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An M/M/1/N Queuing system with Encouraged Arrivals

Bhupender Kumar Som
Associate Professor, JIMS Rohini, Sec -5, New Delhi – 110085

Sunny Seth
Assistant Professor, JIMS Rohini, Sec -5, New Delhi – 110085

Abstract

In this paper, we develop a single-server finite capacity Markovian queuing system with encouraged arrivals. The model is solved in steady-state recursively. Necessary measures of performance are drawn. Economic analysis of the model is presented by developing a cost model. The model is studied numerically and simulated arbitrarily. The term encouraged arrivals emerged from situation that a system experiences after release of offers and discounts by firms. Encouraged arrivals is a new addition to existing customer behaviour in queuing theory.

Keywords: encouraged arrivals, stochastic models, queuing theory

1. INTRODUCTION AND LITERATURE SURVEY

(AK Erlang, 1909) introduced queuing theory as an area of research. In his work he created models to describe the Copenhagen telephone exchange. While developing a solution to the problem he began to realize the applicability of queuing systems in many other fields and began to develop queuing theory further. He is now considered as the father of Queuing theory. In queuing systems, it is important to understand the customer behaviour. Customers generally behave in the following ways: 1) Balking 2) Reneging 3) Jockeying 4) Collusion. Detailed literature on customer behaviour can be seen in the pioneer works of (Haight, 1957, 1959), (Ancker and Gafarian, 1963). From business point of view, dynamic nature of competitive business environment and uncertain customer behaviour drives organisations to study customer behaviour. This helps the organizations in making policies to engage new customers. The margin of error is so low, that a slight compromise with the strategy may result in loss to the business.

In order to engage new customers firms often release various discounts and offers. Let it be online stores or offline stores. Customers also keep on looking for lucrative deals and discounts offered by various organizations and get attracted to visit the particular organization. These attracted customers are termed as *encouraged arrivals* in this paper. The term *encouraged arrivals* contribute to the fundamental queuing literature. Notion of customer mobilisation was introduced by (Jain, et. al., 2014) they called it *reverse balking*. They mentioned that an arriving customer gets attracted towards a system by looking in to large customer base. Whereas reverse balking deals with probability of joining and not joining the system, encouraged arrivals deal with percentage change in customers due to deals and discounts. The phenomenon of encouraged arrivals can also be understood as contrary to discouraged arrivals discussed by (Kumar et. al., 2014). In their work they mentioned that discouraged arrivals occur in accordance to Poisson process defined by parameter $\frac{\lambda}{n+1}$. They mentioned that customers are discouraged to join once they look in to large system size. (Raynolds, 1968) presents multi-server queuing model with discouragement. He obtained equilibrium distribution of queue length and derived other performance measures from it. (Natvig, 1975) studied single server birth-death queuing process with state dependent parameters, $\lambda_n = \frac{\lambda}{n+1}, n \geq 0$ and $\mu_n = \mu, n \geq 1$.

Literature review shows that no researcher has developed a model with encouraged arrivals. To address the above discussed contemporary practical aspect, we develop a single-server Markovian queuing system with encouraged arrivals in this paper. Rest of the paper is organized as per following details. Section 2 deals with mathematical model formulation while section 3 presents solution of the model in steady-state. Measures of performance are derived in section 4. Numerical illustration is presented in section 5. Section 6 deals with economic analysis of the model. Special case of no encouragement is presented in section 7. Conclusion and future scope are given in section 8.

2. MATHEMATICAL MODEL FORMULATION

A single-server Markovian queuing model is formulated under following assumptions:

(i) The arrivals occur one by one in accordance to Poisson process with parameter $\lambda(1+\eta)$, where 'η' represents the percentage change in number of customers calculated from past or observed data. For instance, if in past an organization offered discounts and the percentage change in number of customers was observed + 50% or 120% then $\eta = 0.5$ or $\eta = 1.2$ respectively.

(ii) Service times are exponentially distributed with parameter μ.

(iii) Customers are serviced in order of their arrival i.e. the queue discipline is first come first served.

(iv) There is a single server through which the service is provided.

(v) The capacity of the system is finite say, N.

Differential difference equations governing the model are given by:

$$\frac{d}{dt}P_0(t) = -\lambda(1+\eta)P_0(t) + \mu P_1(t) \quad \dots (1)$$

$$\frac{d}{dt}P_n(t) = \lambda(1+\eta)P_{n-1}(t) - \{\lambda(1+\eta)+\mu\}P_n(t) + \mu P_{n+1}(t) \quad \dots (2)$$

$$\frac{d}{dt}P_N(t) = \lambda(1+\eta)P_{N-1}(t) - \mu P_N(t) \quad \dots (3)$$

In steady state, $as\ t \rightarrow \infty, P_n(t) = P_n$ and therefore, $\frac{d}{dt}P_n(t) = 0\ as\ t \rightarrow \infty$ and hence, equations (1) – (3) become;

$$0 = -\lambda(1+\eta)P_0 + \mu P_1 \quad \dots (4)$$

$$0 = \lambda(1+\eta)P_{n-1} - \{\lambda(1+\eta)+\mu\}P_n + \mu P_{n+1} \quad \dots (5)$$

$$0 = \lambda(1+\eta)P_{N-1} - \mu P_N \quad \dots (6)$$

3. STEADY-STATE SOLUTION

On solving (4) - (6) iteratively we get;

$$P_n = \Pr\{n\ customers\ in\ the\ system\} = \left[\frac{\lambda(1+\eta)}{\mu}\right]^n P_0\ ;\ \ 1 \le n \le N-1 \quad \dots (7)$$

And the probability that system is full is given by:

$$P_N = Pr\{sytem\ is\ full\} = \left[\frac{\lambda(1+\eta)}{\mu}\right]^N P_0 \quad \dots (8)$$

Using condition of normality $\sum_{n=0}^{N} P_n = 1$

$$P_0 = Pr\{system\ is\ empty\} = \left\{1 + \sum_{n=1}^{N}\left[\frac{\lambda(1+\eta)}{\mu}\right]^n\right\}^{-1}$$

$$= \frac{1-\left(\frac{\lambda(1+\eta)}{\mu}\right)}{1-\left(\frac{\lambda(1+\eta)}{\mu}\right)^{N+1}} \quad \dots (9)$$

4. MEASURES OF PERFORMANCE

1. Expected System Size (L_s)

$$L_s = \sum_{n=1}^{N} nP_n$$

$$L_s = \sum_{n=1}^{N} n\left[\frac{\lambda(1+\eta)}{\mu}\right]^n P_0 \qquad \dots(10)$$

2. Expected queue length (L_q)

$$L_q = \sum_{n=1}^{N} (n-1)P_n$$

$$L_q = \sum_{n=1}^{N} (n-1)\left[\frac{\lambda(1+\eta)}{\mu}\right]^n P_0 \qquad \dots(11)$$

5. NUMERICAL ILLUSTRATION

In this section we present numerical illustration of the above model.

Variation in L_s *and* L_q with respect to λ

We take, $N = 10,\ \mu = 3,\ \eta = 0.5$

Table -1

Average rate of arrival (λ)	Expected System Size (Ls)	Expected Queue Length (L_q)
2	5	4.090909
2.1	5.485556	4.555945
2.2	5.935946	4.989909
2.3	6.345059	5.386128
2.4	6.710709	5.741812
2.5	7.033686	6.057179
2.6	7.316723	6.334452
2.7	7.563606	6.576994

2.8	7.778512	6.788639
2.9	7.965578	6.973261
3	8.128659	7.134507
3.1	8.271202	7.275671
3.2	8.396219	7.399649
3.3	8.506293	7.508937
3.4	8.603619	7.605667
3.5	8.690049	7.691644
3.6	8.767142	7.768389

Source: simulated data

Above table shows that with increase in arrival rate the expected system size and expected length of queue, both increases. Following graph explains the phenomenon.

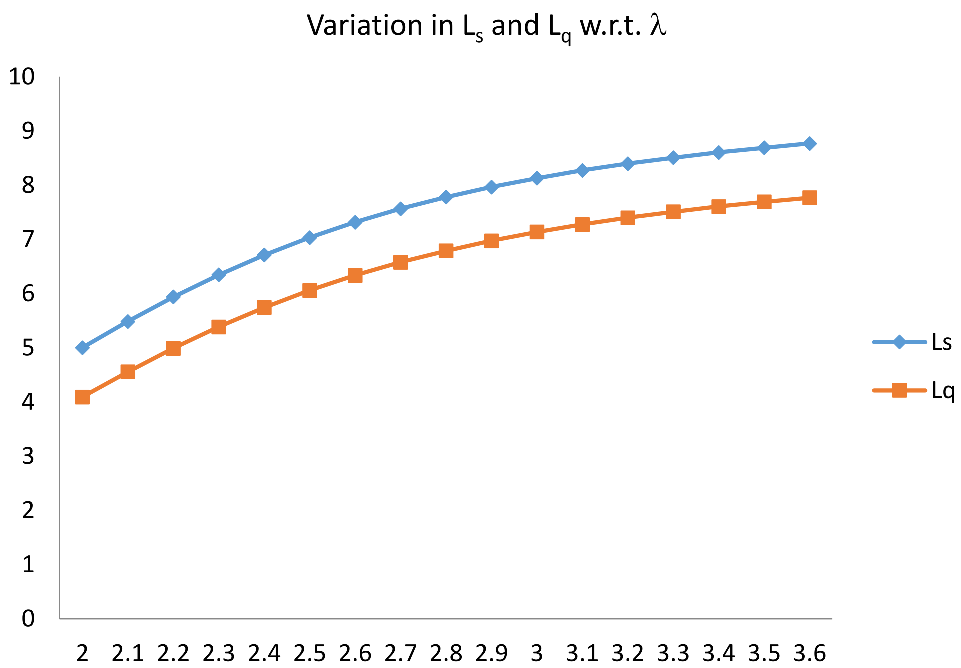

Figure -1

Similarly, the numerical results are obtained by varying service rate

Variation in L_s *and* L_q with respect to μ

We take, $N = 10,\ \lambda = 3,\ \eta = 0.5$

Table -2

Average rate of service (μ)	Expected System Size (Ls)	Expected Queue Length (L_q)
3	8.128659	7.134507
3.1	7.971194	6.978809
3.2	7.803335	6.813117
3.3	7.625211	6.637614
3.4	7.437155	6.452687
3.5	7.239722	6.258936
3.6	7.033686	6.057179
3.7	6.820031	5.848435
3.8	6.599936	5.633906
3.9	6.374734	5.414940
4	6.14588	5.192993
4.1	5.91490	4.969578
4.2	5.683341	4.746220
4.3	5.452724	4.524406
4.4	5.224498	4.305540
4.5	5	4.090909
4.6	4.780427	3.881651

Source: simulated data

Above table shows that with increase in service rate the expected system size decreases and so as expected length of queue. Following graph explains the phenomenon.

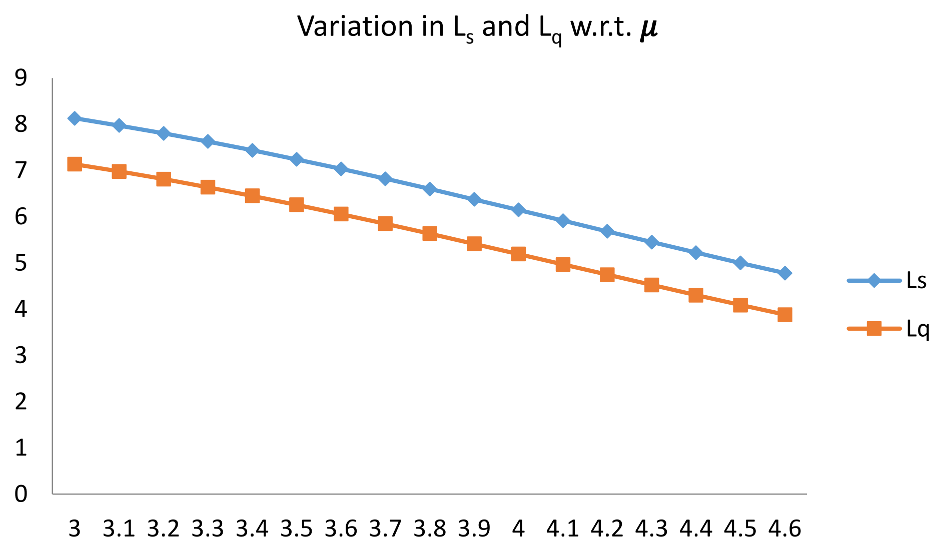

Figure -2

6. ECONOMIC ANALYSIS

Economic analysis of the model is discussed by developing total expected cost (TEC), total expected revenue (TER) and total expected profit (TEP) functions.

Total expected cost of the system (TEC) is given by:

$$TEC = C_s\mu + C_h \sum_{n=1}^{N} n \left[\frac{\lambda(1+\eta)}{\mu}\right]^n P_0 + +C_L\lambda \left[\frac{\lambda(1+\eta)}{\mu}\right]^N P_0$$

Where, $$P_0 = \frac{1-\left(\frac{\lambda(1+\eta)}{\mu}\right)}{1-\left(\frac{\lambda(1+\eta)}{\mu}\right)^{N+1}}$$

Total expected revenue (TER) of the system is given by:

$$TER = R \times \mu \times (1 - P_0)$$

Total expected profit (TEP) of the system is given by;

$$TEP = TER - TEC$$

Where,

C_s= Cost per service per unit time

C_h= holding cost per unit per unit time

C_L= Cost associated to each lost unit per unit time

R= Revenue earned per unit per unit time

Om the cost model formulated above sensitivity analysis is performed for varying rates of arrival and service.

Table -3

Variation in TEC, TER and TEP with respect to λ

We take, $N = 10, \mu = 3, \ \eta = 0.5, C_S = 10, C_L = 15, C_h = 2, R = 200$

Average rate of arrival (λ)	Total Expected Cost (TEC)	Total Expected Revenue (TER)	Total Expected Profit (TEP)
2	42.72727	545.4545	502.7273
2.1	44.58278	557.7667	513.1839
2.2	46.49079	567.6221	521.1313
2.3	48.42219	575.3586	526.9364
2.4	50.35453	581.3377	530.9832
2.5	52.27216	585.9043	533.6321
2.6	54.16531	589.3627	535.1974
2.7	56.02884	591.9675	535.9387
2.8	57.86085	593.9234	536.0625
2.9	59.66164	595.3904	535.7287
3	61.43276	596.4911	535.0584
3.1	63.17648	597.3185	534.142
3.2	64.89534	597.9419	533.0466
3.3	66.59192	598.4133	531.8213
3.4	68.26869	598.7709	530.5022
3.5	69.92793	599.0434	529.1155
3.6	71.57169	599.2520	527.6803

Source: simulated data

Above table shows that the total expected profit increases with increase in average rate of arrival, reaches a maximum value at certain level and starts falling down. This is because of the fact that service rate being fixed, after certain level with increasing load on service, cost increases rapidly than revenue.

Table -4

Variation in TEC, TER and TEP with respect to μ

We take, $N = 10, \lambda = 3,\ \eta = 0.5, C_S = 13, C_L = 15, C_h = 2, R = 200$

Average rate of service (μ)	Total Expected Cost (TEC)	Total Expected Revenue (TER)	Total Expected Profit (TEP)
3	70.43276	596.4911	526.0584
3.1	70.47845	615.2787	544.8003
3.2	70.5197	633.7394	563.2197
3.3	70.55974	651.8136	581.2538
3.4	70.60237	669.4387	598.8363
3.5	70.65192	686.5505	615.8986
3.6	70.71311	703.0851	632.372
3.7	70.79099	718.9814	648.1904
3.8	70.89074	734.1826	663.2919
3.9	71.01751	748.6393	677.6218
4	71.17625	762.3102	691.1339
4.1	71.37158	775.1644	703.7928
4.2	71.60759	787.1818	715.5742
4.3	71.88776	798.3537	726.4660
4.4	72.21486	808.6827	736.4679
4.5	72.59091	818.1818	745.5909
4.6	73.01718	826.8735	753.8563

Source: simulated data

The table shows that the revenue goes high and firms profit keeps on increasing with an improving rate of service.

7. SPECIAL CASE

When, $\eta = 0$,

$$P_n = \Pr\{n\ customers\ in\ the\ system\} = \left(\frac{\lambda}{\mu}\right)^n P_0 \ = \ \rho^n P_0;\ \ 1 \leq n\ \leq N - 1$$

$$P_N = Pr\{sytem\ is\ full\} = \left(\frac{\lambda}{\mu}\right)^N P_0 = \rho^N P_0$$

,

$$P_0 = Pr\{system\ is\ empty\} = \left\{1 + \sum_{n=1}^{N}\left(\frac{\lambda}{\mu}\right)^n\right\}^{-1}$$

$$= \frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} = \frac{1-\rho}{1-\rho^{N+1}}$$

Where $\rho = \frac{\lambda}{\mu} < 1$, is the traffic intensity.

The system reduces to classical single-server queuing model with finite capacity.

8. CONCLUSIONS AND FUTURE SCOPE

This paper studies a single server queuing model with encouraged arrivals. The results of the paper are of immense use for any organization encountering the phenomenon of encouraged customers and load on service. By adopting and implementing this model an effective strategy can be planned the economic analysis of the facility can be measured and financial aspect of the business can also be observed.

Further optimization of service rate and system size can be achieved and the system can be studied in transient state. The system can also be studied for heterogeneous service. A multi-server model can also be developed. While the system can also be studied with infinite capacity.

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