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The Shortest Path Problem on Networks with Intuitionistic Fuzzy Edge Weights

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Abstract

The shortest path problem is a classical and important network optimization problem in a non – fuzzy network. This paper deals with the shortest path problem from a source vertex to the destination vertex on a network with imprecise edge weights namely intuitionistic fuzzy numbers. Two different approaches namely linear programming problem model and the algorithm procedures are proposed for fuzzy shortest path problem. Numerical examples are included to demonstrate the presented approach. Simulation result using C language is added further. Comparison is also made with the existing earlier results.

Keywords: Network, Shortest path problem, Triangular intuitionistic fuzzy numbers, Trapezoidal intuitionistic fuzzy numbers, α- cut ranking technique.

Mathematics Subject Classification (2010): 90B99.

1. INTRODUCTION

Over the past several years, a great deal of attention has been paid to mathematical programs and mathematical models that can be solved through the use of networks. Posing problems on networks not only yields computational advantages, it also serves as a means for visualizing a problem and for developing a better understanding of the problem. It is much easier for a decision maker to draw a picture of what he or she wants than to write down the constraints. A wide variety of network type problems include models such as minimum cost flows, work assignment, project schedules, transportation and shortest path to name a few. It is the purpose of this paper to concentrate on the most basic network problem, the shortest path problem under fuzzy environment. In traditional shortest path problem the arc lengths (edge weights) of a network take precise (crisp) values. But in real life situations the arc length may represent cost, time etc. There are so many cases that a unique number cannot be assigned to edge weights, because of cost, time variation. Hence the focus is on the fuzzy shortest path problem (FSPP). Fuzzy set theory, proposed by Zadeh [15], is frequently utilized to deal with uncertainty problems. The fuzzy shortest path problem was first analyzed by Dubois and Prade [3]. However, the major drawback to this classical fuzzy shortest path problem is, a fuzzy shortest path length is found, but it does not correspond to an actual path in the network. Klein [5] proposed a dynamic programming recursion-based fuzzy algorithm. Liu and Kao [8] and Yu and Wei [14] found the fuzzy shortest path length in a network by means of a fuzzy linear programming approach. Okada and Soper [11] proposed a fuzzy algorithm, which was based on multiple labeling methods to offer non-dominated paths to a decision maker. Nayeem and Pal [10] presented an algorithm which gives a single fuzzy shortest path or a guideline to choose the best fuzzy shortest path on a network with interval valued or triangular fuzzy edge weights according to the decision makers view. Thus numerous papers are published in fuzzy shortest path problems. Few among them are [6,7,9].

The paper is organized as follows: In section 2, the basic definitions which are needed for the proposed approaches are reviewed and a definition is coined for ranking α-cut ranking technique. Section 3 and Section 4, focuses on the linear programming problem model and the algorithmic approaches respectively to obtain the fuzzy shortest path. Section 5, concludes the paper.

2. PRE-REQUISITES

Definition 2.1. Fuzzy set

If X is a universal set, a fuzzy subset μ on a set X is a map $\mu : X \to [0, 1]$.

Definition 2.2. Acyclic digraph

A digraph is a graph (network) each of whose edges are directed. Hence an acyclic digraph is a directed graph without cycle.

Crisp graph with fuzzy edge weights (Type V Fuzzy Graph) [2]

It happens when the graph has known vertices and edges but unknown weights on the edges.

Definition 2.3. Intuitionistic fuzzy set [1]

Let X be an universe of discourse, then an intuitionistic fuzzy set (IFS) $\tilde{A}$ in X is given by $\tilde{A} = \{x, \mu_{\tilde{A}}(x), \gamma_{\tilde{A}}(x)/x \in X\}$, where the functions $\mu_{\tilde{A}}(x) : X \to [0,1]$ and $\gamma_{\tilde{A}}(x) : X \to [0,1]$, determine the degree of membership and non-membership of the element x $\in$X respectively and for every x $\in$ X, $0 \le \mu_{\tilde{A}}(x) + \gamma_{\tilde{A}}(x) \le 1$.

The concept of intuitionistic fuzzy number is of importance for quantifying an ill-known quantity. Intuitionistic fuzzy numbers are the more generalized form of fuzzy numbers involving two independently estimated degrees: degree of acceptance and degree of rejection.

Definition 2.4. Trapezoidal intuitionistic fuzzy number (TrIFN) [12]

The trapezoidal intuitionistic fuzzy number $\tilde{A}$ is denoted by $\tilde{A} = (< a_1, a_2, a_3, a_4 >, < a_1', a_2, a_3, a_4' >)$, where $a_1' < a_1 < a_2 < a_3 < a_4 < a_4'$; $a_1', a_1, a_2, a_3, a_4, a_4' \in \mathbb{R}$. The membership function and the non-membership function are as follows.

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{for } a_1 \le x \le a_2 \\ 1, & \text{for } a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & \text{for } a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_{\tilde{A}}(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'}, & \text{for } a_1' \le x \le a_2 \\ 0, & \text{for } a_2 \le x \le a_3 \\ \dfrac{x - a_3}{a_4' - a_3}, & \text{for } a_3 \le x \le a_4' \\ 1, & \text{otherwise} \end{cases}$$

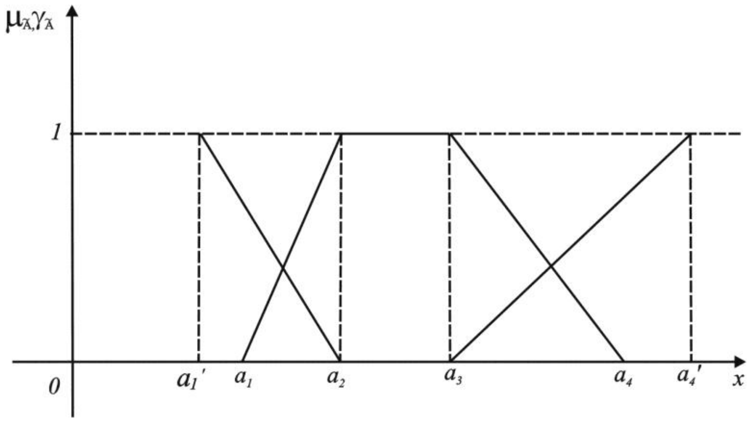


Figure 2.1. Trapezoidal intuitionistic fuzzy number

Definition 2.5. Addition operation on trapezoidal intuitionistic fuzzy numbers

Let $\tilde{A} = (< a_1, a_2, a_3, a_4 >, < a_1', a_2, a_3, a_4' >)$ and $\tilde{B} = (< b_1, b_2, b_3, b_4 >, < b_1', b_2, b_3, b_4' >)$ be two trapezoidal intuitionistic fuzzy numbers then

$$\tilde{A} \oplus \tilde{B} = (< a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 >, < a_1' + b_1', a_2 + b_2, a_3 + b_3, a_4' + b_4' >).$$

Definition 2.6. α-cut ranking technique for trapezoidal and triangular intuitionistic fuzzy number [4]

Let $\tilde{A} = (< a_1, a_2, a_3, a_4 >, < a_1', a_2, a_3, a_4' >)$, where $a_1' < a_1 < a_2 < a_3 < a_4 < a_4'$; $a_1', a_1, a_2, a_3, a_4, a_4' \in \mathbb{R}^+$, be a trapezoidal intuitionistic fuzzy number then the α-cut ranking technique of $\tilde{A}$ is denoted by

$$R(\tilde{A}) = \left(\left\langle \int_0^1 [a_1 + \alpha(a_2 - a_1)]\alpha d\alpha + \int_0^1 [a_4 - \alpha(a_4 - a_3)]\alpha d\alpha \right\rangle, \right.$$
$$\left. \left\langle -\int_0^1 [a_2 - \alpha(a_2 - a_1')]\alpha d\alpha - \int_0^1 [a_3 + \alpha(a_4' - a_3)]\alpha d\alpha \right\rangle \right)$$
$$= \left(\left\langle \frac{a_1 + 2(a_2 + a_3) + a_4}{6} \right\rangle, \; -\left\langle \frac{2(a_1' + a_4') + (a_2 + a_3)}{6} \right\rangle \right)$$
$$= (\alpha\text{ - cut ranking for TrIFN with respect to the membership function,}$$
$$\alpha\text{ - cut ranking for TrIFN with respect to the non - membership function)}$$
$$= (R^{\Delta}(\tilde{A}), R^{\Delta\Delta}(\tilde{A})).$$

If $a_2 = a_3$ then the trapezoidal intuitionistic fuzzy number becomes triangular intuitionistic fuzzy number (TIFN), that is, for $\tilde{A} = (< a_1, a_2, a_2, a_4 >, < a'_1, a_2, a_2, a'_4 >)$, $a'_1 < a_1 < a_2 < a_4 < a'_4;\ a'_1, a_1, a_2, a_4, a'_4 \in \mathbb{R}^+$.

The α-cut ranking technique of $\tilde{A}$ is

$$
\begin{aligned}
R(\tilde{A}) &= \left(\left\langle \frac{a_1 + 4a_2 + a_4}{6} \right\rangle,\ -\left\langle \frac{a'_1 + a_2 + a'_4}{3} \right\rangle \right) \\
&= (\alpha\text{-cut ranking for TIFN with respect to the membership function}, \\
&\qquad \alpha\text{-cut ranking for TIFN with respect to the non-membership function}) \\
&= (R^{\Delta}(\tilde{A}), R^{\Delta\Delta}(\tilde{A})).
\end{aligned}
$$

We have introduced the following ranking method for the above definition 2.6.

Definition 2.7. The ranking based on the α-cut ranking technique

If $\tilde{A}$ and $\tilde{B}$ are triangular or trapezoidal intuitionistic fuzzy numbers then $\tilde{A} \preceq \tilde{B}$ iff $R^{\Delta}(\tilde{A}) \le R^{\Delta}(\tilde{B})$ and $R^{\Delta\Delta}(\tilde{A}) \ge R^{\Delta\Delta}(\tilde{B})$.

Note. In the case of trapezoidal intuitionistic fuzzy numbers

$$
\begin{aligned}
\tilde{A} \preceq \tilde{B} &\Leftrightarrow R^{\Delta}(\tilde{A}) \le R^{\Delta}(\tilde{B}) \text{ and } R^{\Delta\Delta}(\tilde{A}) \ge R^{\Delta\Delta}(\tilde{B}) \\
&\Leftrightarrow (b_1 - a_1) + 2(b_2 - a_2) + 2(b_3 - a_3) + (b_4 - a_4) \ge 0 \\
&\qquad \text{and } 2(b'_1 - a'_1) + 2(b'_4 - a'_4) + (b_2 - a_2) + (b_3 - a_3) \ge 0
\end{aligned}
$$

In the case of triangular intuitionistic fuzzy numbers $\tilde{A} \preceq \tilde{B}$ iff

$(b_1 - a_1) + 4(b_2 - a_2) + (b_4 - a_4) \ge 0$ and $(b'_1 - a'_1) + (b_2 - a_2) + (b'_4 - a'_4) \ge 0$.

3. THE LINEAR PROGRAMMING FORMULATION OF FUZZY SHORTEST PATH PROBLEM USING THE α-CUT RANKING TECHNIQUE

One of the efficient approaches for finding the shortest paths in the networks is the linear programming (LP). Consider a project model G = (V, E) which is a directed acyclic network, where V is the set of n nodes or vertices and E is the set of ((i, j) $\in$ E) arcs or edges. T_{ij} is denoted as the time durations or distances, (i, j) $\in$ E.

The crisp shortest path problem with n nodes is formulated as

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} x_{ij} \tag{3.1}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} 1, & i = 1 \\ -1, & i = n \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \ \forall \ (i, j) \in E$$

Fuzzy Shortest Path Problem

Suppose the time durations T_{ij}, $(i, j) \in E$ are imprecise and can be represented as intuitionistic fuzzy numbers $(\tilde{T}_{ij}, \tilde{t}_{ij})$, $(i, j) \in E = (E_1, E_2)$. Here E_1 and E_2 are the edge weights in the case of membership and non-membership functions. Then the linear programming formulation of the fuzzy shortest path problem is

$$\text{Min } \tilde{Z} = (\min \ \tilde{D}, \max \ \tilde{d}) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{T}_{ij} x_{ij}, \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{t}_{ij} x_{ij} \right) \tag{3.2}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} 1, & i = 1 \\ -1, & i = n \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \ \forall \ (i, j) \in E = (E_1, E_2)$$

where x_{ij} is the decision variable for $(i, j) \in E = (E_1, E_2)$.

Note that the total time duration $\tilde{Z}$ becomes a fuzzy number rather than a crisp one. Consequently, Model (3.2) cannot be solved directly. To deal with this problem, one approach which has been proved to be correct is to transform the fuzzy numbers to crisp ones. The fuzzy time durations are defuzzified into crisp ones by a fuzzy number ranking method. i.e., the transformation is done using α-cut ranking technique. The α-cut ranking technique possesses linearity and additivity properties. Let $\tilde{B}_1$ and $\tilde{C}_1$ be two trapezoidal or triangular intuitionistic fuzzy numbers with respect to the membership function. If $\tilde{Z}_1 = u\tilde{B}_1 + v\tilde{C}_1$, where u and v are constants, then $R^{\Delta}(\tilde{Z}_1) = uR^{\Delta}(\tilde{B}_1) + vR^{\Delta}(\tilde{C}_1)$. Let $\tilde{B}_2$ and $\tilde{C}_2$ be two trapezoidal or triangular intuitionistic fuzzy numbers with respect to the non-membership function. If $\tilde{Z}_2 = a\tilde{B}_2 + b\tilde{C}_2$, where a and b are constants, then $R^{\Delta\Delta}(\tilde{Z}_2) = aR^{\Delta\Delta}(\tilde{B}_2) + bR^{\Delta\Delta}(\tilde{C}_2)$. Consequently on the basis of the α-cut ranking technique, the fuzzy shortest path problem can be transformed to a conventional shortest path problem with crisp time durations.

Crisp Transformation

Case (i): The α-cut ranking technique for TrIFN or TIFN with respect to the membership function

Consider the shortest path problem formulated as Model (3.2) with m paths. Let E_1 be the Edge weights or Arc lengths for the membership function. Let $x_{ij}^{(k)}$, $(i, j) \in E_1$ be the k^{th} basic feasible solution which corresponds to the k^{th} path P_k, k = 1, 2, …, m, in the case of membership function for trapezoidal or triangular intuitionistic fuzzy numbers. Let $\tilde{T}_{ij}$, $(i, j) \in E_1$ be the time durations represented as fuzzy numbers. Then $\tilde{D}^{(k)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{T}_{ij}x_{ij}^{(k)}$ is the fuzzy total time duration of the k^{th} path. Of these m paths, the one with the smallest total time duration $\tilde{D}^{*} = \min\{\tilde{D}^{(k)}, k = 1,2,...,m\}$ is identified as the fuzzy shortest path. According to the α-cut ranking technique for TrIFN or TIFN with respect to the membership function given in Definition 2.7, it suffices to find the smallest of α-cut ranking technique $R^{\Delta}(\tilde{D}^{*}) = \min\{R^{\Delta}(\tilde{D}^{(k)}), k = 1,2,...,m\}$. Furthermore, since the α-cut ranking technique method for TrIFN or TIFN possesses the properties of linearity and additivity in the case of membership function, $R^{\Delta}(\tilde{D}^{(k)}) = R^{\Delta}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{T}_{ij}x_{ij}^{(k)}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta}(\tilde{T}_{ij})x_{ij}^{(k)}$. That is, the minimum fuzzy objective value $\tilde{D}^{*}$ with respect to membership function corresponds to the minimum α-cut ranking technique. $R^{\Delta}(\tilde{D}^{*}) = \min_{1\le k\le m}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta}(\tilde{T}_{ij})x_{ij}^{(k)}\right\}$. Consequently, the shortest path problem with the fuzzy time durations with respect to the membership function can be formulated as follows:

$$R^{\Delta}(\tilde{D}^{*}) = \min\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta}(\tilde{T}_{ij})x_{ij}\right\} \quad (3.3)$$

subject to the constraints

$$\sum_{j=1}^{n}x_{ij} - \sum_{j=1}^{n}x_{ji} = \begin{cases}1, & i = 1\\ -1, & i = n\\ 0, & \text{otherwise}\end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \ \forall\ (i, j) \in E_1.$$

This problem is essentially a conventional linear programming problem, since the coefficients in the objective function $R^{\Delta}(\tilde{T}_{ij})$, for (i, j) $\in E_1$, are the crisp real numbers rather than the fuzzy numbers, in the case of membership function of trapezoidal or triangular intuitionistic fuzzy numbers.

Case (ii): The α-cut ranking technique for TrIFN or TIFN with respect to the non-membership function

Consider the shortest path problem formulated as Model (3.2) with m paths. Let E_2 be the Edge weights or Arc lengths for the non-membership function. Let $x_{ij}^{(k)}$, (i, j) $\in E_2$ be the k^{th} basic feasible solution which corresponds to the k^{th} path P_k, k = 1, 2, …, m, in the case of non-membership function for trapezoidal or triangular intuitionistic fuzzy numbers. Let $\tilde{t}_{ij}$, (i, j) $\in E_2$ be the time durations represented as fuzzy numbers. Then $\tilde{d}^{(k)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{t}_{ij}x_{ij}^{(k)}$ is the fuzzy total time duration of the k^{th} path. Of these m paths, the one with the largest total time duration $\tilde{d}^{*} = \max\{\tilde{d}^{(k)}, k = 1,2,...,m\}$ is identified as the fuzzy shortest path. According to the α-cut ranking technique for TrIFN or TIFN with respect to the non-membership function given in Definition 2.7, it suffices to find the largest of α-cut ranking technique $R^{\Delta\Delta}(\tilde{d}^{*}) = \max\{R^{\Delta\Delta}(\tilde{d}^{(k)}), k = 1,2,...,m\}$. Furthermore, since the α-cut ranking technique method for TrIFN or TIFN possesses the properties of linearity and additivity in the case of non-membership function, $R^{\Delta\Delta}(\tilde{d}^{(k)}) = R^{\Delta\Delta}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{t}_{ij}x_{ij}^{(k)}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta\Delta}(\tilde{t}_{ij})x_{ij}^{(k)}$. That is, the maximum fuzzy objective value $\tilde{d}^{*}$ with respect to non-membership function corresponds to the maximum α-cut ranking technique. $R^{\Delta\Delta}(\tilde{d}^{*}) = \max_{1\le k\le m}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta\Delta}(\tilde{t}_{ij})x_{ij}^{(k)}\right\}$.

Consequently, the shortest path problem with the fuzzy time durations with respect to the non-membership function can be formulated as follows:

$$R^{\Delta\Delta}(\tilde{d}^{*}) = \max\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}R^{\Delta\Delta}(\tilde{t}_{ij})x_{ij}\right\} \tag{3.4}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} 1, & i = 1 \\ -1, & i = n \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \ \forall \ (i, j) \in E_2.$$

This problem is essentially a conventional linear programming problem, since the coefficients in the objective function $R^{\Delta\Delta}(\tilde{t}_{ij})$, for (i, j) $\in E_2$, are the crisp real numbers rather than the fuzzy numbers, in the case of membership function of trapezoidal or triangular intuitionistic fuzzy numbers.

Case (iii): The α-cut ranking technique for TrIFN or TIFN (combination of membership and non-membership function)

The minimum fuzzy objective value $\tilde{Z}^*$ with respect to membership and non-membership function corresponds to the minimum α-cut ranking technique for TrIFN or TIFN with respect to the membership function and maximum α-cut ranking technique for TrIFN or TIFN with respect to the non-membership function. The shortest path problem with the fuzzy time durations with respect to membership and non-membership function can be formulated as follows:

$$R(\tilde{Z}^*) = \left(\min\left(\sum_{i=1}^{n}\sum_{j=1}^{n} R^{\Delta}(\tilde{T}_{ij}) x_{ij} \right), \max\left(\sum_{i=1}^{n}\sum_{j=1}^{n} R^{\Delta\Delta}(\tilde{t}_{ij}) x_{ij} \right) \right) \tag{3.5}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} 1, & i = 1 \\ -1, & i = n \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \ \forall \ (i, j) \in E = (E_1, E_2).$$

Numerical Example 3.1

Construct a fuzzy acyclic network G(V, E) as shown in the Figure 3.1. Here the edge weights are taken as the combination of trapezoidal and triangular intuitionistic fuzzy numbers.

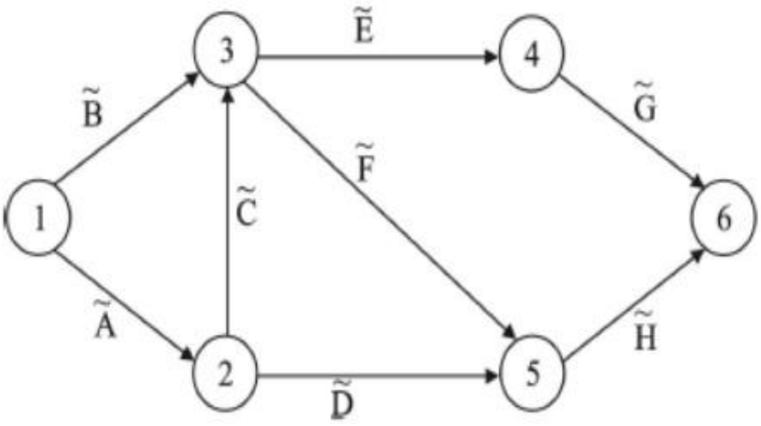


Figure 3.1. A network with fuzzy arc lengths [8,11,14]

The fuzzy arc lengths which denote the distance in terms of kilometres are given below:

$$\tilde{d}_{12} = \tilde{A}(1-2) = (<10,20,20,30>, <8,20,20,35>)$$

$$\tilde{d}_{13} = \tilde{B}(1-3) = (<52,62,65,70>, <47,62,65,74>)$$

$$\tilde{d}_{23} = \tilde{C}(2-3) = (<35,38,40,45>, <30,38,40,50>)$$

$$\tilde{d}_{25} = \tilde{D}(2-5) = (<52,55,60,65>, <46,55,60,70>)$$

$$\tilde{d}_{34} = \tilde{E}(3-4) = (<10,13,17,20>, <9,13,17,25>)$$

$$\tilde{d}_{35} = \tilde{F}(3-5) = (<8,9,9,10>, <6,9,9,15>)$$

$$\tilde{d}_{46} = \tilde{G}(4-6) = (<70,75,85,97>, <64,75,85,100>)$$

$$\tilde{d}_{56} = \tilde{H}(5-6) = (<50,70,80,100>, <47,70,80,105>)$$

The associated problem based on α-cut ranking technique for intuitionistic fuzzy numbers is

$$
\begin{aligned}
R(\tilde{Z}^{*}) = (\min\{ & R^{\Delta}[(10,20,20,30)]x_{12} + R^{\Delta}[(52,62,65,70)]x_{13} \\
& + R^{\Delta}[(35,38,40,45)]x_{23} + R^{\Delta}[(52,55,60,65)]x_{25} \\
& + R^{\Delta}[(10,13,17,20)]x_{34} + R^{\Delta}[(8,9,9,10)]x_{35} \\
& + R^{\Delta}[(70,75,85,97)]x_{46} + R^{\Delta}[(50,70,80,100)]x_{56}\}, \\
\max\{ & R^{\Delta\Delta}[(8,20,20,35)]x_{12} + R^{\Delta\Delta}[(47,62,65,74)]x_{13} \\
& + R^{\Delta\Delta}[(30,38,40,50)]x_{23} + R^{\Delta\Delta}[(46,55,60,70)]x_{25} \\
& + R^{\Delta\Delta}[(9,13,17,25)]x_{34} + R^{\Delta\Delta}[(6,9,9,15)]x_{35} \\
& + R^{\Delta\Delta}[(64,75,85,100)]x_{46} + R^{\Delta\Delta}[(47,70,80,105)]x_{56}\})
\end{aligned}
\tag{3.6}
$$

subject to the constraints

$x_{12} + x_{13} = 1;$

$x_{23} + x_{25} - x_{12} = 0,\ x_{34} + x_{35} - x_{13} - x_{23} = 0,\ x_{46} - x_{34} = 0,\ x_{56} - x_{25} - x_{35} = 0;$

$- x_{46} - x_{56} = - 1;$

$x_{12}, x_{13}, x_{23}, x_{25}, x_{34}, x_{35}, x_{46}, x_{56} = 0$ or 1.

By applying the α-cut ranking technique for membership function and non-membership function, the fuzzy time durations $\tilde{T}_{ij}$ and $\tilde{t}_{ij}$ are calculated as:

$R^{\Delta}(\tilde{T}_{12}) = 20,\ R^{\Delta}(\tilde{T}_{13}) = 62.667,\ R^{\Delta}(\tilde{T}_{23}) = 39.333,\ R^{\Delta}(\tilde{T}_{34}) = 15,$

$R^{\Delta}(\tilde{T}_{25}) = 57.833,\ R^{\Delta}(\tilde{T}_{35}) = 9,\ R^{\Delta}(\tilde{T}_{46}) = 81.167,\ R^{\Delta}(\tilde{T}_{56}) = 75$ and

$R^{\Delta\Delta}(\tilde{t}_{12}) = -21,\ R^{\Delta\Delta}(\tilde{t}_{13}) = -61.500,\ R^{\Delta\Delta}(\tilde{t}_{23}) = -39.667,\ R^{\Delta\Delta}(\tilde{t}_{34}) = -16.333,$

$R^{\Delta\Delta}(\tilde{t}_{25}) = -57.833,\ R^{\Delta\Delta}(\tilde{t}_{35}) = -10,\ R^{\Delta\Delta}(\tilde{t}_{46}) = -81.333,\ R^{\Delta\Delta}(\tilde{t}_{56}) = -75.667.$

Replacing these values in (3.6) results in a conventional shortest path problem which can be easily solved. An optimal solution is found to be $x^{*}_{12} = x^{*}_{23} = x^{*}_{35} = x^{*}_{56} = 1$ and $x^{*}_{13} = x^{*}_{25} = x^{*}_{34} = x^{*}_{46} = 0$ with $R(\tilde{Z}^{*}) = (143.33, -146.33)$ is derived. That is, the shortest path in the fuzzy sense is $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$.

4. AN ALGORITHMIC APPROACH FOR FUZZY SHORTEST PATH PROBLEM USING α-CUT RANKING TECHNIQUE

4.1. Algorithm for fuzzy shortest path problem

Step 1: Construct an acyclic network G(V, E), where V is the set of vertices and E is the set of edges. Let $\tilde{d}_{ij} = (< a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)} >, < a_{ij}'^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}'^{(4)} >)$, denote the trapezoidal intuitionistic fuzzy numbers associated with the edge (i, j), that is the length necessary to traverse from node i to node j. Each length corresponds to the cost, time etc., in practical problems.

Step 2: The temporary edge [0; (n, n)] is assigned to the destination node t = n, where 0 = (<0, 0, 0, 0>, <0, 0, 0, 0>).

Step 3: If t = s (source node) then go to step 5. Otherwise if t ≠ s among all the temporary labels (edges) associated to node t determine the α-cut ranking technique-wise smallest one for membership functions and α-cut ranking technique-wise largest one for non-membership functions. Let it be the k^{th} label (edge) associated with the node t. It is now taken as the permanent edge, discarding the rest of the temporary labels associated with t. Let that path by P_{it} in G(V, E) for every node $i \in V - \{t\}$. Now, let i be the permanent node and (i, t) be the permanent edge.

Step 4: Check, the permanent node whether it is having more than one temporary labels incident to it. The iterations are repeated until they reach the source node s.

Step 5: Once it reaches the source node s, terminate the execution of the algorithm.

Step 6: Find the optimal path from the source node s to the destination node t, which is identified as the fuzzy shortest path.

Step 7: The intuitionistic fuzzy distance or fuzzy time duration along the fuzzy shortest path P is denoted by d(P) and is defined as $d(P) = \sum_{(i,j) \in P} \tilde{l}_{ij}$. It is calculated using the addition operation given in definition 2.5.

Numerical Example 4.1.

Step 1: Construct a fuzzy acyclic network as given in Numerical Example 3.1.

Step 2: The temporary edge [0; (6, 6)] is assigned to the destination node t = n = 6.

Step 3 and Step 4: t = 6 ≠ 1 = s. From node t = 6, the edges (4, 6) and (5, 6) are new temporary edges with [<70, 75, 85, 97>, <64, 75, 85, 100>; (4, 6)] = (81.17, −81.33) and [<50, 70, 80, 100>, <47, 70, 80, 105>; (5, 6)] = (75, −75.67). Edge (5, 6) is determined as α-cut ranking technique-wise smallest for membership function and α-cut ranking technique-wise largest for non-membership function of all the temporary edges. From the permanent node 5, the edges (2, 5) and (3, 5) are temporary edges with [<52, 55, 60, 65>, <46, 55, 60, 70>; (2, 5)] = (57.83, −57.83) and [<8, 9, 9, 10>, <6, 9, 9, 15>; (3, 5)] = (9, −10). Here edge (3, 5) is selected. From the permanent node 3, the edges (1, 3) and (2, 3) are temporary edges with [<52, 62, 65, 70>, <47, 62, 65, 74>; (1, 3)] = (62.67, −61.5) and [<35, 38, 40, 45>, <30, 38, 40, 50>; (2, 3)] = (39.33, −39.67). Here edge (2, 3) is selected. From the permanent node 2, there is only one edge (1, 2).

Step 5: The iteration is stopped since it reaches the source node 1.

Step 6: The optimal path $1 \to 2 \to 3 \to 5 \to 6$ is the fuzzy shortest path.

Step 7: The intuitionistic fuzzy distance along the fuzzy shortest path is

(<103, 137, 149, 185>, <91, 137, 149, 205>).

Note: Algorithm 4.1, is not applicable if the two temporary edges incident to the permanent node have the same α-cut ranking technique for membership and non-membership function because it creates an ambiguous situation to identify the fuzzy shortest path.

The above note motivated us to give the general algorithm to find the fuzzy shortest path.

4.2. General Algorithm for fuzzy shortest path problem

Step 1: Same as given in Algorithm 4.1.

Step 2: Calculate all possible paths P_i, i = 1 to n from the source vertex 's' to the destination vertex 'd' and the corresponding path lengths $\tilde{L}_i$, i = 1 to n using definition 2.5.

Set $\tilde{L}_i = (< a_i, b_i, c_i, d_i >< a'_i, b_i, c_i, d'_i >)$.

Step 3: Calculate the α-cut ranking technique for each possible path lengths $\tilde{L}_i$, i = 1 to n. The path having the smallest α-cut ranking technique for membership function and largest α-cut ranking technique for non-membership function is identified as the fuzzy shortest path.

Numerical Example 4.2.

Step 1: Construct a fuzzy acyclic network as given in Numerical Example 3.1.

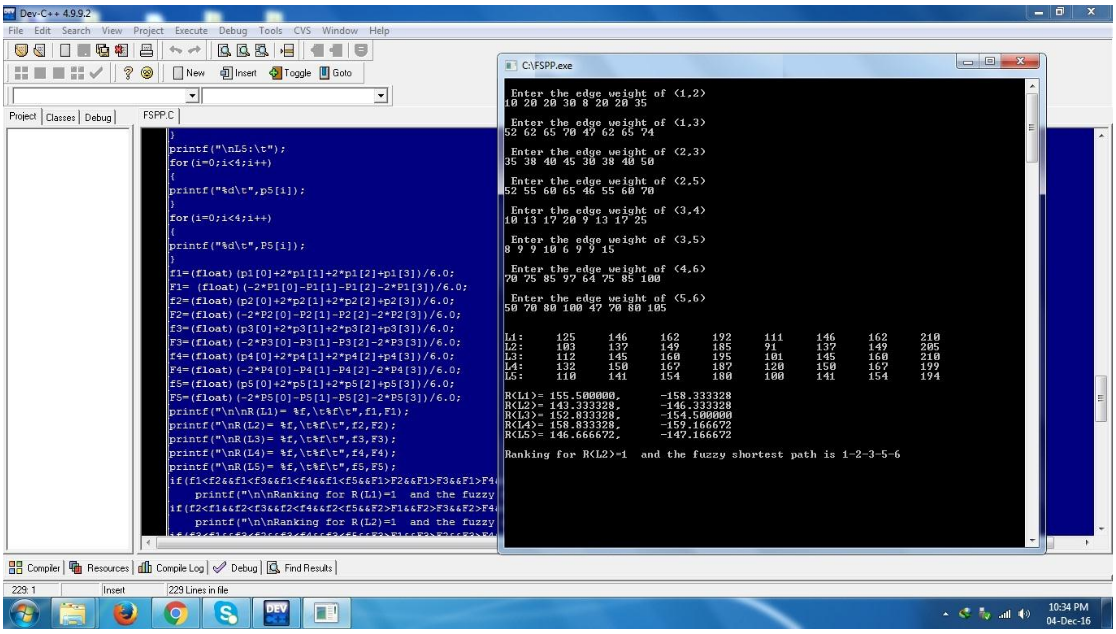
Step 2 and Step 3:

Table: Results of the Network

Paths (P_i)	Path Lengths ($\tilde{L}_i$)	α-cut ranking technique $R(\tilde{L}_i)$	Ranking
P_1 : 1→2→3→4→6	(<125, 146, 162, 192>, <111, 146, 162, 210>)	(155.5, –158.33)	4
P_2 : 1→2→3→5→6	(<103, 137, 149, 185>, <91, 137, 149, 205>)	(143.33, –146.33)	1
P_3 : 1→2→5→6	(<112, 145, 160, 195>, <101, 145, 160, 210>)	(152.83, –154.5)	3
P_4 : 1→3→4→6	(<132, 150, 167, 187>, <120, 150, 167, 199>)	(158.83, –159.17)	5
P_5 : 1→3→5→6	(<110, 141, 154, 180>, <100, 141, 154, 194>)	(146.67, –147.17)	2

Here path P_2 : 1→2→3→5→6 is identified as the fuzzy shortest path.

Simulation result using C program :



Results and Discussions :

One way to verify the solution obtained is to make an exhaustive comparison.

The following authors considered only the membership function for the arc lengths and obtained the following results.

- Okada and Soper [11] developed an algorithm based on the multiple labeling method for a multi criteria shortest path to find a number of non-dominated paths. For Figure 3.1 they obtained two non-dominated paths $1 \to 2 \to 3 \to 5 \to 6$ and $1 \to 3 \to 5 \to 6$.
- Liu and Kao [8] used ranking indices calculated from the Yager method [13] and Yu and Wei [14] used linear multiple objective programming for Figure 3.1. It is found that the path $1 \to 2 \to 3 \to 5 \to 6$ is the fuzzy shortest path.

Hence the methods developed in this paper based on the α-cut ranking technique for membership and non-membership function treated as an improved form of Okada's [11] method and is an alternative form of Liu's [8] and Yu's [14] methods.

5. CONCLUSION

Many researchers have focused on fuzzy shortest path problem in a network since it is significant to various applications. In this paper, we have tried to accumulate most of the existing ideas and proposed new approaches based on LPP model and algorithmic procedure to make a detailed analysis of FSPP. It aims at providing a decision maker (DM) with the shortest path in a network with fuzzy arc lengths. Also the ranking given to the paths in the case of algorithmic procedure helps the DM as they make decision in choosing the best of all the positive path alternatives. Hence it is concluded that the procedures developed in the current work is an alternative method to identify the shortest path in fuzzy environment.

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