Second Hankel determinant obtained with New Integral Operator defined by Polylogarithm Function

S. M. Patil

Department of Applied Sciences,
SSVPS B.S. Deore College of Engineering,
Deopur, Dhule, Maharashtra, India.

S. M. Khairnar

Professor & Head, Department of Engineering Sciences,
MIT Academy of Engineering, Alandi,
Pune-412105, Maharashtra, INDIA.

Abstract

By using polylogarithm function, a new integral operator is introduced. By using this operator a new subclass of analytic functions are introduced for these classes we obtained sharp upper bounds for functional $|a_2 a_4 - a_3^2|$.

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1. Introduction

In 1966, Pommerenke stated the $q^{th}$ Hankel determinant for $q \leq 1$, & $n \leq 0$ as

$$H_q(n) = \left| \begin{array}{cccc}
a_n & a_{n+1} & \cdots & a_{n+q+1} \\
a_{n+1} & \vdots & \ddots & \vdots \\
a_{n+q-1} & \vdots & \ddots & a_{n+2q-2} \\
\end{array} \right| \quad (1.1)$$

where $a_n$'s are the coefficients of various power of $z$ in $f(z)$. 
This determinant has also been considered by several authors. For example Noor determined the rate of growth of \( H_q(n) \) an \( n \to \infty \) for function \( f \), with bounded boundary.

One can easily observe that Fekete and Szegö functional \( H_2(1) \). Fekete and Szegö then further generalized and estimate \( |a_3 - \mu a_2^2| \) where \( \mu \) is real & \( f \in S \).

We consider the Hankel determinant for the case \( q = 2 \) and \( n = 2 \),

\[
H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2| \quad (1.2)
\]

We recall here the definition of well known generalization of the polylogarithm function \( \phi(a, z) \) given by

\[
\phi(a, z) = \sum_{k=1}^{\infty} \frac{z^k}{ka} \quad (a \in \mathbb{N}, z \in \mathbb{E}) \quad (1.3)
\]

Let \( \phi_{\delta}(a, z) \) denote the well known generalization of the Riemann Zeta & polylogarithm function or simply the \( \delta^{th} \) order polylogarithm function given by,

\[
\phi_{\delta}(a, z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+a)^\delta} \quad (1.4)
\]

where any term with \( k+a = 0 \) is excluded.

Using the definition of the Gamma function, a simply transformation produces the integral formula,

\[
\phi_{\delta}(a, z) = \frac{1}{\Gamma(\delta)} \int_0^1 t^{a-1} \left( \log \frac{1}{t} \right)^{\delta-1} f(tz) \, dt \quad \Re a > -1, \Re \delta > 1 \quad (1.5)
\]

Note that \( \phi_{-1}(0, z) = \frac{z}{(1-z)^2} \) is koebe function for more details about polylogarithms in theory of univalent functions see Punnusamy & Sabapathy. Recently Khalifa Alshaqsi introduced a certain Integral Operator \( I_{\delta}^a \) defined by,

\[
I_{\delta}^a f(z) = \frac{(1+a)^\delta}{\Gamma(\delta)} \int_0^1 t^{a-1} \left( \log \frac{1}{t} \right)^{\delta-1} f(tz) \, dt \quad a > 0, \delta > 1, z \in \mathbb{E} \quad (1.6)
\]

We also note that the operator \( I_{\delta}^a f(z) \) defined by \([1]\) can be expressed by the series expansion as follows,

\[
I_{\delta}^a f(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+a}{k+a} \right)^{\delta} a_k z^k \quad (1.7)
\]

obviously, we have for \( (\delta, \lambda \leq 0) \)

\[
I_{\delta}^a (I_{\delta}^\lambda f(z)) = I_{\delta}^{\delta+\lambda} f(z) \quad (1.8)
\]
and

\[ I_α^δ (zf'(z)) = z(I_α^δ f(z))' \]  \hspace{1cm} (1.9)

Moreover from (1.7) it follows that,

\[ z(I_α^{δ+1} f(z))' = (a + 1) I_α^δ f(z) - a I_α^{δ+1} f(z) \]  \hspace{1cm} (1.10)

We note that,

- For \( a = 0 \) and \( δ = n \) (\( n \) is any integer) the multiplier transformation \( I_0^n f(z) = I^n f(z) \) was studied by Flett and Salagean.

- For \( a = 0 \) and \( δ = -n \) (\( n \in \mathbb{N}_0\{0, 1, 2, 3, \ldots\} \)), the differential operator \( I_0^{-n} f(z) = D^a f(z) \) was studied by Salagen.

- For \( a = 1 \) and \( δ = n \) (\( n \) is any integer) the operator \( I_1^n f(z) = I^n f(z) \) was studied by Uralegaddi and Somanatha.

- For \( a = 1 \), the multiplier transformation \( I_1^f f(z) = I^f f(z) \) was studied by Jung et al.

- For \( a = k - 1 \) (\( k \gg 1 \)), the integral operator \( I_{k-1}^δ f(z) \), \( I_{k-1}^δ f(z) \) was studied by Komatu using the operator \( I_α^δ \), we now introduced the following cases.

**Definition 1.1.** We say that a function \( f \in A \) is in the class \( S_{αδ}(b) \) if

\[ \Re\left\{ 1 + \frac{b}{1} \left( \frac{I_α^δ f(z)}{I_α^δ f(z)} - 1 \right)^' \right\} > 0 \hspace{1cm} (a > 0, \delta \leq 0, b \in \mathbb{C}\{0\}; z \in \mathbb{E}) \]  \hspace{1cm} (1.11)

**Definition 1.2.** We say that a function \( f \in A \) is in class \( C_{αδ}(b) \) if

\[ \Re\left\{ 1 + \frac{1}{p} \left( \frac{I_α^δ f(z)}{I_α^δ f(z)} \right)^'' \right\} > 0 \hspace{1cm} (a > 0, \delta \leq 0, b \in \mathbb{C}\{0\}; z \in \mathbb{E}) \]  \hspace{1cm} (1.12)

Note that

\[ f \in C_{αδ}(b) \iff zf' \in S_{αδ}(b) \]  \hspace{1cm} (1.13)

In particular, we have starlike & convex function classes \( S_{α0}(1) = S^* \) and \( C_{α0}(1) = C \) respective.

**Lemma 1.3.** Let \( p \in \mathbb{P} \) then \( |c_k| \leq 2, k = 1, 2, \ldots \) and the inequality is sharp.
Lemma 1.4. Let \( p \in \mathbb{P} \) then
\[
2c_2 = c_1^2 + x(4 - c_1^2) \\
4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2)
\]
for some \( x \) and \( y \) such that \( |x| \leq 1, |y| \leq 1 \).

Theorem 1.5. If \( f \in S_{a\delta}(b) \) then
\[
|a_2a_4 - a_3^2| \leq b^2 \left( \frac{a + 3}{a + 1} \right)^{2\delta}
\]

Proof. By the definition of the class \( S_{a\delta}(b) \), there exist \( p \in P \) such that,
\[
1 + \frac{1}{p} \left( \frac{z(I_a^\delta f(z))'}{I_a^\delta f(z)} - 1 \right) = p(z)
\]
\[
\therefore z(I_a^\delta f(z))' = 1 - b + bp(z)
\]
Let \( I_a^\delta = z + A_2z^2 + a_3z^3 + \ldots \), where,
\[
A_2 = \left( \frac{a + 1}{a + 2} \right)^\delta a_2 \\
A_3 = \left( \frac{a + 1}{a + 2} \right)^\delta a_3 \\
A_4 = \left( \frac{a + 1}{a + 2} \right)^\delta a_4
\]
so that,
\[
\frac{z[1 + 2A_2z + 3A_3z^2 + 4A_4z^3 + \ldots]}{z + A_2z^2 + A_3z^3 + \ldots} = 1 - b + b[1 + c_1z + c_2z^2 + c_3z^3] 
\]
Simplify and equating the coefficient of \( z^2 \) on both side
\[
2A_2 = A_2 - A_2b + A_b + bc_1 \\
\therefore A_2 = bc_1
\]
\[
A_2 = \left( \frac{a + 2}{a + 1} \right)^\delta bc_1
\]
Equating the coefficient of \( z^3 \) on both side
\[
A_3 = \frac{b}{2}[c_2 + 2bc_1^2] \\
\therefore a_3 = \left[ \frac{b}{2}(c_2 + 2bc_1^2) \left( \frac{a + 2}{a + 1} \right)^\delta \right]
\]
Equating the coefficients of $z^4$:

$$3A_4 = \frac{b62c_1c_2}{2} + \frac{b^2c_1^3}{2}b^2c_1c_2 + bc_3$$

$$\therefore A_4 = \frac{3b^2c_1c_2 + b^2c_1^3 + 2bc_3}{6}$$

$$\therefore a_4 = \left(\frac{a + 4}{a + 1}\right)^{\delta} \frac{(3b^2c_1c_2 + b^2c_1^3 + 2bc_3)}{6}$$

It is establish that,

$$|a_2a_4 - a_3^2| = \left|\left(\frac{a + 2}{a + 1}\right)^{\delta} \left(\frac{a + 4}{a + 1}\right)^{\delta} \frac{(3b^2c_2^2c_2 + b^2c_4 + 2bc_1c_3)}{6} - \left(\left(\frac{a + 3^{\delta}}{a + 1}\right)^{\delta} b\right)^2 \left(c_2 + bc_1^2\right)\right|^2$$

By using Lemma,

$$c_2 = c_1^2 + \frac{x(4 - c_1^2)}{2} \text{ for some } |x| \leq 1$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)x$$

for some real value of $z$ with $|z| \leq 1$

$$|a_2a_4 - a_3^2| \leq \left|\left(\frac{a + 2}{a + 1}\right)^{\delta} \left(\frac{a + 4}{a + 1}\right)^{\delta} \frac{1}{6} \left[3b^2c_2^2\left(c_1^2 + \frac{x(4 - c_1^2)}{2}\right) + b^2c_4 + 2bc_1\right] - \left(\left(\frac{a + 3^{\delta}}{a + 1}\right)^{\delta} b\right)^2 \left[c_2 + \frac{x(4 - c_1^2)}{2} + bc_1^2\right]\right|^2$$

Since $|c_1| \leq 2$, $c_1 = c$ assume without restriction, $c \in [0, 2]$, we obtain by using triangle inequality $|x| \leq 1 = \rho$. $c_1 = c$.

$$|a_2a_4 - a_3^2| \leq \left|\left(\frac{a + 2}{a + 1}\right)^{\delta} \left(\frac{a + 4}{a + 1}\right)^{\delta} \frac{1}{6} \left[3b^2c_2^2 + \frac{3b^2c_2^2(4 - c_2^2)\rho}{2} + b^2c_4 + 2b\right]ight|$$

$$\left[c_3^3 + 2c(4 - c_2^2) - \rho^2(4 - c_2^2)\rho\right] + \left(\frac{a + 3^{\delta}}{a + 1}\right)^{\delta^2} b^2 \left[c_2 + \frac{x(4 - c_1^2)}{2} + bc_1^2\right]^2$$

$$\leq F(\rho)$$

$$\text{(1.23)}$$
\[ F'(\rho) = \left\{ \left( \frac{a+2}{a+1} \right)^{\delta} \left( \frac{a+4}{a+1} \right)^{\delta} 1 \left( \frac{3b^2c^2(4-c^2)}{2} \right) + \frac{bc}{2} \left[ 2c(4-c^2) - 2\rho(4-c^2)\rho \right] \right\} \]
\[ + \left( \frac{a+3}{a+1} \right)^{2\delta} b^2 \left[ \frac{\rho(4-c^2)}{2} \right]^2 \right\} \}

(1.24)

\[ F'(\rho) > 0 \text{ for } \rho > 0, \text{ implies that } F \text{ is an increasing function. The upper bound for } (5) \text{ are } c = 0. \]

\[ |a_2a_4 - a_3^2| \leq b^2 \left( \frac{a+3}{a+1} \right)^{2\delta} \]

Corollary 1.6.

\[ S_{a0}(1) = S^* \]

If we put \( \delta = 0, b = 1 \) we get,

\[ |a_2a_4 - a_3^2| \leq 1 \]

this result is coincide the result with Janteng.

Corollary 1.7. If \( f \in c_{a\delta} (b) \) then,

\[ 1 + \frac{1}{b} \left( z(I_{a\delta} f(z)'') \right) = p(z) \]

\[ \therefore \left( \frac{z(I_{a\delta} f(z)''')}{(I_{a\delta} f(z)''')} \right) = bp(z) - b \]

Simplify & equating the coefficients we get,

\[ a_2 = \frac{bc_1}{2} \left( \frac{a+2}{a+1} \right)^{\delta} \]

\[ a_3 = \left( \frac{a+3}{a+1} \right)^{\delta} \left[ \frac{b^2c_1^2 + bc_2}{6} \right] \]

\[ a_4 = \left( \frac{a+4}{a+1} \right)^{\delta} \left[ \frac{3b^2c_1c_2 + b^2c_1^3 + 2bc_3}{24} \right] \]

(1.26)

and calculate the same \( |a_2a_4 - a_3^2| \) then we get the result of convex function. After simplification we put \( \delta = 0, b = 1 \) then we get \( |a_2a_4 - a_3^2| \leq \frac{1}{8} \) results is coincide the result with Janteng.
References


