Lict Double Domination in Graphs

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Abstract

For any graph $G = (V,E)$, lict graph $n(G)$ of a graph $G$ is the graph whose vertex set is the union of the set of edges and the set of cut vertices of $G$ in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of $G$ are incident. A subset $D^d$ of $V[n(G)]$ is double dominating set of $n(G)$ if for every vertex $v \in V[n(G)]$, $|N(v) \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbour in $D^d$ or $v$ is in $V[n(G)] - D^d$ and has at least two neighbours in $D^d$. The lict double dominating number $\gamma_{ddn}(G)$ is a minimum cardinality of lict double dominating set. In this paper many bounds on $\gamma_{ddn}(G)$ are obtained and its exact values for some standard graph are found in terms of parameter of $G$. Also its relationship with other domination parameters is investigated.

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INTRODUCTION

The graphs considered here are simple and finite. Let $G$ be a graph with $V = V(G)$ is the vertex set of $G$ and $E = E(G)$ is the edge set of $G$. The neighbourhood of a vertex $v \in V$ is defined by $N(v) = \{ u \in V/uv \in E \}$. The close neighbourhood of a vertex $v$ is $N[v] = N(v) \cup \{v\}$. The order $|V(G)|$ of $G$ is denoted by $p$. The degree of $v$ is...
The maximum degree of a graph $G$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. The minimum number of color in any colouring of a graph $G$ such that no two adjacent vertices have same color is called the chromatic number of $G$ and is denoted by $\chi(G)$. A vertex cover in a graph $G$ is a set of vertices that covers all the edges of $G$. The vertex covering number $\alpha_0(G)$ is a minimum cardinality of a vertex cover in $G$. An edge cover of a graph $G$ without isolated vertices is a set of edges of $G$ that covers all vertices is a set of edges of $G$ that covers all the vertices of $G$. The edge covering number $\alpha_1(G)$ of a graph $G$ is the minimum cardinality of an edge cover of $G$. A set of vertices/edges in a graph $G$ is said to be an independent set if no two vertices /edges in the set are adjacent. The vertex independent number $\beta_0(G)$ is the maximum cardinality of an independent set of vertices. The edge independent number $\beta_1(G)$ of a graph $G$ is the maximum cardinality of an independent set of edges. A total dominating set of $G$ is a subset $S$ of $V$ such that each vertex in $V$ is adjacent to a vertex of $S$. The total domination number, denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set. The line graph $n(G)$ of a graph $G$ is the graph whose vertex set is the union of the set of edges and the set of cutvertices of $G$ in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of $G$ are incident. Let $G = (V, E)$ be a graph. A set $D$ of vertices in a graph $G$ is called a dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. A set $D$ subset of $V[n(G)]$ is said to be a dominating set of $n(G)$, if every vertex not in $D$ is adjacent to a vertex in $D$ of $n(G)$. The domination number of $n(G)$ is denoted by $\gamma[n(G)]$ is the minimum cardinality of a dominating set. A subset $D^d$ of $V[n(G)]$ is double dominating set of $n(G)$ if for every vertex $v \in V[n(G)]$, $|N(v) \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbour in $D^d$ or $v$ is in $V[n(G)] - D^d$ and has at least two neighbours in $D^d$ and it is denoted by $\gamma_{ddn}(G)$. In this paper many bounds on $\gamma_{ddn}(G)$ are obtained and its exact values for some standard graph are found in terms of parameter of $G$. Also its relationship with other domination parameters is investigated. Further domination related graph valued functions were studied in [4, 5, 6]. We need the following theorems.

**Theorem A [1]** Let $G$ be a connected graph of order $n$, Then $\gamma'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

**Theorem B [2]** For any graph $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. 
Theorem C [3] For any path $P_n$, the edge covering number is $\alpha_1(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

Theorem D [3] For any path $P_n$, the vertex covering number is $\alpha_0(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$

Theorem E [3] For any graph $G$ of order $p$,

(i) $\chi(G) \geq \omega(G)$.

(ii) $\chi(G) \geq \frac{q}{\beta_0}(G)$.

Theorem F[7] Let $G$ be a connected graph, $\chi(G) \leq 1 + \Delta(G)$.

Upper Bounds for $\gamma_{ddn}(G)$

Theorem 1. For any connected $(p, q)$ graph $G$ with $p \geq 3$, $\gamma_{ddn}(G) \leq p - 1$.

Proof. Let $T$ be a spanning tree of $G$. If $p = 2$, then $n(G)$ has an isolated vertex. Hence $p \geq 3$. Let $I = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all end edges of $T$ and $I' = E(T) - I$. Then there exist a maximal independent set of edges $J = \{e_1, e_2, e_3, ..., e_k\} \subseteq I'$, in $I'$ such that $J$ forms an edge dominating set of $T$. Further if $J = \emptyset$, then $J = \{e\} \subseteq I$ forms an edge dominating set of $T$. Now without loss of generality, the corresponding edges of $J$ forms a vertex set $D_1 = \{v_1, v_2, v_3, ..., v_i\}$ in $n(T)$ which is also a dominating set of $n(T)$. Let $V_1 = V[n(T)] - D_1$ and $V_1 \in N(D_1)$. Clearly $D_d = D_1 \cup D_2$ form a double dominating set in $n(T)$, where $D_2 \subseteq V_1$. It follows that $|D_1 \cup D_2| \leq p - 1$ and hence $\gamma_{ddn}(G) \leq p - 1$.

Theorem 2. For any connected $(p, q)$ graph $G$, $\gamma_{ddn} \leq \gamma_t(G) + \Delta(G) - 1$.

Proof. Let $D = \{v_1, v_2, v_3, ..., v_k\}$ be a dominating set of $G$ and $V_1 = V(G) - D$, $V_1 \in N(D)$. Let $H \subseteq V_1$ be the minimum set of vertices which are adjacent to $D$ then $D \cup H$ is a total dominating set of $G$. Let $v \in D \cup H$, $S \subseteq V(G) - DUH$ be the set of all vertices adjacent to $v$. Then $D \cup H \cup S$ becomes a maximal domination set of $G$.

Now, let $S_1 = \{e_1, e_2, e_3, ..., e_j\}$ be the minimal set of edges which are incident to the vertices of $D$. Now without loss of generality, let $D_1 = \{v_1, v_2, v_3, ..., v_i\}$ be a dominating set of $n(G)$. Further if $V_2 = V[n(G)] - D_1$ and $D_2 = \{v_1, v_2, v_3, ..., v_j\} \subseteq$
Then \(D^d = D_1 \cup D_2\) form a double dominating set \(n(G)\). Clearly, it follows that 
\[|D^d| \leq |D \cup H \cup S| \leq |D \cup H| \cup |S| - 1\]
and hence \(\gamma_{ddn} \leq \gamma_1(G) + \Delta(G) - 1\).

**Theorem 3.** For any tree \(T\) with \(p \geq 3\), then 
\(\gamma_{ddn} \leq \left\lfloor \frac{p}{2} \right\rfloor + 1\).

**Proof.** Let \(S = \{e_1, e_2, e_3, \ldots, e_k\}\) be an edge dominating set of \(T\). By definition of \(n(T)\), \(V[n(T)] = E(T) \cup C(T)\), corresponding to the edges of \(S\), we obtain a vertex set \(D_1 = \{v_1, v_2, v_3, \ldots, v_k\}\) which is a dominating set of \(n(T)\), since \(D_1 \subseteq V[n(G)]\). Suppose \(V_1 \subseteq V[n(T)] - D_1\) be the set of vertices which are neighbours of the elements of \(D_1\). Further \(D_2 \subseteq V_2\) and \(D_2 \in N(D_1)\). Then \(D^d = D_1 \cup D_2\) becomes double dominating set of \(n(T)\) such that any vertex \(v \in V[n(T)] - D^d\) has at least two neighbours in \(D_1 \cup D_2\). Also by Theorem A, \(\gamma'(G) \leq \left\lfloor \frac{p}{2} \right\rfloor\) clearly it follows that 
\[|D^d| \leq \left\lfloor \frac{p}{2} \right\rfloor + 1\] and hence \(\gamma_{ddn} \leq \left\lfloor \frac{p}{2} \right\rfloor + 1\).

**Theorem 4.** For any connected \((p, q)\) graph \(G\), 
\(\gamma_{ddn}(G) + \chi(G) \leq p + \Delta(G)\). Equality holds if \(G\) is isomorphic to \(C_4, C_5\).

**Proof.** By Theorem 1, \(\gamma_{ddn}(G) \leq p - 1\) and by Theorem F, \(\chi(G) \leq 1 + \Delta(G)\). Clearly it follows that, \(\gamma_{ddn}(G) + \chi \leq p + \Delta(G)\). If \(G \cong C_4, C_5, C_6\) then \(\gamma_{ddn}(G) = p - 1\) and \(\chi(G) = 3\). Hence \(\gamma_{ddn}(G) + \chi(G) = p + \Delta(G)\).

**Theorem 5.** For any connected \((p, q)\) graph \(G\), 
\(\gamma_{ddn} + \kappa \leq p + \delta - 1\), where \(\kappa\) denotes the connectivity of \(G\). Equality hold if \(G\) is isomorphic to \(C_4, C_5, C_6\) or \(p_3, p_4\).

**Proof.** By Theorem 1, \(\gamma_{ddn}(G) \leq p - 1\) and by Theorem B, \(\kappa \leq \lambda \leq \delta\). Clearly it follows that \(\gamma_{ddn}(G) + \kappa \leq p + \delta - 1\). If \(G \cong C_4, C_5, C_6\) or \(p_3, p_4\), then \(\gamma_{ddn}(G) = p - 1\) and \(k = 2\). Hence \(\gamma_{ddn} + \kappa = p + \delta - 1\).

Now we proceed to construct an upper bound to \(\gamma_{ddn}\) by connecting edge connectivity of a graph \(G\).

**Corollary 1.** For any connected \((p, q)\) graph \(G\), \(\gamma_{ddn}(G) + \lambda \leq p + \delta - 1\), where \(\lambda\) denotes the edge connectivity of \(G\). Equality holds, if \(G\) is isomorphic to \(C_3, C_4, C_5, C_6, P_3, P_4, P_5\).

**Proof.** The result follows from Theorem 1 and Theorem B.

**Theorem 6.** For any \((p, q)\) tree \(T\) with \(p \geq 3\), \(\gamma_{ddn}(T) \leq q\).

**Proof.** Let \(T\) be a tree with \(E = \{e_1, e_2, e_3, \ldots, e_q\}\) and \(C = \{c_1, c_2, c_3, \ldots, c_i\}\) \(i < q\) be the set of edges and cutvertices in \(G\). In \(n(G), V[n(G)] = E(G) \cup C(G)\). Further if
there exists a vertex set \( V_1 \subseteq V[n(G)] - C \), \( \{ V_1 \subseteq E(G) \text{ in } n(G) \} \). Then \( D^d = V_1 \cup C_1 \), \( C_1 < C \) where \( C_1 \) is a set of cutvertices in \( n(G) \). Also every vertex of \( n(G) \) are adjacent to at least two vertices of \( D^d \). Clearly \( D^d \) forms a double dominating set of \( n(G) \). Therefore it follow that \( |D^d| \leq E(G) \). Hence \( \gamma_{ddn}(T) \leq q \).

**Theorem 7.** For any path \( P_n \) of order \( n \), \( \gamma_{ddn} \leq \begin{cases} 2\alpha_0(P_n) - 1 & \text{n is even} \\ 2\alpha_0(P_n) & \text{n is odd} \end{cases} \)

**Proof.** Let \( P_n \) be the path with \( n \geq 3 \) vertices. Consider \( V = \{ v_1, v_2, v_3, ..., v_n \} \) be the vertices and \( E = \{ (v_i, v_{i+1}) \mid i = 1,2,3,... \} \) be the edge set of path \( P_n \). By the Theorem D, we have the following cases.

**Case(i):** Suppose \( n \) is even. Then \( \alpha_0(P_n) = \frac{n}{2} \Rightarrow n = 2\alpha_0(P_n) \). Since \( \gamma_{ddn}(P_n) \leq n - 1 \), we have \( \gamma_{ddn}(P_n) \leq 2\alpha_0(P_n) - 1 \).

**Case(ii):** Suppose \( n \) is an odd. Then \( \alpha_0(P_n) = \frac{n-1}{2} \Rightarrow n - 1 = 2\alpha_0(P_n) - 1 \). Since \( \gamma_{ddn}(P_n) \leq n - 1 \), we have \( \gamma_{ddn}(P_n) \leq 2\alpha_0(P_n) \).

**Theorem 8.** For any path \( P_n \) of order \( n \), \( \gamma_{ddn} \leq \begin{cases} 2\alpha_1(P_n) - 1 & \text{n is even} \\ 2\alpha_1(P_n) - 2 & \text{n is odd} \end{cases} \)

**Proof:** Let \( P_n \) be the path with \( n \geq 3 \) vertices. Consider \( V = \{ v_1, v_2, v_3, ..., v_n \} \) be the vertices and \( E = \{ (v_i, v_{i+1}) \mid i = 1,2,3,... \} \) be the edge set of path \( P_n \). We have the following cases.

**Case(i):** Suppose \( n \) is even, by the Theorem C, we have, \( \alpha_1(P_n) = \frac{n}{2} \Rightarrow n = 2\alpha_1(P_n) \). Since \( \gamma_{ddn}(P_n) \leq n - 1 \), we have \( \gamma_{ddn}(P_n) \leq 2\alpha_1(P_n) - 1 \).

**Case(ii):** Suppose \( n \) is an odd, by Theorem C, we have, \( \alpha_1(P_n) = \frac{n+1}{2} \Rightarrow n + 1 = 2\alpha_1(P_n) - 1 \). Since \( \gamma_{ddn}(P_n) \leq p - 1 \), we have \( \gamma_{ddn} \leq 2\alpha_1(P_n) - 2 \).

**Theorem 9.** For any tree \( T \) with \( k \) number of cutvertices \( \gamma_{ddn}(T) \leq k + 1 \), further equality holds if \( T = K_{1,p} \) \( p \geq 3 \).

**Proof.** Let \( V = \{ v_1, v_2, v_3, ..., v_k \} \subset V(T) \) be the set of all cutvertices of a tree \( T \) with \( |C| = k \), since the number of vertices and the number of pendant vertices. If for every cutvertex \( u \in C \), \( u \neq v \) such that \( u \) is adjacent to \( v \). Otherwise, let \( e_1 \in E(G) \) such that \( e_1 \) is incident with \( C \), so that \( \gamma_{ddn}(T) \leq \{ C \cup e_1 \} = |C| + 1 = k + 1 \). For equality, let \( T = K_{1,p} \) with a cutvertex \( k = 1 \), then \( D^d = \{ K \cup e \} \) is a double dominating set of \( n(T) \) with cardinality \( k + 1 \). Hence \( \gamma_{ddn}(T) \leq k + 1 \).
Lower Bounds for $\gamma_{ddn}(G)$

**Theorem 10.** For any tree $T$ of order $p \geq 3$, $\gamma_{ddn}(T) \geq \chi(T)$, and equality holds for all star graphs $K_{1,p}$.

**Proof.** For any tree $T$, we have $\chi(T) = 2$ and $2 \leq \gamma_{ddn}(T) \leq p - 1$. Hence $\gamma_{ddn}(T) \geq \chi(T)$. For $T = K_{1,p}$, clearly $\chi(T) = 2$ and $\gamma_{ddn}(T) = 2$. Hence the proof.

**Theorem 11.** For any tree $T$ of order $p \geq 3$, $\gamma_{ddn}(T) \geq \omega(T)$.

**Proof.** The result follows from Theorem 10 and Theorem E.

**Theorem 12.** For any tree $T$ of order $p \geq 3$, $\gamma_{ddn}(T) \geq \beta_0(T)$.

**Proof.** For any tree $T$, we have $\chi(T) \geq \beta_0(T)$ and $\gamma_{ddn}(T) \geq \chi(T)$. Hence $\gamma_{ddn}(T) \geq \beta_0(T)$. Hence the proof.

**Theorem 13.** For any graph $G$ of order $p$, $\gamma_{ddn}(T) \geq p - m$, where $m$ is the number of end vertices.

**Proof.** Let $V_1 = \{v_1, v_2, v_3, ..., v_m\}$ be the set of all end vertices in $T$ with $|V_1| = m$. Further $E = \{e_1, e_2, e_3, ..., e_q\}$ and $C = \{c_1, c_2, c_3, ..., c_i\}$ be the set of edges and cut vertices in $G$. In $n(G)$, $V[(G)] = E(G) \cup C(G)$. Let $D^d = \{v_1, v_2, v_3, ..., v_n\} \subseteq V[n(G)]$ be the double dominating set such that $|V[n(G)] - D^d| \geq 1$, then $\{V[n(G)] - D^d\}$ contains at least one vertex which gives $|n - m| \leq |D^d|$. Hence $\gamma_{ddn}(T) \geq p - m$.

**Theorem 14.** For any nontrival tree $(p, q)$ tree $T$ with $k$ number of cut vertices, then $p - k \leq \gamma_{ddn}(T)$.

**Proof.** Let $S = X \cup C$ be the set of vertices of $n(T)$ where $X = \{v_1, v_2, v_3, ..., v_i\}$ and $C = \{c_1, c_2, c_3, ..., c_j\}$, $j < i$ are the vertices of $n(T)$ corresponding the edges and cutvertices of $T$ respectively. Now consider the set $D^d = X' = \{v_1, v_2, v_3, ..., v_i\} \subseteq X \subseteq V[n(T)]$ be the minimal set of vertices which covers all the vertices in $n(T)$. Suppose any vertex $v \in V[n(T)] - X'$ has at least two neighbours in $X'$ then $D^d$ itself is a double dominating set of $n(T)$. Clearly it follows that $|V[n(T)] - C| \leq |D^d|$. Hence $p - k \leq \gamma_{ddn}(T)$.

**Corollary 2.** For any path $P_n$, $n \geq 3$, $\gamma_{ddn}(P_n) \geq p - k$ where $k$ be the cutvertices.

**Proof.** From Theorem 14 the result follows.
**Theorem 15.** For any connected $(p, q)$ graph $G$, $\gamma_{dd}(G) \leq \gamma_{dn}(G)$, equality holds if $G$ a block graph.

**Proof.** Since $V[L(G)] \subseteq V[n(G)]$ by definition, then the result follows. If $G$ is a block, then $V[L(G)] = V[n(G)]$ and $L(G) \cong n(G)$. Hence the equality holds.

**Theorem 16.** If $T$ is a tree which is not a star, then $\gamma_{dn}(T) \geq \beta_0$.

**Proof.** Suppose $T = K_{1,p}, p \geq 3$. Then $\beta_0 = p > \gamma_{dd}(T)$. Let $K = \{v_1, v_2, v_3, ..., v_n\} \subseteq V(G)$ be the maximum set of vertices such that $d(v_i, v_j) \geq 2$ and $N(v_i) \cap N(v_j) = \emptyset$. Let $E_1 = \{e_1, e_2, e_3, ..., e_m\}, E_2 = \{e_1, e_2, e_3, ..., e_n\}, C = \{c_1, c_2, c_3, ..., c_k\}$ be the set of end edges, non-end edges and cut vertices of $G$. By the definition of $n(G)$, $V[n(G)] = E_1 \cup E_2 \cup C$ and each block of $n(G)$ is complete. Suppose $E_1' = \{e_1, e_2, e_3, ..., e_j\} \subseteq E_1, E_2' = \{e_1, e_2, e_3, ..., e_j\} \subseteq E_2$ be the set of vertices and cut vertices corresponding to the edges of $G$. Then $E_1' \cup E_2'$ covers all the vertices of $n(G)$ such that $\forall v_i \in E_1' \cup E_2'$ covers at least two vertices of $V[u(G)] - \{E_1' \cup E_2'\}$. Then $\gamma_{dd}(T)$ set. Otherwise $E_2' \cup C_1$ where $C_1 \subset C$ gives $\gamma_{dd}(T)$ - set. Hence in all the cases with $|E_1' \cup E_2'| \geq |K|$ or $|E_1' \cup C_1| \geq |K|$ gives $\gamma_{dd}(T) \geq \beta_0(T)$.

**Theorem 17.** For any connected graph $G$, $n(G) \neq k_n, n > 4$ vertices $\gamma_{dd}(G) \geq \left\lceil \frac{n}{2} \right\rceil$.

**Proof.** We consider the following cases.

Case(i): Suppose $G$ is a tree with $V = \{v_1, v_2, v_3, ..., v_n\}$ be the set of all vertices in $T$. Then $V_t = \{v_1, v_2, v_3, ..., v_i\}$ be the set of all end vertices in $T$ and let $E_1 = \{e_1, e_2, e_3, ..., e_j\}$ be the set of all non-end edges in $T$ and also $E_2 = \{e_1, e_2, e_3, ..., e_j\}$ be the set of all end edges of $T$. Let $C$ be the set of all cut vertices in $T$. $V[n(T)] = E(T) \cup C(T) = E_1 \cup E_2 \cup C$. Suppose $D^d$ be a $\gamma_{dd}(T)$ - set of $T$ such that $D^d = E_2' \cup E_1'$ where $E_2' \subset E_2, E_1' \cup E_1$ which gives $|E_2' \cup E_1'| = \gamma_{dd}(T) \geq \frac{|V|}{2}$ implies that $\gamma_{dd}(G) \geq \left\lceil \frac{n}{2} \right\rceil$.

Case(ii): Suppose $G$ is not a tree. Then there exists at least one edge joining two distinct vertices of a tree $T$, which from a cycle. From case(i) $|V[n(G)]| \geq |E_2' \cup E_1' \cup C_1| + 1$, where $C_1 \subset C$, it follows that $|E_2' \cup E_1'| + 1 \geq \left\lceil \frac{|V|}{2} \right\rceil + 1$, which implies $\gamma_{dd}(G) \geq \left\lceil \frac{n}{2} \right\rceil$. 


Theorem 18. For any connected \((p, q)\) graph, \(\gamma_{ddn} \geq \left\lceil \frac{p}{\Delta(G)} \right\rceil\). Equality holds if \(G \cong K_{1,p}, p \geq 2\).

Proof. Let \(D\) be a dominating set of \(n(G)\) and \(V_1 = V[n(G)] - D\) such that \(V_1 \in N(D)\). Let \(D_2 \subseteq V_1\) and \(D_2 \in N(D)\), then \(D^d = D_1 \cup D_2\) is a double dominating set of \(n(G)\). Further, let \(C = \{v_1, v_2, v_3, \ldots, v_k\}\) be the set of all non-end vertices in \(G\), then there exists at least one vertex \(v\) of maximum degree \(\Delta(G)\) in \(C\), such that \(|D_1 \cup D_2| \cdot \Delta(G) \geq p\). It follows that \(\gamma_{ddn} \geq \left\lceil \frac{p}{\Delta(G)} \right\rceil\). Suppose \(G\) is isomorphic to \(K_{1,p}, p \geq 1\). Then \(n(G) \cong K_{p+1}\), clearly \(\gamma_{ddn}(G) = 2\). Since for any given graph \(G \cong K_{1,p}, p = \Delta(G) + 1\) and \(\left\lceil \frac{p}{\Delta(G)} \right\rceil = 2\). Hence it follows that \(, \gamma_{ddn} = \left\lceil \frac{p}{\Delta(G)} \right\rceil\).

REFERENCES


