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Fractional Differential Equation Associated with an Integral Operator with the $\overline{H}$-function in the Kernel

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Abstract

In this paper, we investigate an integral operator associated with the $\overline{H}$-function in its kernel. First, we give solution to a fractional differential equation involving Hilfer derivative operator and the integral operator associated with the $\overline{H}$-function in its kernel. Due to the general nature of $\overline{H}$-function, a number of special cases of our main result can be obtained by specialising the parameters occurring therein. We record here two such special cases involving polylogarithm function and generalised Wright hypergeometric function. Using some of the special cases of our main findings, graphical illustrations are presented. The results derived in this paper generalize the results obtained by Srivastava et al. [14] and Srivastava and Tomovski [15].

AMS subject classification: 33C60, 44A10.
Keywords: $\overline{H}$-Function, Laplace Transform, Integral Operator, Fractional differential equation.

1. Introduction

Fractional differential equations involving known integral operators have been studied earlier by a large number of authors (see, for details, [14], [15] and [16]) and have diverse applications.

Motivated by above mentioned work and the references cited therein, we make use of the following functions and fractional integral operators:
The $\overline{H}$-function occurring in the present paper was introduced by Inayat Hussain [9] and

studied by Bushman and Srivastava [1] and others, it is defined and represented in the following manner:

$$\overline{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(e_j,E_j;\in_j)_{1,n}, & (e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m}, & (f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right]=\frac{1}{2\pi\omega}\int_{\mathfrak{L}}\overline{\Theta}(\xi)z^{\xi}d\xi \tag{1.1}$$

where, $\omega=\sqrt{-1}$, $z\in\mathbb{C}\setminus\{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(f_j-F_j\xi)\prod_{j=1}^{n}\left\{\Gamma(1-e_j+E_j\xi)\right\}^{\in_j}}{\prod_{j=m+1}^{q}\left\{\Gamma(1-f_j+F_j\xi)\right\}^{\Im_j}\prod_{j=n+1}^{p}\Gamma(e_j-E_j\xi)} \tag{1.2}$$

and

$$1\leqq m\leqq q \quad \text{and} \quad 0\leqq n\leqq p \qquad (m,q\in\mathbb{N}=\{1,2,3,\cdots\};\ n,p\in\mathbb{N}_0=\mathbb{N}\cup\{0\}), \tag{1.3}$$

The nature of contour $\mathfrak{L}$ in (1.1) and various conditions on its parameters can be seen in the paper by Gupta, Jain and Agarwal [4].

In this paper we make use of the Riemann-Liouville fractional integral operator I_{a+}^{p} and the Riemann-Liouville fractional derivative operator D_{a+}^{p}, which are defined by (see, for details, [10], [11] and [12]):

$$(I_{a+}^{\mu}f)(x)=\frac{1}{\Gamma(\mu)}\int_a^x\frac{f(t)}{(x-t)^{1-\mu}}\,dt \qquad \big(\Re(\mu)>0\big) \tag{1.4}$$

and

$$(D_{a+}^{\mu}f)(x)=\left(\frac{d}{dx}\right)^n(I_{a+}^{n-\mu}f)(x) \qquad \big(\Re(\mu)>0;\ n=[\Re(\mu)]+1\big), \tag{1.5}$$

where $[x]$ denotes the greatest integer in the real number x. Hilfer [7] generalized the operator in (1.5) and defined a general fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0<\mu<1$ and type $0\leqq\nu\leqq1$ with respect to x as follows:

$$(D_{a+}^{\mu,\nu}f)(x)=\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)}f\right)\right)(x). \tag{1.6}$$

Eq. (1.6) yields the classical Riemann-Liouville fractional derivative operator D_{a+}^{μ} when $\nu=0$ and for $\nu=1$ it reduces to the fractional derivative operator introduced by Joseph Liouville (1809–1882) in 1832, which is called the Liouville-Caputo fractional derivative operator (see [3], [10] and [16]).

Now, the Laplace transform $\mathcal{L}[f(x)](s)$ of the function $f(x)$ is defined as follows:

$$\mathcal{L}[f(x)](s) = \int_0^\infty e^{-sx}\ f(x)dx \qquad \big(\Re(s) > 0\big), \tag{1.7}$$

provided that the integral exists, we recall the following known result (see, for details, [15] and [16]):

$$\mathcal{L}[(D_{0+}^{\mu,\nu} f)(x)](s) = s^{\mu}\mathcal{L}[f(x)](s) - s^{-\nu(1-\mu)}\left(I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+)$$
$$\big(\Re(s) > 0;\ 0 < \mu < 1\big), \tag{1.8}$$

where the initial-value term:

$$\left(I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+)$$

involves the Riemann-Liouville fractional integral (1.4) (*with* $a = 0$) of the function $f(t)$ of order

$$\mu \mapsto (1-\nu)(1-\mu) \tag{1.9}$$

evaluated in the limit as $x \to 0+$.

2. An Integral Operator Involving $\overline{H}$-function

In our present investigation we make use of an integral operator with $\overline{H}$-function in its kernel defined as follows:

$$\left(\overline{\mathcal{H}}_{a+;p,q;\beta}^{w;m,n;\gamma}\ \varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\ \overline{H}_{p,q}^{m,n}[w(x-t)^{\gamma}]\varphi(t)dt \tag{2.1}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\backslash\{0\};\ 1 \leqq m \leqq q;\ 0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\right).$$

If we take $w = 1, m = 1$ and $a = 0$ in (2.1) we obtain an integral operator introduced by Harjule (see for details [6, p.80, Eq.(5.1.10)]).

For $a = 0$, by using the *Convolution Theorem* for the Laplace Transform in (1.7), we find from the definition (2.1) that

$$\begin{aligned}
&\mathcal{L}\left[\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\ \varphi\right)(x)\right](s)\\
&\quad = \mathcal{L}\left[x^{\beta-1}\ \overline{H}_{p,q}^{m,n}[wx^{\gamma}]\right](s)\cdot\mathcal{L}[\varphi(x)](s)\\
&\quad = s^{-\beta}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\ \middle|\ \begin{matrix}(1-\beta,\gamma;1), & (e_j,E_j;\epsilon_j)_{1,n},(e_j,E_j)_{n+1,p}\\ & (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right]\Phi(s)
\end{aligned} \tag{2.2}$$

$$\left(\Re(s) > 0;\ \gamma > 0;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\right)$$

where,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad \big(\Re(s) > 0\big).$$

In its special case when $\varphi(x) \equiv 1$, (2.2) immediately yields

$$\begin{aligned}&\mathcal{L}\left[\left(\overline{\mathcal{H}}^{w;m,n;\gamma}_{0+;p,q;\beta}\ 1\right)(x)\right](s)\\ &= s^{-\beta-1}\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\ \middle|\ \begin{matrix}(1-\beta,\gamma;1), & (e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ & (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right]\end{aligned} \tag{2.3}$$

$$\left(\Re(s) > 0;\ \gamma > 0;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\right)$$

3. A General Family of Fractional Differential Equations

In this section, a general family of fractional differential equations [16, p.803, Eq. (3.7)] given by (3.1), was introduced in [8] for dielectric relaxation in glasses but its general solution was not given, though the laplace transformed relaxation function and the corresponding dielectric susceptibility were calculated. Therefore, in this section we proceed to find its general solution.

$$a\left(D^{\alpha_1,\beta_1}_{0+}\ y\right)(x) + b\left(D^{\alpha_2,\beta_2}_{0+}\ y\right)(x) + cy(x) = g(x) \tag{3.1}$$

where

$$\left(0 < \alpha_1 \leqq \alpha_2 < 1;\ 0 \leqq \beta_1, \beta_2 \leqq 1 \text{ and } a, b, c \in \mathbb{R}\right)$$

in the space of Lebesgue integrable functions (see [3, 15]) $y \in L(0,\infty)$ with the initial conditions:

$$\left(I^{(1-\beta_i)(1-\alpha_i)}_{0+}\ y\right)(0+) = C_i \qquad (i = 1, 2), \tag{3.2}$$

where, without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2).$$

if $C_1 < \infty$, then $C_2 = 0$ unless $(1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2)$.

Now, in (3.1) if we take $\alpha_1 = \alpha_2 = \alpha, \beta_1 \neq \beta_2, c = 0$ and $g(x) = \lambda\left(\overline{\mathcal{H}}^{w;m,n;\gamma}_{0+;p,q;\beta}\ 1\right)(x) + f(x)$, we obtain a fractional differential equation using which we formulate the following Theorem.

3.1. Main Theorem

Theorem 3.1. The following fractional differential equation:

$$a\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) + b\left(D_{0+}^{\alpha,\beta_2}\, y\right)(x) = \lambda\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\, 1\right)(x) + f(x) \tag{3.3}$$

$$\Bigg(0 < \alpha < 1; 0 \leqq \beta_1, \beta_2 \leqq 1; \Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ 1 \leqq m \leqq q;$$

$$0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\Bigg)$$

with the initial condition:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha)}\, y\right)(0+) = C_i \qquad (i = 1, 2), \tag{3.4}$$

has its solution in the space $L(0, \infty)$ given by

$$\begin{aligned} y(x) =& \frac{a\,C_1}{(a+b)}\,\frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\,C_2}{(a+b)}\,\frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ &+ \frac{\lambda}{(a+b)} x^{\alpha+\beta}\ \overline{H}_{p+1,q+1}^{m,n+1}\left[wx^{\gamma}\,\middle|\,\begin{matrix} A^* \\ \\ B^* \end{matrix}\right] \\ &+ \frac{1}{(a+b)\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\ f(t)dt, \end{aligned} \tag{3.5}$$

where $A^* = (1-\beta, \gamma; 1), (e_j, E_j; \in_j)_{1,n}, (e_j, E_j)_{n+1,p}$,
$B^* = (f_j, F_j)_{1,m}, (f_j, F_j; \Im_j)_{m+1,q}, (-\beta-\alpha, \gamma; 1)$ and C_1, C_2, λ are arbitrary constants and the function f is suitably prescribed.

Proof. We denote by $Y(s)$ the Laplace transform of the function $y(x)$, which is given as in (1.7). Then, by applying the Laplace transform operator $\mathcal{L}$ to each side of (3.3), and using the formulas (1.8) and (2.3) and the initial condition (3.4), we find that

$$\begin{aligned} &a(s^{\alpha}Y(s) - C_1 s^{-\beta_1(1-\alpha)}) + b(s^{\alpha}Y(s) - C_2 s^{-\beta_2(1-\alpha)}) \\ &= \lambda s^{-\beta-1}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1), (e_j, E_j; \in_j)_{1,n}, (e_j, E_j)_{n+1,p} \\ \\ (f_j, F_j)_{1,m}, (f_j, F_j; \Im_j)_{m+1,q}\end{matrix}\right] + F(s) \end{aligned} \tag{3.6}$$

which readily yields

$$\begin{aligned} Y(s) =& \frac{a\,C_1}{(a+b)} s^{-\alpha-\beta_1(1-\alpha)} + \frac{b\,C_2}{(a+b)} s^{-\alpha-\beta_2(1-\alpha)} \\ &+ \frac{\lambda}{(a+b)} s^{-\beta-\alpha-1}\ \overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\,\middle|\,\begin{matrix} C^* \\ \\ D^* \end{matrix}\right] + \frac{s^{-\alpha}}{(a+b)}\ F(s) \end{aligned} \tag{3.7}$$

where $C^* = (1-\beta, \gamma; 1), (e_j, E_j; \in_j)_{1,n}, (e_j, E_j)_{n+1,p}$ and
$D^* = (f_j, F_j)_{1,m}, (f_j, F_j; \Im_j)_{m+1,q}$.
Now, by taking the inverse Laplace transformation of each side of (3.7), we get

$$\begin{aligned} y(x) =& \frac{a\,C_1}{(a+b)}\,\frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\,C_2}{(a+b)}\,\frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ &+ \frac{\lambda}{(a+b)}\left(\frac{1}{2\pi i}\int_{\mathfrak{L}} \overline{\Theta}(\mathfrak{s}) w^{\mathfrak{s}}\,\Gamma(\beta+\alpha\mathfrak{s})\mathcal{L}^{-1}\left[s^{-\alpha-\beta-\gamma^{\mathfrak{s}}-1}\right](x)\,d\mathfrak{s}\right) \\ &+ \frac{1}{(a+b)\Gamma(\alpha)} x^{\alpha-1} f(x) \qquad (3.8) \\ =& \frac{a\,C_1}{(a+b)}\,\frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\,C_2}{(a+b)}\,\frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ &+ \frac{\lambda}{(a+b)} x^{\alpha+\beta}\,\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\,\middle|\,\begin{matrix}E^*\\ \\ F^*\end{matrix}\right] \\ &+ \frac{1}{(a+b)\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,f(t)dt \qquad (3.9) \end{aligned}$$

where $E^* = (1-\beta, \gamma; 1), (e_j, E_j; \in_j)_{1,n}, (e_j, E_j)_{n+1,p}$ and
$F^* = (f_j, F_j)_{1,m}, (f_j, F_j; \Im_j)_{m+1,q}, (-\beta-\alpha, \gamma; 1)$,
which completes our proof of Theorem under the various already-stated parametric constraints. ■

Remark 3.2. If we consider $a = 1, b = 0$ and reduce $\overline{H}$-function to the Mittag-Leffler function (see [13] and [15]) in the integral operator on the right-hand side of (3.3), we get the result obtained by Srivastava and Tomovski [15, p.207, Theorem 8]. Again, if we take $a = 1, b = 0$ and reduce $\overline{H}$-function to the Fox's H-function [6, p.10, Eq.(1.1.42)] in the integral operator on the right-hand side of (3.3), we get a result obtained by Srivastava et al. [14, p.115, Theorem 2].

Remark 3.3. In order to obtain corollary 1 we reduce $\overline{H}$-function to the polylogarithm function of order η [2, p.30] in the integral operator on the right-hand side of (3.3) and define the integral operator as

$$\left(\mathcal{F}^{w;1,1;\gamma}_{a+;1,2;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\,F[w(x-t)^{\gamma}, \eta]\varphi(t)dt \qquad (3.10)$$

$$\Big(\Re(\beta) > 0;\;\; w \in \mathbb{C}\setminus\{0\}\Big),$$

provided that the integral exists.

Corollary 3.4. The following fractional differential equation:

$$a\left(D^{\alpha,\beta_1}_{0+}\,y\right)(x) + b\left(D^{\alpha,\beta_2}_{0+}\,y\right)(x) = \lambda\left(\mathcal{F}^{w;1,1;\gamma}_{0+;1,2;\beta}\,1\right)(x) + f(x) \qquad (3.11)$$

$$\left(0 < \alpha < 1; 0 \leqq \beta_1; \beta_2 \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}\right)$$

with the initial condition (3.4) has its solution in the space $L(0, \infty)$ given by

$$\begin{aligned} y(x) =& \frac{a\,C_1}{(a+b)} \, \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\,C_2}{(a+b)} \, \frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ & - \frac{\lambda}{(a+b)} x^{\alpha+\beta} \overline{H}_{2,3}^{1,2}\left[-wx^{\gamma} \,\middle|\, \begin{matrix} (1-\beta, \gamma; 1), (1, 1; \eta+1) \\ (1, 1), (0, 1; \eta), (-\beta-\alpha, \gamma; 1) \end{matrix}\right] \\ & + \frac{1}{(a+b)\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \end{aligned} \tag{3.12}$$

where C_1, C_2 and λ are arbitrary constants and the function f is suitably prescribed.

Remark 3.5. If we reduce $\overline{H}$-function to the generalised Wright hypergeometric function [5, p.271, Eq.(7)] in the integral operator on the right-hand side of (3.3) and define the integral operator as

$$\left(\overline{\Psi}_{a+;q;\beta}^{w;p;\gamma}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\ {}_p\overline{\Psi}_q \left[\begin{matrix} (e_j, E_j; \in_j)_{1,p} \\ (f_j, F_j; \Im_j)_{1,q} \end{matrix} ; w(x-t)^{\gamma}\right] \varphi(t)dt \tag{3.13}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\right),$$

provided that the integral exists then the Theorem can be specialized to the following form.

Corollary 3.6. The following fractional differential equation:

$$a\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) + b\left(D_{0+}^{\alpha,\beta_2}\, y\right)(x) = \lambda\left(\overline{\Psi}_{0+;q;\beta}^{w;p;\gamma}\, 1\right)(x) + f(x) \tag{3.14}$$

$$\left(0 < \alpha < 1; 0 \leqq \beta_1; \beta_2 \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\right)$$

with the initial condition (3.4) has its solution in the space $L(0, \infty)$ given by

$$\begin{aligned} y(x) =& \frac{a\,C_1}{(a+b)} \, \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\,C_2}{(a+b)} \, \frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ & + \frac{\lambda}{(a+b)} x^{\alpha+\beta}\ {}_{p+1}\overline{\Psi}_{q+1}\left[\begin{matrix} (1-e_j, E_j; \in_j)_{1,p}, (\beta, \gamma; 1) \\ (1-f_j, F_j; \Im_j)_{1,q}, (1+\beta+\alpha, \gamma; 1) \end{matrix} ; \omega x^{\gamma}\right] \\ & + \frac{1}{(a+b)\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \end{aligned} \tag{3.15}$$

where C_1, C_2 and λ are arbitrary constants and the function f is suitably prescribed.

4. Numerical Examples and Graphical Representation

First of all, by letting $\omega \to 0$ in Corollary 2, the generalised Wright hypergeometric function occuring in (3.15) reduces to 1. Further, in order to obtain numerical examples from Corollary 2 we consider $f(x) = x^{\rho}$ where $\Re(\rho) > -1$.

Example 4.1. If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.5$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (3.15), we easily arrive at the following result

$$\delta_{0.5}(x) = \frac{C_1}{2\Gamma(0.625)x^{0.375}} + \frac{C_2}{2\Gamma(0.75)x^{0.25}} + \frac{\lambda x}{2} + \frac{\Gamma(2)}{2\Gamma(2.5)}x^{1.5} \tag{4.1}$$

Example 4.2. If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.6$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (3.15), we easily arrive at the following result

$$\delta_{0.6}(x) = \frac{C_1}{2\Gamma(0.7)x^{0.3}} + \frac{C_2}{2\Gamma(0.8)x^{0.2}} + \frac{\lambda x^{1.1}}{2} + \frac{\Gamma(2)}{2\Gamma(2.6)}x^{1.6} \tag{4.2}$$

Example 4.3. If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.7$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (3.15), we easily arrive at the following result

$$\delta_{0.7}(x) = \frac{C_1}{2\Gamma(0.775)x^{0.225}} + \frac{C_2}{2\Gamma(0.85)x^{0.15}} + \frac{\lambda x^{1.2}}{2} + \frac{\Gamma(2)}{2\Gamma(2.7)}x^{1.7} \tag{4.3}$$

Example 4.4. If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.9$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (3.15), we easily arrive at the following result

$$\delta_{0.9}(x) = \frac{C_1}{2\Gamma(0.925)x^{0.075}} + \frac{C_2}{2\Gamma(0.95)x^{0.05}} + \frac{\lambda x^{1.4}}{2} + \frac{\Gamma(2)}{2\Gamma(2.9)}x^{1.9} \tag{4.4}$$

Example 4.5. If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 1$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (3.15), we easily arrive at the following result

$$\delta_1(x) = \frac{1}{2}(C_1 + C_2) + \frac{\lambda x^{1.5}}{2} + \frac{\Gamma(2)}{2\Gamma(3)}x^2 \tag{4.5}$$

Example 4.6. If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.5$ and $\rho = 0$ in equation (3.15), we easily arrive at the following result

$$y_{0.5}(x) = \frac{C_2}{\sqrt{\pi x}} + \lambda x^{0.25} + 1 \tag{4.6}$$

Example 4.7. If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.65$ and $\rho = 0$ in equation (3.15), we easily arrive at the following result

$$y_{0.65}(x) = \frac{C_2}{\Gamma(0.65)x^{0.35}} + \lambda x^{0.25} + 1 \tag{4.7}$$

Example 4.8. If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.9$ and $\rho = 0$ in equation (3.15), we easily arrive at the following result

$$y_{0.9}(x) = \frac{C_2}{\Gamma(0.9)x^{0.1}} + \lambda x^{0.25} + 1 \tag{4.8}$$

Example 4.9. If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.95$ and $\rho = 0$ in equation (3.15), we easily arrive at the following result

$$y_{0.95}(x) = \frac{C_2}{\Gamma(0.95)x^{0.05}} + \lambda x^{0.25} + 1 \tag{4.9}$$

Example 4.10. If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 \to 1$ and $\rho = 0$ in equation (3.15), we easily arrive at the following result

$$y_1(x) = C_2 + \lambda x^{0.25} + 1 \tag{4.10}$$

Example 4.11. If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.5$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (3.15), we easily arrive at the following result

$$\eta_{0.5}(x) = \frac{C_1}{\sqrt{\pi x}} + \lambda x^{0.5} + x^2 \tag{4.11}$$

Example 4.12. If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.7$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (3.15), we easily arrive at the following result

$$\eta_{0.7}(x) = \frac{C_1}{\Gamma(0.7)x^{0.3}} + \lambda x^{0.5} + x^2 \tag{4.12}$$

Example 4.13. If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.9$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (3.15), we easily arrive at the following result

$$\eta_{0.9}(x) = \frac{C_1}{\Gamma(0.9)x^{0.1}} + \lambda x^{0.5} + x^2 \tag{4.13}$$

Example 4.14. If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.95$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (3.15), we easily arrive at the following result

$$\eta_{0.95}(x) = \frac{C_1}{\Gamma(0.95)x^{0.05}} + \lambda x^{0.5} + x^2 \tag{4.14}$$

Example 4.15. If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 \to 1$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (3.15), we easily arrive at the following result

$$\eta_1(x) = C_1 + \lambda x^{0.5} + x^2 \tag{4.15}$$

The following graphs (see Figure 1, Figure 2 and Figure 3) are obtained by using MATLAB. Figure 1 exhibits a comparison between the behaviours of the solutions $\delta_\alpha(x)$ given by Eqs. (4.1), (4.2), (4.3), (4.4) and (4.5) for different values of the parameter α.

On the other hand, Figure 2 illustrates a comparison between the behaviours of the solutions $y_{\beta_2}(x)$ given by Eqs. (4.6), (4.7), (4.8), (4.9) and (4.10) for different values of the parameter β_2. Similarly, Figure 3 illustrates a comparison between the behaviours of the solutions $\eta_{\beta_1}(x)$ given by Eqs. (4.11), (4.12), (4.13), (4.14) and (4.15) for different values of the parameter β_1.

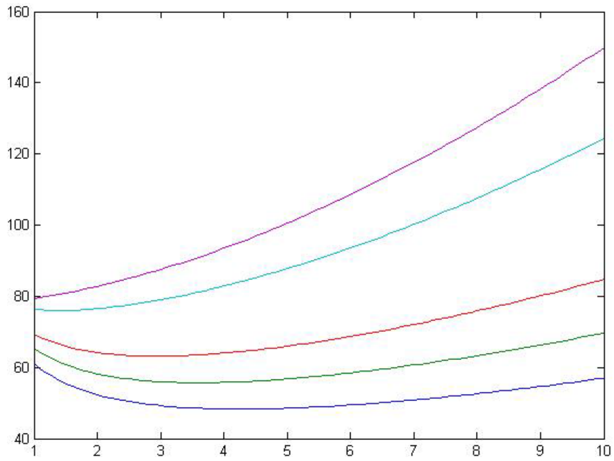

Figure 1: Solutions $\delta_\alpha(x)$ for different values of α when $C_1 = 88.4$, $C_2 = 66.6$ and $\lambda = 3$ [Here $\delta_{0.5}(x)$ is the lowermost graph]

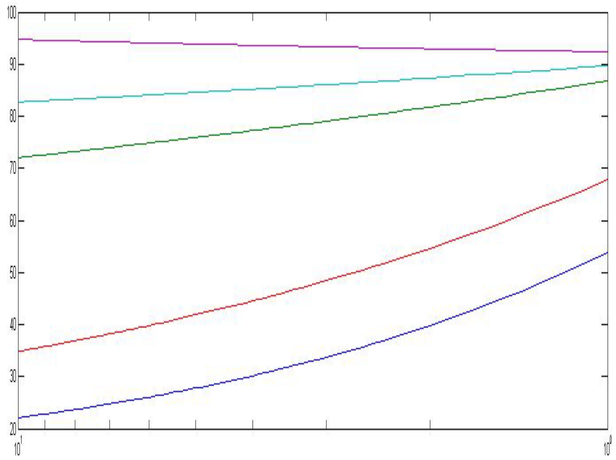

Figure 2: Solutions $y_{\beta_2}(x)$ for different values of β_2 when $C_2 = 88.4$ and $\lambda = 3$ [Here $y_1(x)$ is the uppermost graph and $y_{\beta_2}(x)$ is approaching $y_1(x)$ as $\beta_2 \to 1$]

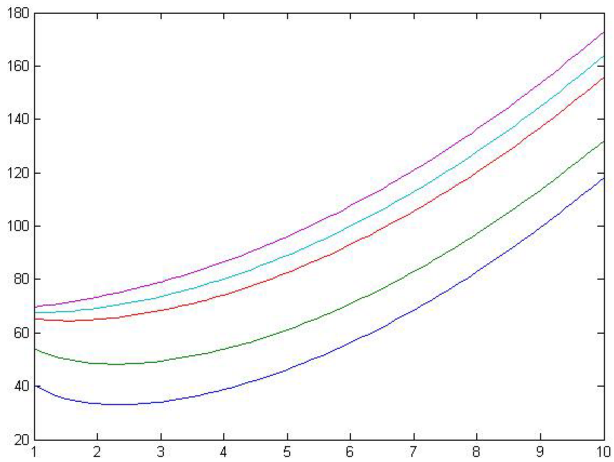

Figure 3: Solutions $\eta_{\beta_1}(x)$ for different values of β_1 when $C_1 = 66.4$ and $\lambda = 2$ [Here $\eta_1(x)$ is the lowermost graph and $\eta_{\beta_1}(x)$ is approaching $\eta_1(x)$ as $\beta_1 \to 1$]

5. Conclusion and Observations

In this paper, a general integral operator involving $\overline{H}$-function in its kernel was investigated. A solution of a fractional differential equation was obtained using this integral operator. Our main result generalizes the results obtained recently by Srivastava et al. [14] and Srivastava and Tomovski [15]. Further, corollaries and examples of the main result have been derived. By using some of these corollaries and examples, graphical illustrations are presented and it is found that the graphs (see Figure 2 and Figure 3) given here are quite comparable to the corresponding physical phenomena involving ordinary calculus, especially when the parameters $\beta_1 > 0$, $\beta_2 > 0$ get closer and closer to an integer. It can be concluded from the graphs that fractional calculus approach leads us to study a broader spectrum of area in any physical phenomenon as compared to the corresponding physical processes in ordinary calculus.

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