Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 3561–3575

http://www.ripublication.com/gjpam.htm

On (p, q)-Frames In Banach Spaces

Khole Timothy Poumai

Department of Mathematics,
Motilal Nehru College,
University of Delhi, Delhi-110021, India.

Ghansyam Singh Rathore

Department of Mathematics and statistics,
Faculty of Science,
M.L.S University of Udaipur Rajasthan, INDIA.

Sunayana Bhati

Department of Mathematics and statistics,
Faculty of Science,
M.L.S University of Udaipur Rajasthan, INDIA.

Abstract

The notion of (p, q)-frame is defined. A necessary and sufficient condition for the existence q-frames is obtained. Also, we discuss conditions for q-frames to be p-Riesz bases. Characterizations of (p, q)-frames have been given and dual properties of (p, q)-frames in Banach spaces have been proved. Further, we show that (p, q)-frames are compression of p-Riesz bases. Finally, we prove that the (p, q)-frame coefficients $\{f_n(x)\}$ have minimal ℓ_p-norm among all sequences representing x in Banach spaces with a given condition.

AMS subject classification: 42C15; 42C30; 42C05; 46B15.
Keywords: p-Frames, p-Riesz Basis, q-Frames.

1. Introduction

Frames were introduced in 1952 by Duffin and Schaeffer [4]. They infact abstracted Gabor's [7] method to define frames for Hilbert spaces. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle ., . \rangle$.

Definition 1.1.

(a) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ if $\{x_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A \sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2 \leq B \sum_{n=1}^{\infty} |a_n|^2, \text{ for each finite sequence } \{a_n\}_{n=1}^{\infty}$$

(b) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. If $A = B$, then $\{x_n\}$ is called an *A-tight frame* and if $A = B = 1$, then $\{x_n\}$ is called a *Parseval frame*. The inequality in (1.1) is called the *frame inequality* of the frame. For more details related to frames and Riesz bases, one may refer to [2, 9]. Feichtinger and Gröcheing [6] extended the notion of frames to Banach space and defined the notion of atomic decomposition. Gröcheing [8] introduced a more general concept for Banach spaces called Banach frame. In [12], P. A. Terekhin defined the notion of frames in Banach spaces which are different from that of atomic decomposition and Banach frames. Aldroubi et al. [1] introduced the notion of p-frames in Banach spaces. Further, Christensen and Stoeva [3] studied *p-frames* and *p-Riesz basis* in Banach spaces. Y. C. Zhu [13] introduced the notion of q-frames in Banach spaces.

Definition 1.2. [1, 3] Let E be a Banach space. A countable family $\{f_n\}_{n=1}^{\infty} \subset E^*$ is a *p-frame* for E $(1 < p < \infty)$ if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \left(\sum_{n=1}^{\infty} |f_n(x)|^p \right)^{1/p} \leq B\|x\|, \text{ for all } x \in E. \tag{1.2}$$

$\{f_i\}$ is a *p-Bessel sequence* if at least the upper *p-frame* condition is satisfied.

Definition 1.3. [3, 13] Let E be a Banach space. A countable family $\{x_n\}_{n=1}^{\infty} \subset E$ is a *p-Riesz basis* for E $(1 < p < \infty)$ if $\overline{span}\{x_n\} = E$ and there exist constants $A, B > 0$ such that for $\{d_n\} \in \ell_p$,

$$A \left(\sum_{n=1}^{\infty} |d_n|^p \right)^{1/p} \leq \| \sum d_n x_n \|_E \leq B \left(\sum_{n=1}^{\infty} |d_n|^p \right)^{1/p} \tag{1.3}$$

Yu Can Zhu defined *q-frames*, *q-Besselian frames* and studied in Banach spaces.

Definition 1.4. [13] Let E be a Banach space. A sequence $\{x_n\}_{n=1}^{\infty} \subset E$ is called a *q-frame* for $E(1 < q < \infty)$, if there are two positive constants A and B such that

$$A\|f\| \leq \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{1/q} \leq B\|f\|, \text{ for all } f \in E^*. \tag{1.4}$$

$\{x_n\}$ is a *q-Bessel sequence* if at least the upper *q-frame* condition is satisfied. A q-frame $\{x_n\} \subset E$ is called a *q-Besselian frame* for E, if whenever $\sum_{n=1}^{\infty} a_n x_n$ converges, then $\{a_n\} \in l_p$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.5. [11] Let X, Y be Banach spaces and $S \in B(X, Y)$. Then the following are equivalent.

(a) S has a pseudoinverse operator $S^\dagger$. i.e. $S^\dagger$: $SS^\dagger S = S$.

(b) There exist closed subspaces W, Z of X, Y respectively, such that

$$X = kerS \oplus W, \ Y = S(X) \oplus Z.$$

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E. Further, $\{e_n\}_{n=1}^{\infty}$ is a sequence of canonical unit vectors which is a Schauder basis of sequence spaces ℓ_p and ℓ_q. $\pi : E \longrightarrow E^{**}$ is the natural canonical projection from E onto E^{**}.

2. Main Results

We begin this section by defining (p, q)-frames in Banach spaces.

Definition 2.1. Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ and $1/p + 1/q = 1$ with $1 < p, q < \infty$. A pair (x_n, f_n) is called (p, q)-frame if

1. $\{x_n\}$ is q-Bessel sequence for E and $\{f_n\}$ is p-Bessel sequence for E.

2. $x = \sum_{n=1}^{\infty} f_n(x) x_n$, for all $x \in E$.

Remark 2.2. Let (x_n, f_n) is called a (p, q)-frame for E. Then, $\{f_n\}$ is a p-frame for E and $\{x_n\}$ is a q-frame for E. Let D be an upper bound of p-Bessel sequence $\{f_n\}$ and B

be an upper bound of q-Bessel sequence $\{x_n\}$. Now, let $f \in E^*$ and consider

$$\begin{aligned}
\|f\| &= \sup_{x\in E,\|x\|=1} |f(x)| = \sup_{x\in E,\|x\|=1} \left|\sum_{n=1}^{\infty} f_n(x) f(x_n)\right| \\
&\le \sup_{x\in E,\|x\|=1} \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{1/q} \\
&\le D\left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{1/q}
\end{aligned}$$

So, $\{x_n\}$ is q-frame for E. Also, let us consider

$$\begin{aligned}
\|x\| &= \sup_{f\in E^*,\|f\|=1} |f(x)| = \sup_{f\in E^*,\|f\|=1} \left|\sum_{n=1}^{\infty} f_n(x) f(x_n)\right| \\
&\le \sup_{f\in E^*,\|f\|=1} \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{1/q} \\
&\le B\left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{1/p}
\end{aligned}$$

In the following result, we give a necessary and sufficient condition for the existence of q-frames in Banach space E.

Theorem 2.3. A sequence $\{x_n\}$ in E is a q-frame for E if and only if there exists a bounded linear operator $T : \ell_p \to E$ from ℓ_p onto E such that $T(e_n) = x_n$, for $n \in \mathbb{N}$ and $1/p + 1/q = 1$.

Proof. Let A and B be lower and upper bounds of the q-frame $\{x_n\}$ respectively and $\{c_n\} \in l_p$. Also, let $n, m \in \mathbb{N}$ with $n \le m$. Then

$$\begin{aligned}
\left\|\sum_{k=n}^{m} c_k\, x_k\right\| &= \sup_{f\in E^*,\|f\|=1} \left|\sum_{k=n}^{m} c_k\, f(x_k)\right| \\
&\le \sup_{f\in E^*,\|f\|=1} \left(\sum_{k=n}^{m} |c_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=n}^{m} |f(x_k)|^q\right)^{\frac{1}{q}} \\
&\le \sup_{f\in E^*,\|f\|=1} B\, ||f||\, \left(\sum_{k=n}^{m} |c_k|^p\right)^{\frac{1}{p}} \\
&= B\left(\sum_{k=n}^{m} |c_k|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence $T : l_p \to E$ given by

$$T(\{c_n\}) = \sum_{n=1}^{\infty} c_n \, x_n, \{c_n\} \in l_p$$

is well-defined bounded operator from l_p into E. Moreover, $T(e_n) = x_n$, for all $n \in \mathbb{N}$. Also, let $f \in E^*$. Consider

$$T^*(f)(e_n) = f(T(e_n)) = f(x_n), \text{ for all } n \in \mathbb{N}.$$

Then $T^*(f) = \{f(x_n)\}$ and $||T^*(f)||_{l_q} = \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{\frac{1}{q}}$.
Using frame inequality, we have

$$A \, ||f||_{E^*} \leq ||T^* f||_{l_q} \leq B \, ||f||_{E^*}.$$

Thus, T^* is one-one and $T^*(E^*)$ is closed. Therefore by [Theorem 4.15, p 103, 10], T is onto.

Conversely, let $T : l_p \to E$ be a well defined bounded linear operator from l_p onto E with $T(e_n) = x_n$, for all $n \in \mathbb{N}$. So, T^* is one-one and $T^*(E^*)$ is closed by [Theorem 4.15, p 103, 10]. Then by [Lemma 1, p 487, 13], there exists a constant $C > 0$ such that

$$||f|| \leq C \, ||T^* f||, \text{ for all } f \in E^*.$$

Also, $\{f(x_n)\} = T^* f \in l_q$, for all $f \in E^*$. So, $||f||_{E^*} \leq C \, ||T^* f||_{l_q} = C \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{\frac{1}{q}}$.
Further, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{\frac{1}{q}} &= ||\{f(x_n)\}||_{l_q} = ||T^* f||_{l_q} \\ &\leq ||T^*|| \, ||f|| = ||T|| \, ||f||.\end{aligned}$$

Hence $\{x_n\}$ is a q-frame for E. ■

Remark 2.4. We observe that, $\{x_n\} \subset E$ is q-Bessel sequence for E if and only if there exists a bounded linear operator $T : l_p \to E$ such that

$$T(e_n) = x_n, \text{ for all } n \in \mathbb{N}.$$

The operator T is called the synthesis operator of q-Bessel sequence and the operator $T^* : E^* \to l_q$ given by $T^*(f) = \{f(x_n)\}$, $f \in E^*$ is called the analysis operator of q-Bessel sequence.

In the following, we give the unconditional convergence of q-frame.

Theorem 2.5. Let $\{x_n\}$ be a q-frame for E with bounds A and B. Then

(a) For $\{a_n\} \in \ell^p$, $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally.

(b) $\| \sum_{k=1}^{n} a_k x_k \| \leq B \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p}$, for any finite sequence of scalars $a_1, a_2, a_3, \ldots, a_n$.

Proof. (a) Let $\{a_n\} \in \ell_p$ and consider

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| = \left\| T \left(\sum_{n=1}^{\infty} a_n e_n \right) \right\| \leq \|T\| \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$$

From here it clearly follows.

(b) Follows from (a). ■

Next, we prove that a p-Riesz basis is a q-frame with same bounds.

Proposition 2.6. Let $\{x_n\}$ be a p-Riesz basis with bounds A and B. Then, $\{x_n\}$ is a q-frame for E with bounds A and B, where $1/p + 1/q = 1$.

Proof. It is clear that a map $T : l_p \to E$ given by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n\, x_n$, $\{a_n\} \in l_p$ is well-defined and $T(e_n) = x_n$, for all $n \in \mathbb{N}$.
Clearly, $\{f(x_n)\} = T^* f \in l_q$.
So,

$$\begin{aligned} \left(\sum_{n=1}^{\infty} |f(x_n)|^q \right)^{\frac{1}{q}} &= ||T^* f|| \leq ||T^*||\, ||f|| \\ &= ||T||\, ||f|| \leq B\, ||f||. \end{aligned}$$

Also,

$$\begin{aligned} ||f|| &= ||(T^*)^{-1} T^*(f)|| \\ &\leq ||(T^*)^{-1}||\, ||T^*(f)|| \\ &\leq A^{-1} \left(\sum_{n=1}^{\infty} |f(x_n)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Thus, $\{x_n\}$ is a q-frame with bounds A and B. ■

Remark 2.7. Let $\{x_n\}$ be a sequence in Banach space E. Then $\{x_n\}$ is a p-Riesz basis if and only if there exists $T : l_p \to E$, an isomorphism from l_p onto E such that $T(e_n) = x_n$, for all $n \in \mathbb{N}$.

Next, we give characterization of p-Riesz basis for a Banach space E in terms of q-frame for E.

Theorem 2.8. Let $\{x_n\}$ be a q-frame for E and T be its synthesis operator. Then, the following conditions are equivalent.

(a) $\{x_n\}$ is a p-Riesz basis for E, where $1/p + 1/q = 1$.

(b) T is one-one.

(c) $\{x_n\}$ is a Schauder basis.

(d) $\{x_n\}$ has unique biorthogonal system $\{f_n\} \subseteq E^*$.

(e) If $\sum_{n=1}^{\infty} a_n\, x_n = 0$, for some $\{a_n\} \in l_p$, then $a_n = 0$, for all $n \in \mathbb{N}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(f) $\{x_n\}$ is minimal, that is $x_j \notin \overline{span}\{x_n\}_{n \neq j}$.

Proof. (a) $\Longleftrightarrow$ (b) follows from Remark 2.7.
(b) $\Longrightarrow$ (c) Since $\{e_n\}$ is a Schauder basis for l_p and T is an isomorphism from l_p onto E. So, $\{x_n\} = \{T(e_n)\}$ is a Schauder basis for E.
(b) $\Longrightarrow$ (d) Let $\{l_n\}$ be a sequence of coordinate functionals on l_p. Now, define $\{f_n\} \subseteq E^*$ by

$$f_n = (T^{-1})^*\, l_n \in E^*, \text{ for all } n \in \mathbb{N}.$$

Then, for $n,\, j \in \mathbb{N}$, we have

$$\begin{aligned} f_n(x_j) &= (T^{-1})^* l_n(x_j) \\ &= l_n T^{-1} x_j = l_n T^{-1} T e_j \\ &= l_n(e_j) = \delta_{nj}. \end{aligned}$$

(d) $\Longrightarrow$ (c) Let $\{f_n\} \subset E^*$ be the unique sequence biorthogonal system to $\{x_n\}$. Note that for any $x \in E$, there exist $\{\alpha_n\} \in l_p$ such that $x = \sum_{n=1}^{\infty} \alpha_n\, x_n$. Therefore $\alpha_n = f_n(x)$, for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a Schauder basis for E.
(c) $\Longrightarrow$ (d) is obvious.
(d) $\Longleftrightarrow$ (f) is obvious.
(b) $\Longleftrightarrow$ (e) Suppose T is one-one and let $\sum_{n=1}^{\infty} \alpha_n\, x_n = 0$, for $\{\alpha_n\} \in l_p$. Then,

$$T\left(\sum_{n=1}^{\infty} \alpha_n\, e_n\right) = 0.$$

This gives $\alpha_n = 0$, for all $n \in \mathbb{N}$.
Conversely, let $T(\{\alpha_n\}) = 0$, for all $\{\alpha_n\} \in l_p$. Then, by the given condition, we have $\alpha_n = 0$, for all $n \in \mathbb{N}$. This gives that T is one-one. ■

In the following result, we show the existence of (p, q)-frame from p-Riesz basis.

Theorem 2.9. Let $\{x_n\}$ be a p-Riesz basis for E. Then there exists a sequence $\{f_n\} \subset E^*$ such that the pair (x_n, f_n) is a (p, q)-frame for E.

Proof. Let A and B respectively be the lower and upper bounds of p-Riesz basis $\{x_n\}$. We know that the synthesis operator T is an isomorphism l_p onto E. Now, we claim that $\{x_n\}$ is a q-frame for E with bounds A and B. Take $f_n = (T^{-1})^* l_n$, for $n \in \mathbb{N}$, where $\{l_n\}$ is a sequence of coordinate functionals on E_d. Then, we have

$$f_n(x) = (T^{-1})^* l_n(x) = l_n(T^{-1}x), x \in E.$$

So, we obtain $\{f_n(x)\} = T^{-1}x \in l_p$.
Therefore

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{\frac{1}{p}} &= ||T^{-1}x||_{l_p} \\ &\leq ||T^{-1}||\ ||x|| \leq A^{-1}\ ||x||, \text{ for all } x \in E.\end{aligned}$$

Thus, $\{f_n\}$ is a p-Bessel sequence for E.
Also, we have

$$\begin{aligned}x &= T(T^{-1}x) = T(\{f_n(x)\}) \\ &= \sum_{n=1}^{\infty} f_n(x)\ x_n.\end{aligned}$$

Hence (x_n, f_n) is a (p, q)-frame. ■

We give the characterizations of (p, q)-frames from q-frames in Banach spaces.

Theorem 2.10. Let $\{x_n\}$ be a q-frame for E and T be its synthesis operator. Then the following statements are equivalent.

(i) There exists a sequence $\{f_n\} \subseteq E^*$ such that (x_n, f_n) is a (p, q)-frame for E.

(ii) T has pseudoinverse $T^{\dagger}$.

(iii) $KerT$ is complemented subspace of l_p.

(iv) $T^*(E^*)$ is complemented subspace of l_q.

(v) T^* has pseudoinverse $T^{*\dagger}$.

(vi) There exist a complemented subspace M of l_p and positive constants $0 < A \leq B < \infty$ such that

$$A\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{n}\alpha_n\, x_n\right\|_E \leq B\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{\frac{1}{p}}, \text{ for all } \alpha_n \in M.$$

Proof. (i) $\Longrightarrow$ (ii)
By hypothesis, there exists a p-Bessel sequence $\{f_n\}$ for E such that

$$x = \sum_{n=1}^{\infty} f_n(x)\, x_n, \text{ for all } x \in E.$$

Let $U : E \longrightarrow l_p$ be analysis operator of p-Bessel sequence $\{f_n\}$, then

$$U(x) = \{f_n(x)\}, \text{ for all } x \in E.$$

So, $TU(x) = I_E$. Thus, $TUT = T$.
This shows that T has pseudoinverse U.
(ii) $\Longrightarrow$ (i)
Let T has pseudoinverse $T^{\dagger}$.
Then, $TT^{\dagger}$ is a projection from E onto $T(l_p) = E$.
So, $TT^{\dagger} = I_E$. Now, take $f_n = (T^{\dagger})^{*}\, l_n, n \in \mathbb{N}$, where $\{l_n\} \subset l_q$ is a sequence of coordinate functionals on l_p. Therefore, we have

$$f_n(x) = (T^{\dagger})^{*}\, l_n(x) = l_n(T^{\dagger}(x))$$

which gives us $\{f_n(x)\} = T^{\dagger}x \in l_p$.
Moreover,

$$\left(\sum_{n=1}^{\infty}|f_n(x)|^p\right)^{\frac{1}{p}} = ||T^{\dagger}(x)|| \leq ||T^{\dagger}||\; ||x||.$$

This gives that $\{f_n\}$ is p-Bessel sequence.
Further, consider

$$x = TT^{\dagger}(x) = T(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)\, x_n, \text{ for } x \in E.$$

Thus, we conclude that (x_n, f_n) is a (p,q)-frame for E.
(ii) $\Longleftrightarrow$ (v) is obvious.
(ii) $\Longleftrightarrow$ (iii)
Since $T(l_p) = E$, by Lemma 1.5, it follows.
(iv) $\Longleftrightarrow$ (v)

Since $KerT^* = \{0\}$, by Lemma 1.5, it follows.
(iii) $\Longrightarrow$ (vi)
Let $l_p = M \oplus KerT$, where M is a closed subspace of l_p and let $T_1 : M \longrightarrow E$ be restriction of T on M. By [Theorem 6.3, p 29, 11], T_1 is an isomorphism from M onto E. Now, suppose that $\{a_n\} \in M$. Then, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} &= ||T_1^{-1} T_1(\{a_n\})||_{l_p} \\ &\leq ||T_1^{-1}|| \left|\left|\sum_{n=1}^{\infty} a_n\, x_n\right|\right|_E.\end{aligned}$$

Further,

$$\begin{aligned}\left|\left|\sum_{n=1}^{\infty} a_n\, x_n\right|\right|_E &= ||T_1\{a_n\}|| \\ &\leq ||T_1|| \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}, \text{ for all } \{a_n\} \in M.\end{aligned}$$

(vi) $\Longrightarrow$ (ii)
Let $T_1 : M \longrightarrow E$ be the restriction of T on M. Then, we have

$$T_1(\{a_n\}) = \sum_{n=1}^{\infty} a_n\, x_n, \text{ for } \{a_n\} \in M.$$

By the given hypothesis, T_1 is invertible.
Let P be a projection from l_p onto M.
Then, $T = T_1 P$ and $T T_1^{-1} T(\alpha) = T_1 P T^{-1} T_1 P(\alpha) = T_1 P^2(\alpha) = T_1 P(\alpha) = T(\alpha)$.
Thus, T has pseudoinverse. ■

Next, we discuss the dual property of (p, q)-frames in Banach spaces.

Theorem 2.11. Let (x_n, f_n) be (p, q)-frame for E. Then, there exists a sequence $\{\phi_n\} \subseteq E^{**}$ such that (f_n, ϕ_n) is a (p, q)-frame for E^*.

Proof. By given hypothesis, we have

$$x = \sum_{n=1}^{\infty} f_n(x)\, x_n, \text{ for all } x \in E.$$

Let U be analysis operator of p-Bessel sequence $\{f_n\}$ and T be synthesis operator of q-Bessel sequence $\{x_n\}$.

Then, $I = TU$. From here, we get $U^*T^* = I_{E^*}$.
Thus, $U^* : l_q \longrightarrow E^*$ is surjective. Moreover,

$$\begin{aligned} U^*(e_n)(x) &= e_n(U(x)) \\ &= e_n(\{f_n(x)\}) = f_n(x). \end{aligned}$$

This gives $f_n = U^*(e_n)$. For $\{a_n\} \in l_q$, we have

$$U^*(\{a_n\}) = \sum_{n=1}^{\infty} a_n\ f_n.$$

Moreover,

$$U^*T^*U^* = U^*.$$

Thus, U^* has pseudoinverse T^*. By Theorem 2.10, there exists a sequence $\{\phi_n\} \subset E^{**}$ such that (f_n, ϕ_n) is (p, q)-frame. ■

In the following, we discuss the relationship between (p, q)-frames and atomic decompositions in Banach spaces.

Theorem 2.12. If (x_n, f_n) is a (p, q)-frame for E, then $(f_n, \pi(x_n))$ is an atomic decomposition for E^* with respect to l_q. Further, if $\{f_n\}$ is p-Bessel sequence and there exists a sequence $\{x_n\} \subseteq E$ such that $(f_n, \pi(x_n))$ is an atomic decomposition, then (x_n, f_n) is (p, q)-frame for E.

Proof. By given hypothesis, $\{x_n\}$ is a q-frame for E. Also, let A and B be bounds of $\{x_n\}$. Then

$$A\ ||f|| \leq \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{\frac{1}{q}} \leq B||f||.$$

Let U be analysis operator of $\{f_n\}$ and T be synthesis operator of $\{x_n\}$. Then, $TU = I$.
So, $U^*T^* = I_{E^*}$.
By Theorem 2.11, we have

$$f = U^*T^*(f) = U^*(\{f(x_n)\}) = \sum_{n=1}^{\infty} f(x_n)\ f_n.$$

Thus, $(f_n, \pi(x_n))$ is an atomic decomposition for E^* with respect to l_q.

Coversely, since $(f_n, \pi(x_n))$ is an atomic decomposition for E^* with respect to l_q. Then,

$$A\ ||f|| \leq \left(\sum_{n=1}^{\infty} |f(x_n)|^q\right)^{\frac{1}{q}} \leq B||f||^2.$$

So, $\{x_n\}$ is a q-frame for E with bounds A and B. Also, $f = \sum_{n=1}^{\infty} f(x_n)\, f_n$, for all $f \in E^*$. By the given condition, $\{f_n\}$ is p-Bessel sequence. Let $N \in \mathbb{N}$ and $x \in E$ and we have

$$\begin{aligned}\left\|x - \sum_{k=1}^{N} f_k(x)\, x_k\right\| &= \sup_{\substack{f \in E^* \\ ||f|| = 1}} \left|\sum_{k=n+1}^{\infty} f_k(x)\, f(x_k)\right| \\ &\leq \sup_{\substack{f \in E^* \\ ||f|| = 1}} \left(\sum_{k=N+1}^{\infty} |f_k(x)|^p\right)^{\frac{1}{p}} \left(\sum_{k=N+1}^{\infty} |f(x_k)|^q\right)^{\frac{1}{q}} \\ &\leq B\left(\sum_{k=N+1}^{\infty} |f_k(x)|^p\right)^{\frac{1}{p}} \longrightarrow 0 \text{ as } N \longrightarrow \infty.\end{aligned}$$

Thus, $x = \sum_{n=1}^{\infty} f_n(x)\, x_n$, for all $x \in E$. ■

Lemma 2.13. Let (x_n, f_n) be (p, q)-frame for E, U be analysis operator of p-Bessel sequence $\{f_n\}$ and T be synthesis operator of q-Bessel sequence $\{x_n\}$, then,

$$\ell_p = U(E) \oplus kerT.$$

Proof. By given hypothesis, we have $TU + I_E$. So, $TUT = T$, $UTU = U$ and UT is a projection from ℓ_p onto $U(E)$. Therefore, we have $\ell_p = U(E) \oplus kerUT$. Now, let $\alpha \in kerUT$, then $UT(\alpha) = 0$. That is $TUT(\alpha) = 0$. So, $T(\alpha) = 0$ and therefore $\alpha \in kerT$. Again, let $a \in kerT$, then $T(a) = 0$. So, $UT(\alpha) = 0$. This gives us $a \in kerUT$. Hence, $\ell_p = U(E) \oplus kerT$. ■

Next, we show that (p, q)-frames are compressions of of p-Riesz bases in Banach spaces.

Theorem 2.14. Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$. Then, (x_n, f_n) is (p, q)-frame for E if and only if there exist a Banach space Z with E as its complemented subspace and p-Riesz basis $\{y_n\}$ for Z such that $P_E(y_n) = x_n$, for all $n \in \mathbb{N}$, where P_E is a projection from Z onto E.

Proof. By Lemma 2.13, we have $\ell_p = U(E) \oplus kerT$, where U is analysis operator of p-Bessel sequence and T is the synthesis operator of q-Bessel sequence. Moreover, UT is a projection from ℓ_p onto $U(E)$. Take $Z = E \oplus kerT$. Define a map $J : \ell_p \to E \oplus kerT$ as

$$J(\alpha) = T(\alpha) + (I_E - UT)(\alpha), \text{ for } \alpha \in \ell_p.$$

We will show J is an isomorphism from ℓ_p onto Z. Let $J(\alpha) = 0$, then $UT(\alpha) = \alpha$ and $\alpha \in kerT$. That is $\alpha \in U(E)$ and $\alpha \in kerT$. Thus, $\alpha = 0$. So, J is one-one. To show J is onto. Let $x + \alpha_0 \in E \oplus kerT$, for $x \in E$ and $\alpha_0 \in kerT$. Since, T is surjective, so there exists $\xi \in \ell_p$ such that $x = T(\xi)$. Now, choose $\alpha = UT(\xi) + \alpha_0$. From here we have $UT(\alpha) = UTUT(\xi) = UT(\xi)$. So,

$$\alpha_0 = \alpha - UT(\xi) - \alpha - UT(\alpha) = (I_E - TU)(\alpha).$$

and

$$x = T(\xi) = TUT(\xi) = TUT(\alpha) = T(\alpha).$$

Thus, we obtain

$$J(\alpha) = T(\alpha) + (I_E - UT)(\alpha) = x + \alpha_0.$$

Hence, J is onto. Take, $y_n = J(e_n)$, for $n \in \mathbb{N}$. Therefore, $\{y_n\}$ is a Schauder basis and $\overline{span}\{y_n\} = Z$. Let $\{\alpha_n\} \in \ell_p$, then we have

$$\left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p} = \|J^{-1}J(\alpha)\|_{\ell_p} \leq \|J^{-1}\| \|\sum_{n=1}^{\infty} \alpha_n y_n\|.$$

Also, we have

$$\|\sum_{n=1}^{\infty} \alpha_n y_n\| = \|J(\alpha)\| \leq \|J\| \|\alpha\| = (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p}.$$

Thus, $\{y_n\}$ is a p-Riesz basis for Z. Let P_E be a projection from Z onto E. By the construction of J, we have $T = P_E J$ and for $n \in \mathbb{N}$ we have

$$x_n = T(e_n) = P_E J(e_n) = P_E(y_n), \text{ for all } n \in \mathbb{N}.$$

Conversely, let $Z = E \oplus W$, where W is a closed subspace of Z. Let $\{y_n\}$ be p-Riesz basis for Z with bounds A and B. and $P_E(y_n) = x_n$. By Remark 2.7, there exists an isomorphism $J : \ell_p \to Z$ from ℓ_p onto Z such that $J(e_n) = y_n$, for all $n \in \mathbb{N}$. Define $T : \ell_p \to E$ as $T = P_E J$. Clear that T is surjective and

$$T(e_n) = P_E J(e_n) = P_E(y_n) = x_n, \text{ for all } n \in \mathbb{N}.$$

By Theorem 2.3, $\{x_n\}$ is q-frame for E and T is the synthesis operator. Let $S : E \to \ell_p$ be restriction of J^{-1} to E. Then,

$$TST = P_E J J^{-1} P_E J = P_E^2 J = P_E J = T.$$

Thus, T has pseudo inverse. Hence, by Theorem 2.10, there exists a p-Bessel sequence $\{f_n\}$ such that (x_n, f_n) is a (p, q)-frame for E. ■

Next, we show the (p, q)-frame coefficients $\{f_n(x)\}$ have minimal ℓ_p-norm among all sequences representing x in Banach spaces with a given condition. A closed subspace F in a Banach space E is said to be 1-*complemented* or *constrained*, if F is the range of a norm one projection on E.

Theorem 2.15. Let (x_n, f_n) is (p, q)-frame for E, then the following conditions are equivalent.

(a) $U(E)$ is 1-complemented subspace of ℓ_p, where U is the analysis operator of p-Bessel sequence $\{f_n\}$.

(b) Among all $\{\alpha_n\} \in \ell_p$ for which the representation $x = \sum_{n=1}^{\infty} \alpha_n x_n$ holds, the sequence $\{f_n(x)\}$ of the coefficient expansion $x = \sum_{n=1}^{\infty} f_n(x) x_n$ has the minimal ℓ_p-norm given by $(\sum_{n=1}^{\infty} |f_n(x)|^p)^{1/p} \leq (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p}$.

Proof. $(b) \Rightarrow (a)$ Let T be synthesis operator of q-Bessel sequence $\{x_n\}$. By given hypothesis we have have $TU = I_E$, $TUT = T$, $UTU = U$ and $\ell_p = U(E) \oplus kerT$. Note that $Q = UT$ is a projection from ℓ_p onto $U(E)$. Let $\alpha = \{\alpha_n\} \in \ell_p$, then we have $\alpha = UT(\alpha) + \alpha_0$, for $UT(\alpha) \in U(E)$ and $\alpha_0 \in kerT$. Indeed, there exists some $x \in E$ such that $UT(\alpha) = U(x)$. Also, we have

$$x = TU(x) = TUT(\alpha) = T(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

From above, we obtain

$$\|UT(\alpha)\| = \|U(x)\| = \left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p} = \|\alpha\|.$$

And hence $\|UT\| = 1$.
$(a) \Rightarrow (b)$ UT is a projection from ℓ_p onto $U(E)$ and $\|UT\| = 1$. Let $\alpha = \{\alpha_n\} \in \ell_p$ such that

$$x = \left(\sum_{n=1}^{\infty} \alpha_n x_n = T(\alpha)\right), \text{ for } x \in E.$$

One can write $\alpha = UT(\alpha) + \alpha_0$, for $UT(\alpha) \in U(E)$ and $\alpha_0 \in kerT$. Indeed, there exists some x_0 in E such that $UT(\alpha) = U(x_0)$. Also, we have

$$x_0 = TU(x_0) = TUT(\alpha) = T(\alpha) = x.$$

Thus, we obtain $UT(\alpha) = U(x)$. Finally, we get

$$\left(\sum_{n=1}^{\infty} |f_n(x)|^p\right)^{1/p} = \|U(x)\| = \|UT(\alpha\| \leq \|UT\| \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p}.$$

■

References

[1] A. Aldroubi, Q. Sun and W. Tang, *p-Frames And Shift Invariant Subspaces of L^p*, J. Fourier Anal. Appl. 7(1)(2001), 1–22.

[2] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäurer, Bostan-Basel-Berlin (2003).

[3] O. Christensen and Diana T. Stoeva, *p-Frames In Seperable Banach Spaces*, Advances In Computational Mathematics, © 2003 Khuwer Academic Publishers, Vol. 18, 117–126.

[4] R. J. Duffin and A. C. Schaeffer, *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc., Vol. 72(1952), 341–366.

[5] N. Dunford and J. Schwartz, *Linear operators*, I, Interscience, New York, 1963.

[6] Feichtinger and K. Gröchenig, *A unified approach to atomic decompositions via integrable group representations*, In: Proc. Conf. Function Spaces and Applications, Lecture Notes Math. 1302, Berlin-Heidelberg-New York: Springer (1988), 52–73.

[7] D. Gabör, *Theory of communications*, J. IEE,London, 93, No. 3(1946).

[8] K. Gröchenig, *Describing functions: Atomic decompositions versus frames*, Monatsh. Math., Vol. 112(1991), 1–41.

[9] Heil, C., (1998), A basis theory primer, Birkhäuser (expanded edition)

[10] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, second edition, 1991.

[11] Taylor, A. E. and Lay, D. C., Introduction to Functional Analysis, Wiley, New York, 1980.

[12] P. A. Terekhin, *Frames in Banach spaces*, Functional Analysis and its Application, Vol. 44(3)2010, 199–208.

[13] Zhu, Y. C., *q-Besselian Frames in Banach Spaces*, Acta Mathematica Sinica, English Series, Sep., Vol. 23(9)(2007), 1707–1718.