More exactly, O'Neill defined tensors $\mathcal{A}$ and $\mathcal{T}$ for vector fields E and F on M by

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.11}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.12}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [8], and ∇ is Riemannian connection on M. On the other hand, from equations (2.11) and (2.12), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.13}$$

$$\nabla_X U = \mathcal{H}\nabla_X U + \mathcal{T}_X U, \tag{2.14}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{2.15}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{2.16}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If U is basic, then $\mathcal{A}_X U = \mathcal{H}\nabla_X U$.

It is Simply seen that for $q \in M$, $X \in \mathcal{V}_q$ and $U \in \mathcal{H}_q$ the linear opretors

$$\mathcal{A}_U,\ \mathcal{T}_X : T_q M \to T_q M,$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_U E, F) = -g_M(E, \mathcal{A}_U F) \text{ and } g_M(\mathcal{T}_X E, F) = -g_M(E, \mathcal{T}_X F), \tag{2.17}$$

for each $E, F \in T_q M$. We have also defined the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{\mathcal{V}\times\mathcal{V}}$ is precisely the second fundamental form of the fibres of f. Since $\mathcal{T}_V$ is skew-symmetric we get: f has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$. For the special event when f is horizontally conformal we have the following proposition.

Proposition 2.2. ([11] (2.1.2)) Let f be horizontal conformal submersion between Riemannian manifolds (M, g_M) and (N, g_N) with dilation λ and U, V be horizontal vector fields, then

$$A_U V = \frac{1}{2}\left\{\mathcal{V}[U, V] - \lambda^2 g_M(U, V) grad_{\mathcal{V}}\left(\frac{1}{\lambda^2}\right)\right\}, \tag{2.18}$$

We know that the skew-symmetric part of $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

We also memorize the concept of harmonic maps between Riemannian manifolds. Let $f : (M, g_M) \to (N, g_N)$ is a smooth map between Riemannian manifolds. Then the differential f_* of f can be observed a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the bundle which has fibres $(f^{-1}TN)_x = T_{f(x)}N$ has a connection ∇ induced from the Riemannian connection ∇^M and the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(U, V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V), \tag{2.19}$$

for vector fields $U, V \in \Gamma(TM)$, where ∇^f is the pullback connection. We know that the second fundamental form is symmetric. A smooth map f between Riemannian manifolds is said to be harmonic if $trace(\nabla f_*) = 0$. On the extra need, the tension field of f is the section $\tau(f)$ of $\Gamma(f^{-1}TN)$ defined by

$$\tau(f) = div f_* = \sum_{i=1}^{m} (\nabla f_*)(e_i, e_i), \tag{2.20}$$

where $\{e_i, \ldots, e_m\}$ is orthonormal frame on M. Then it follows that f is harmonic if and only if $\tau(f) = 0$, for facts [2].

Lastly, we recollect the subsequent lemma from [2].

Lemma 2.3. Let (M, g_M) and (N, g_N) are two Riemannian manifolds. If $f : (M, g_M) \to (N, g_N)$ horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields U, V and vertical vector fields X, Y we have

$$\begin{aligned}
\text{(i)}\ \nabla df(U, V) &= U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U, V)df(gradln\lambda);\\
\text{(ii)}\ \nabla df(X, Y) &= -df(\mathcal{A}^{\mathcal{V}}_X Y);\\
\text{(iii)}\ \nabla df(U, X) &= -df(\nabla^M_U X) = df((\mathcal{A}^{\mathcal{H}})^*_U X).
\end{aligned}$$

where $(\mathcal{A}^{\mathcal{H}})^*_X$ is the adjoint of $(\mathcal{A}^{\mathcal{H}}_X)$ characterized by

$$\langle (\mathcal{A}^{\mathcal{H}})^*_U E, F\rangle = \langle E, \mathcal{A}^{\mathcal{H}}_U F\rangle, \qquad (\text{for} \quad E, F \in \Gamma(TM)).$$

3. Conformal anti-invariant submersions admitting vertical structure vector field

In this section, we define conformal anti-invariant submersions from an almost contact metric manifold onto Riemannian manifolds.

Definition 3.1. Let $(M, \phi, \xi, \eta, g_M)$ be a almost contact metric manifold and (N, g_N) be a Riemannian manifold, where $dimM = m$ and $dimN = n$. A horizontally conformal submersion $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is called a conformal anti-invariant submersion if the distribution $\ker f_*$ is anti-invariant with respect to ϕ i.e., $\phi(\ker f_*) \subseteq (\ker f_*)^\perp$. We have $\phi(\ker f_*)^\perp \cap \ker f_* \neq \{0\}$. We denote the complementary orthonormal distribution to $\phi(\ker f_*)$ in $(\ker f_*)^\perp$ by μ. Then we have

$$(\ker f_*)^\perp = \phi(\ker f_*) \oplus \mu.$$

It is clear that μ is an invariant distribution of $(\ker f_*)^\perp$ under the endomorphism ϕ.

Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If a map $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ horizontally conformal submersion admitting vertical structure vector field i.e., $(\xi \in \ker f_*)$.

Then we have

$$(\ker f_*)^\perp = \phi(\ker f_*) \oplus \mu. \tag{3.1}$$

It is clear that μ is an invariant distribution of $(\ker f_*)^\perp$, under the endomorphism ϕ.

Thus, for any $U \in \Gamma(\ker f_*)^\perp$, we have

$$\phi U = BU + CU, \tag{3.2}$$

where $BU \in \Gamma(\ker f_*)$ and $CU \in \Gamma(\mu)$. On the additional point, since $f_*(\Gamma(\ker f_*)^{\perp}) = TN$ and f is a conformal submersion, for every $X \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equation (3.2) we get $\frac{1}{\lambda^2} g_N(f_*\phi U, f_*CX) = 0$, which denotes that

$$TN = f_*(\phi(\ker f_*)) \oplus f_*(\mu). \tag{3.3}$$

For any vector field $X \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^{\perp}$, using equations (2.1), (2.2), (3.1) and (3.2), we get

$$C^2V = -V - \phi BV, BCV = 0, \eta(BV) = 0,$$

$$B\phi X = -X + \eta(X)\xi, C\phi X = 0, C^3V + CV = 0.$$

Lemma 3.2. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then we get

$$T_X\xi = -\phi X, A_V\xi = -CV,$$
$$g_M(CV, \phi X) = 0, \tag{3.4}$$

and

$$g_M(\nabla_U^M CV, \phi X) = -g_M(CV, \phi \mathcal{A}_U X) + \eta(X) g_M(U, CV), \tag{3.5}$$

for $X \in \Gamma(\ker f_*)$ and $U, V, \phi X \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, since $BV \in \Gamma(\ker f_*)$ and $\phi X \in (\Gamma(\ker f_*)^{\perp})$, using equations (3.2) and (2.4), we have

$$g_M(CV, \phi X) = 0.$$

Now, using equations (2.6), (2.14) and (3.4), we get

$$\begin{aligned} g_M(\nabla_U CV, \phi X) &= -g_M(CV, \nabla_U \phi X) \\ &= -g_M(CV, \phi \mathcal{A}_U X) + \eta(X) g_M(U, CV), \end{aligned}$$

since $\phi \mathcal{V} \nabla_U X \in \Gamma(\phi(\ker f_*))$. Therefore, we obtain the result. ∎

Theorem 3.3. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:

(i) $(\ker f_*)^{\perp}$ is integrable,

(ii) $\frac{1}{\lambda^2} g_N(\nabla_V^f f_*CU - \nabla_U^f f_*CV, f_*\phi X)$

$$\begin{aligned} = \; & g_M(A_U BV - A_V BU, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ & + g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) - 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V), \end{aligned}$$

for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi V \in (\Gamma \ker f_* \oplus \mu)$ and $\phi X \in (\Gamma(\ker f_*)^\perp)$. Using equations (2.1), (2.4), (2.6) and (3.2) , we get

$$\begin{aligned} g_M([U,V],X) &= g_M(\phi\nabla_U V, \phi X) + \eta(X)\eta(\nabla_U V) - g_M(\phi\nabla_V U, \phi X) - \eta(X)\eta(\nabla_V U), \\ &= g_M(\nabla_U \phi V, \phi X) - g_M(\nabla_V \phi U, \phi X) + g_M([U,V],\xi)\eta(X), \\ &= g_M(\nabla_U BV, \phi X) + g_M(\nabla_U CV, \phi X) - g_M(\nabla_V BU, \phi X), \\ &\quad - g_M(\nabla_V CU, \phi X) + g_M([U,V],\xi)\eta(X). \end{aligned}$$

Since f is a conformal submersion, using equations (2.14) and (2.15) we get

$$\begin{aligned} g_M([U,V],X) \;\; = \;\; & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \frac{1}{\lambda^2} g_N(f_*\nabla_U CV, f_*\phi X) \\ & -\frac{1}{\lambda^2} g_N(f_*\nabla_V CU, f_*\phi X) + g_M([U,V],\xi)\eta(X). \end{aligned}$$

Using equations (2.23), (3.4) and lemma $1(i)$, we get

$$\begin{aligned} g_M([U,V],X) = {} & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ & + g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) \\ & - 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V) \; - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU \\ & - \nabla^f_U f_* CV, f_*\phi X) + g_M([U,V],\xi)\eta(X). \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. ■

Theorem 3.4. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:

(i) $(\ker f_*)^\perp$ is integrable,

(ii) f is horizontally homothetic,

(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$,
for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using theorem (2), we get

$$\begin{aligned} g_M([U,V],X) = {} & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) \\ & + g_M(V, \phi X) g_M(\mathcal{H} gradln\lambda, CU) \\ & - 2 g_M(CU, V) g_M(\mathcal{H} gradln\lambda, \phi X) \\ & - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X) + g_M([U,V],\xi)\eta(X). \end{aligned}$$

Now, using conditions (i) and (ii), we get (iii)

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X).$$

Similarly, one can obtain the other assertions. ■

Remark 3.5. Let f be a conformal anti-invariant submersion is conformal Lagrangian submersion, if $\phi(\ker f_*) = (\ker f_*)^\perp$. Then (3.3), we have $TN = f_*(\phi(\ker f_*)^\perp)$.

Corollary 3.6. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the following assertions are equivalent to each other:

(i) $(\ker f_*)^\perp$ is integrable,

(ii) $\mathcal{A}_U \phi V = \mathcal{A}_V \phi U$,

(iii) $(\nabla f_*)(V, \phi U) = (\nabla f_*)(U, \phi V)$, for $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For any $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, from definition (2), $\phi X \in (\Gamma(\ker f_*)^\perp)$ and $\phi V \in \Gamma(\phi(\ker f_*))$. From theorem (2), we have

$$\begin{aligned} g_M([U, V], X) = {} & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ & + g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) \\ & - 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V) \\ & - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) + \eta(X) g_M([U, V], \xi). \end{aligned}$$

Since f conformal Lagrangian submersion, we have

$$g_M([U, V], X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \eta(X) g_M([U, V], \xi),$$

which implies $(i) \Leftrightarrow (ii)$. On the other hand using definition (2) and equation (2.15), we get

$$\begin{aligned} & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) \\ & = g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{A}_V BU, \phi X), \\ & = \frac{1}{\lambda^2} g_N(f_* \mathcal{A}_U BV, f_* \phi X) - \frac{1}{\lambda^2} g_N(f_* \mathcal{A}_V BX, f_* \phi X), \\ & = \frac{1}{\lambda^2} g_N(f_*(\nabla_U BV), f_* \phi X) - \frac{1}{\lambda^2} g_N(f_*(\nabla_V BU), f_* \phi X). \end{aligned}$$

Now, using equation (2.23) we have

$$\begin{aligned}
&\frac{1}{\lambda^2} g_N(f_*(\nabla_U BV), f_*\phi X) - \frac{1}{\lambda^2} g_N(f_*(\nabla_V BU), f_*\phi X) \\
&= \frac{1}{\lambda^2} g_N(-(\nabla f_*)(U, BV) + \nabla_U^f f_* BV, f_*\phi X) \\
&\quad - \frac{1}{\lambda^2} g_N(-(\nabla f_*)(V, BU) + \nabla_V^f f_* BU, f_*\phi X), \\
&= \frac{1}{\lambda^2} [g_N((\nabla f_*)(V, BU) - (\nabla f_*)(U, BV), f_*\phi X)],
\end{aligned}$$

which proves that $(ii) \Leftrightarrow (iii)$. ■

Theorem 3.7. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,

(ii) $$\frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_*\phi X) = -g_M(\mathcal{A}_U BV, \phi X) + g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV)$$
$$- g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X),$$

for any $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For any $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using equations (2.4), (2.6), (2.15), (2.16) and (3.2), we have

$$\begin{aligned}
g_M(\nabla_U V, X) &= g_M(\nabla_U \phi V, \phi X) + \eta(X)\eta(\nabla_U V), \\
&= g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X) + \eta(X)\eta(\nabla_U V).
\end{aligned}$$

Since f is conformal submersion, using equation (2.23), lemma $1(i)$, definition (2) and equation (3.4), we get

$$\begin{aligned}
g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) &- g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\
&+ \eta(X)\eta(\nabla_U V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) \\
&+ \frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_*\phi X),
\end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. ■

Theorem 3.8. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,

(ii) f is horizontally homothetic,

(iii) $g_M(\mathcal{A}_U BV, \phi X) = -\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X)$, for any $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using theorem (4), we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ &\quad + \eta(X)\eta(\nabla_U V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X). \end{aligned}$$

Using conditions (i) and (ii), we get (iii)

$$g_M(\mathcal{A}_U BV, \phi X) = -\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X).$$

Similarly, one can obtain the other assertions. ■

Corollary 3.9. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the followings are equivalent to each other:

(i) $(\ker f_*)^{\perp}$ defines a totally geodesic foliation on M,

(ii) $\mathcal{A}_U \phi V = 0$,

(iii) $(\nabla f_*)(U, \phi V) = 0$, for $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, from definition (2), $\phi V \in \Gamma(\phi(\ker f_*))$ and $\phi X \in \Gamma((\ker f_*)^{\perp})$. Using theorem (4), we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ &\quad - \eta(X)\eta(\nabla_U V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) + \eta(X)\eta(\nabla_U V), \\ &= g_M(\mathcal{A}_U \phi V, \phi X) + \eta(X)\eta(\nabla_U V), \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$.

Further, using equation (2.15), we get

$$g_M(\mathcal{A}_U BV, \phi X) = g_M(\nabla_U BV, \phi X).$$

Since f is conformal submersion, we get

$$g_M(\mathcal{A}_U BV, \phi X) = \frac{1}{\lambda^2} g_N(f_* \nabla_U BV, f_* \phi X).$$

Using equation (2.23), we get

$$\begin{aligned} g_M(\mathcal{A}_U BV, \phi X) &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, BV), f_* \phi X), \\ &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, \phi V), f_* \phi X). \end{aligned}$$

which shows $(ii) \Leftrightarrow (iii)$. ■

Theorem 3.10. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) $-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU)$
$= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) + \eta(Y) g_M(X, \phi U)$,

for any $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For any $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using equations (2.4), (2.6), (2.14) and (3.2), we get

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\phi \nabla_X Y, \phi U) + \eta(\nabla_X Y)\eta(U) \\ &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H} \nabla_X \phi Y, CU) + \eta(Y) g_M(X, \phi U). \end{aligned}$$

Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\nabla_{\phi Y} X, CU) + \eta(Y) g_M(X, \phi U),$$

using equations (2.4) and (2.6), we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\nabla_{\phi Y} \phi X, \phi CU) + \eta(Y) g_M(X, \phi U),$$

here we have used μ is invariant. Since f is conformal submersion, using equation (2.23) and Lemma $1(i)$, we get

$$\begin{aligned} g_M(\nabla_X Y, U) = {} & g_M(\mathcal{T}_X \phi Y, BU) - \frac{1}{\lambda} g_M(\mathcal{H} gradln\lambda, \phi Y) g_N(f_* \phi X, f_* \phi CU) \\ & - \frac{1}{\lambda} g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* \phi Y, f_* \phi CU) \\ & + \frac{1}{\lambda} g_M(\phi Y, \phi X) g_N(f_* \mathcal{H} gradln\lambda, f_* \phi CU) \\ & + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) + \eta(Y) g_M(X, \phi U). \end{aligned}$$

Next, using definition (2) and (3.4), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) + \eta(Y) g_M(X, \phi U), \end{aligned}$$

which shows $(i) \Leftrightarrow (ii)$. ∎

Theorem 3.11. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) λ is constant on $\Gamma(\mu)$,

(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU)$

$= -g_M(\mathcal{T}_X \phi Y, \phi U) + \eta(Y) g_M(X, \phi U)$,

for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, from theorem (6), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) + \eta(Y) g_M(X, \phi U). \end{aligned}$$

Now, using conditions (i) and (iii), we have

$$g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) = 0.$$

From above equation λ is constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. ∎

Corollary 3.12. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the following statements are equivalent to each other:

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) $T_X \phi Y = -\eta(Y) X$, or $T_X \phi Y = -\eta(Y) X$ is parallel to ξ for $X, Y \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$, from theorem (6), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) + \eta(Y) g_M(X, \phi U). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, \phi U) + \eta(Y) g_M(X, \phi U),$$

which proves $(i) \Leftrightarrow (ii)$. ∎

4. Harmonicity of conformal anti-invariant submersion admitting vertical structure vector field

In this section, we investigate the necessary and sufficient conditions for a conformal anti-invariant submersions to be harmonic. We also find the necessary and sufficient conditions for such submersions to be totally geodesic.

Theorem 4.1. Let $(M^{2k+2r+1}, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N^{k+2r}, g_N) be a Riemannian manifold. If $(M^{2k+2r+1}, \phi, \xi, \eta, g_M) \rightarrow (N^{k+2r}, g_N)$ be a conformal anti-invariant submersion, then the tension field τ of f is

$$\tau(f) = -kf_*(\mu^{\ker f_*}) + (2-k-2r)f_*(\mathcal{H}gradln\lambda), \tag{4.1}$$

where $\mu^{\ker f_*}$ is the mean curvature vector field of the distribution of $\ker f_*$.

Proof. Let $\{e_1, e_2, \ldots, e_k, \xi, \phi e_1, \ldots, \phi e_k, \mu_1, \ldots, \mu_r, \phi\mu_1, \ldots, \phi\mu_r\}$ be an orthonormal basis of $\Gamma(TM)$ such that $\{e_1, e_2, \ldots, e_k, \xi\}$ is orthonormal basis of $\Gamma(\ker f_*)$, $\{\phi e_1, \ldots, \phi e_k\}$ is orthonormal basis of $\Gamma(\phi \ker f_*)$ and $\{\mu_1, \ldots, \mu_r, \mu_{r+1}, \ldots, \mu_{2r}\}$ is orthonormal basis of $\Gamma(\mu)$.

Then the trace of fundamental form (restriction of $\ker f_* \times \ker f_*$) is given by

$$trace^{(\ker f_*)^\perp}(\nabla f_*) = \sum_{i=1}^{k}(\nabla f_*)(\phi e_i, \phi e_i) + \sum_{j=1}^{2r}(\nabla f_*)(\mu_j, \mu_j).$$

Using lemma $1(i)$, we obtain

$$\begin{aligned} trace^{(\ker f_*)^\perp}(\nabla f_*) &= \sum_{i=1}^{k} 2g_M(gradln\lambda, \phi e_i)f_*(\phi e_i) - kf_*(gradln\lambda) \\ &\quad + \sum_{i=1}^{2r} 2g_M(gradln\lambda, \mu_j)f_*(\mu_j) - 2rf_*(gradln\lambda). \end{aligned}$$

Since f is conformal anti-invariant submersion, for $x \in M$, and $1 \leq i \leq k$, $1 \leq h \leq r$ $\{\frac{1}{\lambda(x)}f_{*x}(\phi e_i), \frac{1}{\lambda(x)}f_{*x}(\mu_h)\}$ is an orthonormal basis of $T_{f(x)}N$; thus we obtain

$$\begin{aligned} trace^{(\ker f_*)^\perp}(\nabla f_*) &= \sum_{i=1}^{k} 2g_N(f_*(gradln\lambda), \frac{1}{\lambda}f_*(\phi e_i))\frac{1}{\lambda}f_*(\phi e_i) - kf_*(gradln\lambda) \\ &\quad + \sum_{i=1}^{2r} 2g_N(f_*(gradln\lambda), \frac{1}{\lambda}f_*(\mu_j))\frac{1}{\lambda}f_*(\mu_j) - 2rf_*(gradln\lambda) \end{aligned}$$

$$trace^{(\ker f_*)^\perp}(\nabla f_*) = (2-k-2r)f_*(\mathcal{H}gradln\lambda). \tag{4.2}$$

In a similarly, we get

$$trace^{(\ker f_*)}(\nabla f_*) = \sum_{i=1}^{k}(\nabla f_*)(e_i, e_i) + (\nabla f_*)(\xi, \xi).$$

Using equation (2.22) and (2.13), we find

$$trace^{(\ker f_*)}(\nabla f_*) = -kf_*(\mu^{\ker f_*}). \tag{4.3}$$

From equations (4.2) and (4.3), we get

$$\tau(f) = -kf_*(\mu^{\ker f_*}) + (2 - k - 2r)f_*(\mathcal{H}gradln\lambda).$$

Therefore, we obtain the result. ■

Theorem 4.2. Let $(M^{2k+2r+1}, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N^{k+2r}, g_N) be a Riemannian manifold. If $(M^{2k+2r+1}, \phi, \xi, \eta, g_M) \to (N^{k+2r}, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:

(i) f is harmonic,

(ii) The fibres are minimal,

(iii) f is a horizontally homothetic map.

Proof. Taking equation (4.1), we have

$$\tau(f) = -kf_*(\mu^{\ker f_*}) + (2 - k - 2r)f_*(\mathcal{H}gradln\lambda).$$

Now, using conditions (i) and (ii), then f is a horizontally homothetic map. ■

Corollary 4.3. Let $(M^{2k+2r+1}, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N^{k+2r}, g_N) be a Riemannian manifold. Let $(M^{2k+2r+1}, \phi, \xi, \eta, g_M) \to (N^{k+2r}, g_N)$ be a conformal anti-invariant submersion. If $k + 2r = 2$, then f is harmonic if and only if the fibres are minimal.

Next, we find necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We memorize that a differentiable map f between two Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(V, W) = 0, \text{ for all } V, W \in \Gamma(TM).$$

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 4.4. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is a totally geodesic map if and only if

$$\begin{aligned} -\nabla^f_V f_* W &= f_*(\phi \mathcal{A}_V \phi W_1 + \phi \mathcal{V} \nabla_V BW_2 + \phi \mathcal{A}_V CW_2 + C\mathcal{H}\nabla_V \phi W_1 \\ &\quad + C\mathcal{A}_V BW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_1)CV), \end{aligned}$$

for any $V, W \in \Gamma(TM)$, where $W = W_1 + W_2$, $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Proof. Taking equation (2.22) and using equations (2.1) and (2.6), we get

$$(\nabla f_*)(V, W) = \nabla^f_V f_* W + f_*(\phi \nabla_V \phi W + \eta(W)\phi V - \eta(\nabla_V W)\xi),$$

for any $V, W \in \Gamma(TM)$.

Now, using equations (2.15) and (3.2), we get

$$\begin{aligned} (\nabla f_*)(V, W) &= \nabla^f_V f_* W + f_*(\phi \mathcal{A}_V \phi W_1 + B\mathcal{H}\nabla_V \phi W_1 + C\mathcal{H}\nabla_V \phi W_1 \\ &\quad + B\mathcal{A}_V BW_2 + C\mathcal{A}_V BW_2 + \phi \mathcal{V}\nabla_V BW_2 + \phi \mathcal{A}_V CW_2 \\ &\quad + B\mathcal{H}\nabla_V CW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_1)BV \\ &\quad + \eta(W_1)CV - \eta(\nabla_V W)\xi), \end{aligned}$$

for $W = W_1 + W_2 \in \Gamma(TM)$, where $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Thus taking into account the vertical terms, we get

$$\begin{aligned} (\nabla f_*)(V, W) &= \nabla^f_V f_* W + f_*(\phi(\mathcal{A}_V \phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) \\ &\quad + C(\mathcal{H}\nabla_V \phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) + \eta(W_1)CV). \end{aligned}$$

Thus

$$\begin{aligned} (\nabla f_*)(V, W) &= 0 \Leftrightarrow \\ -\nabla^f_V f_* W &= f_*(\phi(\mathcal{A}_V \phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) \\ &\quad + C(\mathcal{H}\nabla_V \phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) + \eta(W_1)CV). \end{aligned}$$

Therefore, we obtain the result. ■

Definition 4.5. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi \ker f_*, \mu)-$totally geodesic map provided

$$(\nabla f_*)(\phi X, U) = 0, \text{ for } X \in \Gamma(\ker f_*) \text{ and } U \in (\Gamma(\ker f_*)^{\perp}).$$

Theorem 4.6. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi \ker f_*, \mu)-$totally geodesic map if and if f is horizontally homothetic map.

Proof. For $X \in \Gamma(\ker f_*)$ and $U \in \Gamma(\mu)$, using lemma $1(i)$, we get

$$(\nabla f_*)(\phi X, U) = \phi X(ln\lambda) f_*(U) + U(ln\lambda) f_*(\phi X) - g_M(\phi X, U) f_*(gradln\lambda).$$

From above equation, if f is a horizontally homothetic, then $(\nabla f_*)(\phi X, U) = 0$. Conversely, if $(\nabla f_*)(\phi X, U) = 0$,we find

$$\phi X(ln\lambda) f_*(U) + U(ln\lambda) f_*(\phi X) = 0. \tag{4.4}$$

Taking inner product in above equation with $f_*(\phi X)$ and since f is conformal submersion, we have

$$g_M(\mathcal{H}gradln\lambda, \phi X) g_N(f_* U, f_*\phi X) + g_M(\mathcal{H}gradln\lambda, U) g_N(f_*\phi X, f_*\phi X) = 0.$$

Above equation shows that λ is a constant $\Gamma(\mu)$.

On the other hand taking inner product in equation (4.4) with $f_* X$, we get

$$g_M(\mathcal{H}gradln\lambda, \phi X) g_N(f_* U, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, U) g_N(f_*\phi X, f_* U) = 0.$$

From above equation shows that λ is a constant on $\Gamma(\phi(\ker f_*))$. Thus λ is a constant on $\Gamma((\ker f_*)^{\perp})$. ■

Theorem 4.7. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion. Then f is totally geodesic map if and only if

(i) $\mathcal{T}_X \phi Y = -\eta(Y) X$ and $\mathcal{H}\nabla_Y \phi X \in \Gamma(\phi \ker f_*)$,

(ii) f is horizontally homothetic map,

(iii) $\widehat{\nabla}_X BU = -\mathcal{T}_X CU$ or parallel to ξ and $\mathcal{T}_X BU + \mathcal{H}\nabla_X CU \in \Gamma(\phi \ker f_*)$, for all $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\mu)$.

Proof. For $X, Y \in \Gamma(\ker f_*) - \{\xi\}$, using equations (2.1), (2.6) and (2.22), we get

$$(\nabla f_*)(X, Y) = f_*(\phi(\nabla_X \phi Y) + \eta(Y)\phi X - \eta(\nabla_X Y)\xi).$$

Now, using equations (2.14) and (3.2), we get

$$(\nabla f_*)(X, Y) = f_*(\phi \mathcal{T}_X \phi Y + C\mathcal{H}\nabla_X \phi Y + \eta(Y)\phi X).$$

Thus shows that $\mathcal{T}_X \phi Y + \eta(Y) X = 0$ and $\mathcal{H}\nabla_X \phi Y \in \Gamma(\phi \ker f_*)$.

On the other hund using lemma $1(i)$, we get

$$(\nabla f_*)(U, V) = U(ln\lambda) f_*(V) + V(ln\lambda) f_*(U) - g_M(U, V) f_*(gradln\lambda),$$

for $U, V \in \Gamma(\mu)$. It is obvious that if f is horizontally homothetic, it follows that $(\nabla f_*)(U, V) = 0$. Conversely, if $(\nabla f_*)(U, V) = 0$, taking $V = \phi U$ in above equation, we have

$$U(ln\lambda) f_*(\phi U) + \phi U(ln\lambda) f_*(U) = 0. \tag{4.5}$$

Taking inner product in (4.5) with $f_*\phi U$, we get

$$g_M(\mathcal{H}gradln\lambda, U)g_N(f_*\phi U, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*U, f_*\phi U) = 0.$$

From above equation λ is constant on $\Gamma(\mu)$. On other hand, for $X, Y \in \Gamma(\ker f_*)$, from lemma $1(i)$, we get

$$(\nabla f_*)(\phi X, \phi Y) = \phi X(ln\lambda)f_*(\phi Y) + \phi Y(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi Y)f_*(gradln\lambda),$$

Again if f is horizontally homothetic, then $(\nabla f_*)(\phi X, \phi Y) = 0$. Conversely, if $(\nabla f_*)(\phi X, \phi Y) = 0$, putting $X = Y$ in above equation, we get

$$2\phi X(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi X)f_*(gradln\lambda) = 0.$$

Taking inner product in above equation with $f_*\phi X$ and since f is conformal submersion, we have

$$g_M(\phi X, \phi X)g_M(gradln\lambda, \phi X) = 0.$$

From above equation, λ is constant on $\Gamma(\phi \ker f_*)$. Thus λ is constant on $\Gamma((\ker f_*)^{\perp})$.

Now, for $X \in \Gamma(\ker f_*)$ and $U \in \Gamma((\ker f_*)^{\perp})$, using equations (2.1), (2.6) and (2.23) we get

$$(\nabla f_*)(X, U) = f_*(\phi(\nabla_X \phi U) - \eta(\nabla_X U)\xi).$$

Now, again using (2.14) and (3.2), we get

$$(\nabla f_*)(X, U) = f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU).$$

Thus $(\nabla f_*)(X, U) = 0 \Leftrightarrow f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU) = 0.$ Therefore, we obtain the result. ■

5. Decomposition Theorems for a conformal anti-invariant submersion admitting vertical structure vector field

In this section, we obtain decomposition theorems by using the existence of conformal anti-invariant submersions. Initial, we memorise the following results from [20]. Let g_B be a Riemannian metric tensor on the manifold $B = M \times N$ and assume that the canonical foliations D_M and D_N intersect perpendicularly everywhere. Then g_B is the metric tensor of

(i) a twisted product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is totally umbilical foliation,

(ii) a warped product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel.

We note in this case, from [3] we have

$$\nabla_X U = X(\ln F)U,$$

for $X \in \Gamma(TM)$ and $U \in \Gamma(TN)$, where ∇ is the Riemannian connection on $M \times N$,

(iii) a usual product of Riemannian manifolds if and only if D_M and D_N are totally geodesic foliations.

Next, we found a decomposition theorem related to the concept of twisted product manifold. However, we first memorise the adjoint map of a map. Let $f : (M, g_M) \to (N, g_N)$ be a map between Riemannian manifolds (M, g_M) and (N, g_N). Then the adjoint map ${}^*f_*$ of f_* is characterized $g_M(X, f_{*p}Y) = g_N({}^*f_{*p}X, Y)$ by $X \in T_pM$, $Y \in T_{f(p)}N$ and $p \in M$. Considering f_*^h at each $p \in M$ as a linear transformation

$$f_{*p}^h : ((\ker f_*)_{(p)}^{\perp}, g_{M(p)((\ker f_*)_p^{\perp})}) \to (rangef_{*(q)}, g_{N(q)(rangef_*)(q)}),$$

we will denote the adjoint $f_{*(p)}^h$ by ${}^*f_{*(p)}^h$. Let $f_{*(p)}^h$ be the adjoint of $f_{*(p)}^h : (T_pM, g_{M(p)}) \to (T_{(q)}N, g_{N(q)})$. The linear transformation $({}^*f_{*p})^h : (rangef_{*(p)}) \to (\ker f_*)_{(p)}^{\perp}$ defined $({}^*f_{*(p)})^h Y ={}^* f_{*(p)}^h Y$, where $Y \in (ranrgef_{*(p)})$, $q = f(p)$, is an isomorphism and $(f_{*(p)}^h)^{-1} = ({}^*f_{*p})^h ={}^* f_{*(p)}^h$.

Our first decomposition theorem for a conformal anti-invariant submersion comes from theorem (4) and theorem (6) in terms of the second fundamental forms of such submersions.

Theorem 5.1. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion. Then f is a locally product manifold if and only if

$$\begin{aligned} &-\frac{1}{\lambda^2} g_N(\nabla_U^f f_*CV, f_*\phi X) \\ &\quad = g_M(\mathcal{A}_U BV, \phi X) - g_M(gradln\lambda, CV) g_M(U, \phi X) \\ &\qquad + g_M(gradln\lambda, \phi X) g_M(U, CV), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &-\frac{1}{\lambda^2} g_N(\nabla_{\phi Y}^f f_*\phi X, f_*\phi CU) \\ &\quad = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\qquad + \eta(Y) g_M(X, BU), \end{aligned} \tag{5.2}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V, W \in (\Gamma(\ker f_*)^{\perp})$.

Again, from Corollary (2) and Corollary (3), we have the following theorem.

Theorem 5.2. Let $(M, \phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is a locally product manifold if and only if $\mathcal{A}_U \phi V = 0$ and $\mathcal{T}_X \phi Y = -\eta(Y)X$, for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Theorem 5.3. Let f be a conformal anti-invariant submersion from a Sasakian manifolds $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then M is a locally twisted product manifold of the form $M_{(\ker f_*)} \times M_{(\ker f_*)^{\perp}}$ if and only if

$$\begin{aligned} &-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi C U) \\ &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi C U) \\ &\quad + \eta(Y) g_M(X, \phi U), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} g_M(U, V)H &= -B\mathcal{A}_U BV + CV(ln\lambda)BU \\ &\quad -B(\mathcal{H} gradln\lambda) g_M(U, CV) \\ &\quad -\phi^* f_*(\nabla^f_U f_* CV), \end{aligned} \tag{5.4}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, where $M_{(\ker f_*)}$ and $M_{(\ker f_*)^{\perp}}$ are integral manifolds of the distributions $(\ker f_*)^{\perp}$ and $(\ker f_*)$ and H is the mean curvature vector field of $M_{(\ker f_*)^{\perp}}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.4), (2.6), (2.14) and (3.2), we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_X \phi Y, CU) + \eta(Y) g_M(X, \phi U).$$

Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_{\phi Y} X, CU) + \eta(Y) g_M(X, \phi U).$$

Using equations (2.4), (2.6) and (2.16), we have

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\nabla_{\phi Y} \phi X, \phi C U) + \eta(Y) g_M(X, \phi U).$$

Since f is conformal submersion, using equation (2.23) and lemma 1(i), we find

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, \phi Y) g_N(f_* \phi X, f_* \phi C U) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* \phi Y, f_* \phi C U) \\ &\quad + \frac{1}{\lambda^2} g_M(\phi X, \phi Y) g_N(f_* \mathcal{H} gradln\lambda, f_* \phi C U) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi C U) + \eta(Y) g_M(U, \phi X). \end{aligned}$$

Next, using definition (2) and equation (3.2), we obtain

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V \phi W, BX) + g_M(\phi V, \phi W) g_M(\mathcal{H} gradln\lambda, \phi CX) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi W} f_* \phi V, f_* \phi CX) + \eta(W) g_M(X, \phi V). \end{aligned}$$

Thus shows that $M_{(\ker f_*)}$ is totally geodesic if and only if

$$\begin{aligned} & -\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) \\ = \; & g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi X, \phi Y) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ & +\eta(Y) g_M(U, \phi X). \end{aligned}$$

On the other hand for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.4), (2, 4), (2.15), (2.16) and (3.2), we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H} \nabla_U CV, \phi X).$$

Since f is conformal submersion, using equation (2.23) and lemma $1(i)$, we obtain that

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{T}_U BV, \phi X) - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, U) g_N(f_* CV, f_* \phi X) \\ &\quad -\frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, CV) g_N(f_* U, f_* \phi X) \\ &\quad +\frac{1}{\lambda^2} g_M(U, CV) g_N(f_* \mathcal{H} gradln\lambda, f_* \phi X) \\ &\quad +\frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_* \phi X) + \eta(X)\eta(\nabla_U V). \end{aligned}$$

Moreover, using definition (2) and equation (3.4), we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{T}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) + \eta(X)\eta(\nabla_U V) \\ &\quad + g_M(U, CV) g_N(\mathcal{H} gradln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_* \phi X). \end{aligned}$$

Then, we have

$$\begin{aligned} g_M(U, V) H &= -B\mathcal{A}_U BV + CV(ln\lambda) BU - B(\mathcal{H} gradln\lambda) g_M(U, CV) \\ &\quad -\phi f_*(\nabla^f_U f_* CV) + \eta(\mathcal{A}_U V)\xi, \end{aligned}$$

which proves. ■

6. Example

Note that given an Euclidean space $(x_1, \ldots, x_{2m}, x_{2m+1})$ with coordinates we can canonically choose an almost contact structure ϕ on R^{2m+1} as follows:

$$\begin{aligned} &\phi(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots + a_{2m-1}\frac{\partial}{\partial x_{2m-1}} + a_{2m}\frac{\partial}{\partial x_{2m}} + a_{2m+1}\frac{\partial}{\partial x_{2m+1}}) \\ = &(-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \cdots - a_{2m}\frac{\partial}{\partial x_{2m-1}} + a_{2m-1}\frac{\partial}{\partial x_{2m}}) \end{aligned}$$

where $\xi = \frac{\partial}{\partial x_{2m+1}}$ and $a_1, a_2, \ldots, a_{2m}, a_{2m+1}$ are $C^\infty-$real valued functions in R. Let $\eta = dx_{2m+1}$ and $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{2m}}, \frac{\partial}{\partial x_{2m+1}})$ is orthogonal basis of vector fields on R^{2m+1}.

Example 6.1. Define a map $f : R^5 \to R^2$ by

$$f(x_1, \ldots, x_5) = (e^{x_1}\sin x_2, e^{x_1}\cos x_2)$$

Then we have

$$\ker f_* =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_1}$.

Example 6.2. Define a map $f : R^5 \to R^2$ by

$$f(x_1, \ldots, x_5) = (e^{x_3}\cos x_4, e^{x_3}\sin x_4)$$

Then we have

$$\ker f_* =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_3}$.

References

[1] M. A. Akyol and B. Sahin, Conformal anti-invariant submersions from almost Hermitian manifolds. Turk J. Math., 40 (2016), 43–70.

[2] P. Baird and J. C. Wood, Harmonic Morphisms Between Riemannian Manifolds. London Mathematical Society Monographs, 29, Oxford University Press, The Clarendon Press. Oxford, (2003).

[3] RL. Bishop and B. O'Neill, Manifolds of negative curvature, Trans Amer Math Soc. 145 (1969), 1–49.

[4] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhiiuser, Boston, (2002).

[5] J. P. Bourguignon and H. B. Lawson, A Mathematician's visit to Kaluza-Klein theory, Rend. Semin. Mat. Torino Fasc. Spec. (1989), 143–163.

[6] B. Y. Chen, Geometry of slant submanifolds, Katholieke Universiteit Leuven, Leuven, (1990).

[7] D. Chinea, Almost contact metric submersions, Rend. Circ. Mat. Palermo, 34(1) (1985), 89–104.

[8] M. Falcitelli, S. Ianus and A. M. Pastore, Riemannian submersions and related topics, World Scientific Publishing Co., (2004).

[9] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier (Grenoble) 28 (1978), 107–144.

[10] S. Gudmundsson, The Geometry of Harmonic Morphisms, PhD Thesis, University of Leeds, Leeds, UK, (1992).

[11] D. Gromoll, W. Klingenberg and W. Meyer, Riemannsche Geometrie im Groβen, Lecture Notes in Mathematics 55, Springer (1975).

[12] S. Ianus, A. M. Ionescu, R. Mazzocco and G. E. Vilcu, Riemannian submersions from almost contact metric manifolds, arXiv: 1102.1570v1 [math. DG].

[13] S. Ianus and M. Visinescu, Kaluza-Klein theory with scalar fields and generalized Hopf manifolds, Class. Quantum Gravity 4 (1987), 1317–1325.

[14] S. Ianus and M. Visinescu, Space-time compactification and Riemannian submersions, In: Rassias, G.(ed.), The Mathematical Heritage of C. F. Gauss, (1991), 358–371, World Scientific, River Edge.

[15] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto. Univ.19 (1979), 215–229.

[16] J. C. Marrero and J. Rocha, Locally conformal Kähler submersions, Geom. Dedicata, 52(3), (1994), 271–289.

[17] M. T. Mustafa, Applications of harmonic morphisms to gravity, J. Math. Phys., 41(10) (2000), 6918–6929.

[18] C. Murathan and I. K. Erken, Anti-invariant Riemannian submersions from Cosymplectic manifolds, Filomat 29(7) (2015), 1429–1444.

[19] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J., 13, (1966), 458–469.

[20] R. Ponge and H. Reckziegel, Twisted products in pseudo-Riemannian geometry, Geom Dedicata 48 (1993), 15–25.

[21] B. Sahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, Cent. Eur. J. Math. 8(3), (2010) 437–447.

[22] B. Sahin, Slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome 54(102) No 1, (2011), 93–105.

[23] B. Sahin, Semi-invariant submersions from almost Hermitian manifolds, Canad. Math. Bull., 56(1), (2013), 173–183.

[24] B. Watson, G, G$^{|}$-Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., 57 (1983), 324–349, Teubner, Leipzig.

[25] B. Watson, Almost Hermitian submersions, J. Differential Geometry, 11(1), (1976), 147–165.