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Reduction of Domination Parameter in Fuzzy Graphs

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Abstract

In this paper, the cobondage set and cobondage number $b_C(G)$ of a fuzzy graph G are defined. The upper bound of a fuzzy graph with p nodes is given as $min(p - (1 + \triangle_S(G)), \triangle_S(G) + 1)$. The exact values of $b_C(G)$ for some standard fuzzy graphs are found. Some results on $b_C(G)$ are also discussed. A numerical example is also illustrated.

AMS subject classification: 03E72, 05C40, 05C72.
Keywords: Strong arcs, cobondage set and cobondage number.

1. Introduction

Euler first introduced the concept of graph theory, in the year 1736. He created the first graph to simulate a real time place and situation to solve a problem which is known as Seven Bridges of Konigsberg. The study of dominating sets in graphs was started by Ore and Berge [1, 11]. The domination number and the independence number was introduced by Cockayne and Hedetniemi [4]. In 1996, the concept of the cobondage number in graphs was introduced by Janakiram and Kulli [6].

Fuzzy logic and the theory of fuzzy sets have been applied widely in areas like information theory, pattern recognition, clustering, expert systems, database theory, control theory, robotics, networks and nanotechnology. A Mathematical framework to describe the phenomena of uncertainty in real life situation is first suggested by L.A. Zadeh [15] in 1965. Zadeh published his seminal paper on Fuzzy Sets which described fuzzy set theory and consequently fuzzy logic. His aim was to develop a mathematical theory to deal with uncertainty and imprecision.

Based on Zadeh's fuzzy relations,the first definition of a fuzzy graph was given by Kaufmann in 1973. But it was Ariel Rosenfeld [12] who considered fuzzy relations on fuzzy sets and introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields. The concept of complete fuzzy graph was investigated by Sunitha and Vijayakumar in 2002. In the year 2003, Bhutani and Rosenfeld [3] introduced the concept of strong arcs, fuzzy end nodes and geodesics in fuzzy graph. The study on $\mathrm{M}-$ Strong fuzzy graphs was done by Bhutani and Battou [2]. Samanta and Madhumangal Pal [13] gave a new approach to social networks using fuzzy graphs in the year 2014. Rashmanlou, Samanta, Madhumangal Pal and Borzooeia[5] made a study on Bipolar fuzzy graphs in the year 2015.

The concept of domination in fuzzy graphs using effective edges was introduced by A.Somasundram and S.Somasundram [14] in 1998. Then Nagoor Gani and Chandrasekaran [7] discussed domination in fuzzy graph using strong arcs. Nagoor Gani and Vadivel [10] discussed domination, independent domination and irredundance in fuzzy graphs using strong arcs. Nagoor Gani and Prasanna Devi [9] discussed edge domination and edge independence in fuzzy graphs. Nagoor Gani, Prasanna Devi and Muhammed Akram [8] also introduced bondage number and non-bondage number of a fuzzy graph. The purpose of this paper is to reduce the domination number of a fuzzy graph by introducing new arcs to it. In this paper we discussed about addition of strong arcs to a fuzzy graph and also about cobondage set and cobondage number of a fuzzy graph with examples. We also discuss about the cobondage number of specific standard fuzzy graphs.

2. Preliminaries

The *cobondage number of a graph G* is the minimum cardinality of a set of edges whose addition to G results in a graph with domination number lesser than that of G.

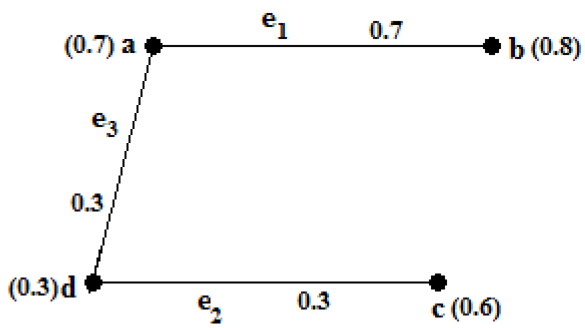
A *fuzzy graph* $G =< \sigma, \mu >$ is a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, where for all $x, y \in V$, we have $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. The *underlying crisp graph* of a fuzzy graph $G =< \sigma, \mu >$ is denoted by $G^* =< \sigma^*, \mu^* >$, where $\sigma^* = \{v_i \in V/\sigma(v_i) > 0\}$ and $\mu^* = \{(v_i, v_j) \in V \times V/\mu(v_i, v_j) > 0\}$. An edge in G is called an *isolated edge* if it is not adjacent to any edge in G. A node in G is called an *isolated node* if it is not adjacent to any node in G. A *path* with n vertices in a fuzzy graph is denoted as P_n. A fuzzy graph $G =< \sigma, \mu >$ is a *complete* *fuzzy graph* if $\mu(v_i, v_j) = \sigma(v_i) \wedge \sigma(v_j)$ for all $v_i, v_j \in \sigma^*$. An arc (x,y) in a fuzzy graph $G =< \sigma, \mu >$ is said to be *strong* if $\mu^\infty(x, y) = \mu(x, y)$. *The maximum cardinality of strong neighbourhood* $\triangle_S(G) = max\{| N_S(u) | /u \in V(G)\}$ where $N_S(u) = \{v \in V : (u, v)$ is a strong arc$\}$. A subset D of V is called a *dominating set* of a fuzzy graph G if for every $v \in V - D$, there exist u $\in$ D such that u dominates v. The *domination number,γ(G)*, is the smallest number of nodes in any dominating set of G.

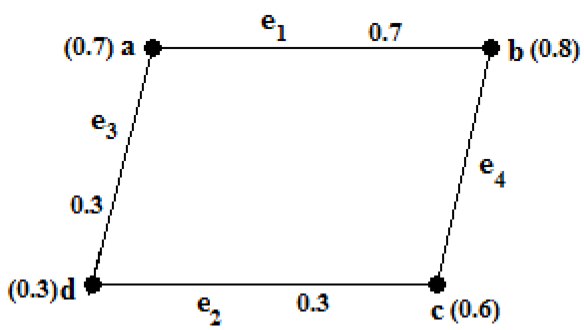
3. Addition of strong arcs

In this section we discuss about the addition of strong arcs. If we add an arc e to the fuzzy graph G then the resulting fuzzy graph will be denoted as G+e.

Definition 3.1. An arc e is added to the fuzzy graph is said to be ***addition of a strong arc*** if e is a strong arc in the resulting fuzzy graph G+e and $\mu(e) \leq \sigma(u) \wedge \sigma(v)$, where u and v are end nodes of e i.e., $e = (u, v)$.

Example 3.2.





(G) **Fig. 3.1(a)**

(G+e_4) **Fig.3.1(b)**

Here $0 < \mu(e_4) \leq 0.6$
If $\mu(e_4)$ lies between [0.3, 0.6] i.e., $0.3 \leq \mu(e_3) \leq 0.6$, then e_4 is a strong arc.

Remark 3.3. In the above definition, addition of a strong arc, if only the condition $\mu(e) \leq \sigma(u) \wedge \sigma(v)$, $e = (u, v)$ is satisfied then we say it as **addition of an arc**.

Theorem 3.4. If an arc e with membership value equal to the maximum membership value among all the arcs in G, is addition of an arc to the fuzzy graph G, then e is a strong arc in G+e.

In otherwords, if an arc e with $\mu(e) = max\{\mu(e^{'})/e^{'} \in \mu^{*})\}$, is addition of an arc to the fuzzy graph G then e is a strong arc in G+e.

Proof. Addition of an arc e to the fuzzy graph G with $\mu(e) = max\{\mu(e^{'})/e^{'} \in \mu^{*}\}$. Let $e = (u, v)$. Then $\mu(e) \leq \sigma(u) \wedge \sigma(v)$.

W.k.t $\mu^{\infty}(u, v)$ is the strength of strongest path between the nodes u and v. The arc e is a strongest path between u and v because $\mu(e) = max\{\mu(e^{'})/e^{'} \in \mu^{*}\}$.
And $\mu^{\infty}(u, v) \ngtr \mu(u, v)$, for $u, v \in V$ because e has the greatest membership value in G. Therefore $\mu(e) = \mu(u, v) = \mu^{\infty}(u, v)$.
$\Longrightarrow$ e is a strong arc in G+e. ■

Theorem 3.5. Let e be the addition of an arc to the fuzzy graph G and (G+e)* has exactly one cycle C formed by e. The arc e is not a strong arc if and only if e is the only one weakest arc in C.

Proof. Let C is the only one cycle in (G+e)* formed by $e = (u, v)$.
Assume that e is the only one weakest arc in C.
Then there exist 2 paths between u and v, one containing e and another not containing e. Since e is the only one weakest arc in C, the path between u and v not containing e is the strongest path with strength greater than $\mu(e)$. i.e., $\mu^{\infty}(u, v) \neq \mu(e) = \mu(u, v)$.
Therefore e is not a strong arc in G+e.
Conversely, assume that e is not a strong arc.
C is the cycle formed by e in (G+e)* then $\mu^{\infty}(u, v) > \mu(e)$ that is, there is a path not containing e whose strength is greater than $\mu(e)$.
Thus e is the only one weakest arc in the cycle C. ■

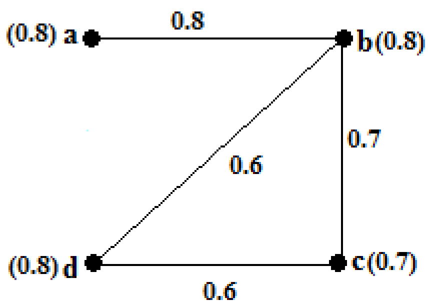
Theorem 3.6. If e is the addition of an arc to the fuzzy graph G which does not forms a cycle then e is a strong arc.

Proof. Let e be an arc added to the fuzzy graph G and it does not forms a cycle in (G+e)*. Let $e = (u, v)$. Since e does not forms a cycle in (G+e)* there exist only one path between the vertices u and v containing e. Thus e should be a strong arc in G+e. ■

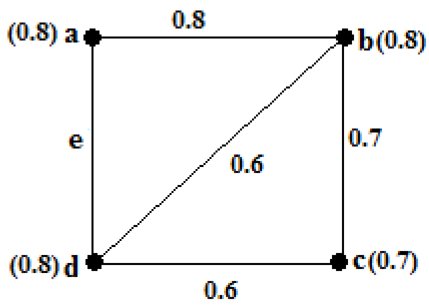
Remark 3.7.

1. If the set of arcs X is added to the fuzzy graph G then it is denoted as G+X.

2. Addition of a strong arc to a fuzzy graph G may change (increase or decrease) or unchange the domination number of G.



(G)
Fig. 3.2



(G+e)
Fig. 3.3

Here $\gamma(G) = 1$.
If $\mu(e) = 0.6$, then $\gamma(G + e) = 1$
If $\mu(e) = 0.8$, then $\gamma(G + 2) = 2$

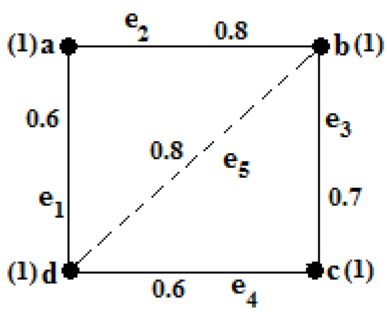
4. Cobondage number of a fuzzy graph

In this section we discuss about cobondage set and cobondage number of a fuzzy graph. The upper bound for the cobondage number of a fuzzy graph is also given.

Definition 4.1. The ***cobondage set*** of a fuzzy graph G is the set of strong arcs X whose addition to G reduces the domination number of G. i.e., $\gamma(G) > \gamma(G + X)$.

The ***cobondage number,*** $b_C(G)$, of a fuzzy graph G is the smallest number of arcs in any cobondage set of G.

Example 4.2.



(G)
Fig. 4.1

Here $\gamma(G) = 2$ and $\gamma(G + e_5) = 1$.
Thus $\{e_5\}$ is a cobondage set and $b_C(G) = 1$.

Theorem 4.3. For any fuzzy graph G, the upper bound of the cobondage number is $p-(1+\triangle_S(G))$, where p is the number of vertices in G. i.e., $b_C(G) \leq p-(1+\triangle_S(G))$.

Proof. Without loss of generality, choose a node $u \in V$ such that $\mid N_S(u) \mid = \triangle_S(G)$. Then u dominates $\triangle_S(G)$ nodes and itself. Thus the number of nodes dominated by u is $1 + \triangle_S(G)$ and the number of nodes not dominated by u is $p - (1 + \triangle_S(G))$
Therefore, $b_C(G) \leq p - (1 + \triangle_S(G))$. ∎

Theorem 4.4. If a fuzzy graph G has isolated nodes then $b_C(G) = 1$.

Proof. Let G be a fuzzy graph with isolated nodes. Let u be an isolated node in G.

Then u belongs to every minimum dominating set D of G. Let $v \in D - \{u\}$. Now addition of a strong arc e between u and v then $D - \{u\}$ is a dominating set of G+e with domination number $\gamma(G) - 1$. Therefore $b_C(G) = 1$. ∎

Corollary 4.5. If a fuzzy graph G has an isolated edge then $b_C(G) \leq 2$.

Proof. Let G be a fuzzy graph with an isolated edge $e = (u, v)$. Then u (or v) belongs to every minimum dominating set D of G. Now addition of two strong arcs from u and v to a node $w \in D$, $w \neq u$ and $w \neq v$ then w dominates both u and v. Thus $D - \{u\}$ (or $D - \{v\}$) is a dominating set of G+$\{uw, vw\}$.
$\Longrightarrow \gamma(G + \{uw, vw\}) < \gamma(G)$. Therefore $b_C(G) \leq 2$. ∎

Theorem 4.6. For any fuzzy graph G, the upper bound of the cobondage number is $\triangle_S(G) + 1$, i.e., $b_C(G) \leq \triangle_S(G) + 1$.

Proof. Let G be a fuzzy graph and D be a minimum dominating set of G.
Let $v \in D$ and S $\subseteq V - D$ such that for each vertex $w \in$ S, $N_S(w) \cap$ D $= \{v\}$.

If v has no strong neighbour in D, add strong arcs from each node in S$\cup\{v\}$ to some other node $u \in$ D$-\{v\}$ then the resulting fuzzy graph has the domination number $\gamma(G) - 1$. Thus $b_C(G) \leq\mid S \cup \{v\} \mid\leq \triangle_S(G) + 1$.

If v has a strong neighbour u in D, then add strong arcs from each node in S to u then we get the resulting fuzzy graph with domination number $\gamma(G) - 1$. Thus $b_C(G) \leq\mid S \mid\leq \triangle_S(G) - 1 \leq \triangle_S(G) + 1$. ∎

Corollary 4.7. For any fuzzy graph G,the upper bound for cobondage number is $min\{p-(1 + \triangle_S(G)), \triangle_S(G) + 1\}$. i.e.,$b_C(G) \leq min\{p - (1 + \triangle_S(G)), \triangle_S(G) + 1\}$.

Results 4.8.

1. For any fuzzy graph G, $b_C(G)+\gamma(G) \leq p+1$, where p is the number of vertices in G.

2. Let G be a fuzzy graph with n nodes, $n \geq 4$ and G^* be a cycle. Then

$$b_C(G) = \begin{matrix} 1, & if & n \equiv 1(mod \quad 3) \\ 2, & if & n \equiv 2(mod \quad 3) \\ 3, & & otherwise \end{matrix}$$

3. If G is a complete fuzzy graph then $b_C(G) = 0$.
4. If G is a fuzzy graph and G^* is a star then $b_C(G) = 0$.
5. If G is a complete bipartite fuzzy graph then $b_C(G) = min\{| V_1 |, | V_2 |\} - 1$, where V_1 and V_2 are the partitions of V.

5. Application

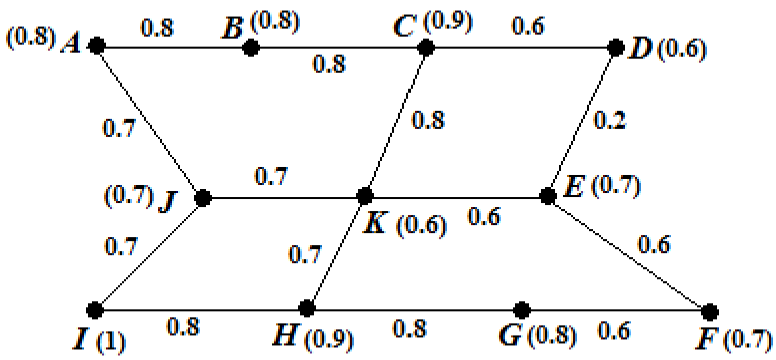
Many candidates will compete in a election and they will start their election campaign. Election campaign is the period of time immediately before an election when politicians try to persuade people to vote for them. The message of the campaign contains the ideas that the candidate wants to share with the voters.

A candidate will make the people convince to support him/him with the help of election campaign. In the election, the candidate wants the message of campaign to reach each and every individual. But in reality, its hard to achieve. So, the candidate prefer to pass the message to all, through a selective set of people. He/She likes to make the set in minimum, which makes the candidates work in a easier way i.e., he/she wants the selective set of people to pass the message to each and every individual in their particular area.

If there is a area where the candidate has a set of m selective people to pass his/her message to all the other people in that particular area. The candidate wants to minimize the set of m selective people. For which, we are using the concept of cobondage set with the help of addition of strong arcs.

5.1. Numerical Example

If there is a place with 11 families of voters as shown in the following figure.



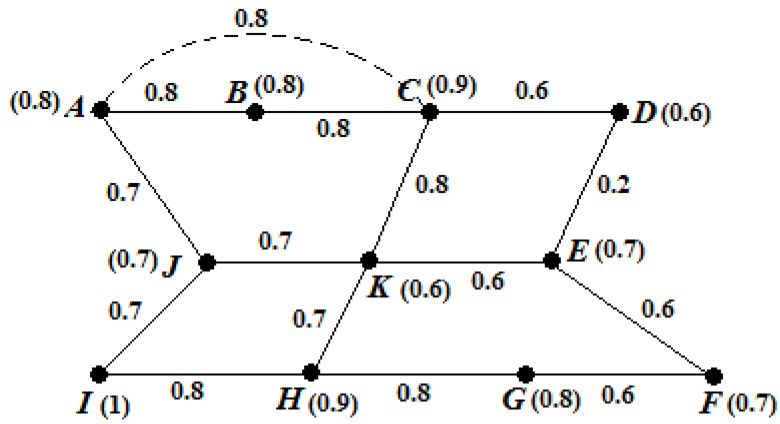
(G)
Fig. 5.1

Here $A, B, C, D, E, F, G, H, I, J$ and K represents the 11 families of voters. The membership values of each node represents the influence of them in that particular area.

Now the candidate X will pass the message of the campaign to all the 11 families by selecting a selective set of persons. In that set, the candidate X didn't want the persons from the families D and K since these both families has the least influence in that area i.e., these two nodes has least membership value in the given fuzzy graph.

For the above fuzzy graph G, $\{A, C, F, I\}$ forms a dominating set which is also a minimum dominating set. Thus $\gamma(G) = 4$.

The candidate X will select persons from each family of $D = \{A, C, F, I\}$ who will pass the message to all the other persons in their family and also to the persons in other families. The candidate X choose the set D because D is a minimal dominating set and the set contains no members from family D and K.



(G+AC)
Fig. 5.2

Now we use the concept of cobondage number, for the above fuzzy graph G, $\gamma(G + AC) = 3$ where AC is a addition of strong arc to G(as shown in the figure). Thus $b_C(G) = 1$.

Therefore the candidate will ask the selected person from the family C to pass the message to the members in the family A also. Thus now the candidate X has a 3 selective members from the families C, F and I respectively who will pass the message of the campaign to all the persons in that particular area.

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