Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 3049-3065

http://www.ripublication.com

General solution and Ulam-Hyers Stability of Duodeviginti Functional Equations in Multi-Banach Spaces

R. Murali [1] **, M. Eshaghi-Gordji** [2] **, A. Antony Raj** [3]

[1,3] *Department of Mathematics, Sacred Heart College, Tirupattur-635 601, TamilNadu, India.*

[2] *Department of Mathematics, Semnan University, Iran.*

Abstract

In this current work, we carry out the general solution and Hyers-Ulam stability for a new form of Duodeviginti functional equation in Multi-Banach Spaces by using fixed point technique.

1. INTRODUCTION

The stability problem of functional equations has a long History. When we say the functional equation is stable, if for every approximate solution, there exists an exact solution near it. In 1940, famous Ulam [22] gave a wide-ranging talk before the mathematics club of the university of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms, which was first solved by D.H.Hyers [7], in 1941. Thereafter, several stability problems for different type of functional equations generalized by many authors for functions with more general domains and co-domains ([1], [3], [7], [8], [10]). In this stability results have many applications in physics, economic theory and social and behavioral science [2].

In last few years, some authors have been estabilished the stability of different type of functional equations in Multi-Banach spaces [6], [11],[20], [21],[23], [24].

In recently M Eshaghi, S Abbaszadeh [12], [13], [14] have been estabilished the stability problems in Banach Algebra. And also John M. Rassias, M. Arunkumar, E. Sathya and T. Namachivayam [9] established the general solution and also proved Stability of Nonic Functional Equation in Felbin's type fuzzy normed space and intuitionistic fuzzy normed space using direct and fixed point method.

In 2016, the general solution and generalized Hyers-Ulam stability of a decic type functional equation in Banach spaces, generalized 2-normed spaces and random normed spaces by using direct and fixed point methods was discussed by Mohan Arunkumar, Abasalt Bodaghi, John Michael Rassias and Elumalai Sathiya [15].

In K. Ravi, J.M. Rassias and B.V. Senthil Kumar [18] estabilished Ulam-Hyers stability of undecic functional equation in quasi β-normed space by Fixed point method.

In K. Ravi, J.M. Rassias, S. Pinelas and S.Suresh [19] estabilished the general solution and prove the stability of this quattuordecic functional equation in quasi β-normed spaces by Fixed point method.

Very recently, R. Murali, A. Antony Raj discussed the Hexadecic functional equation and its stability in Multi-Banach Spaces.

In this current work, we carry out the general solution and Hyers-Ulam stability for a new form of Duodeviginti functional equation

$$\begin{aligned}\mathcal{G}\varphi(u,v) &= \varphi(u+9v)-18\varphi(u+8v)+153\varphi(u+7v)-816\varphi(u+6v)+3060\varphi(u+5v)\\ &-8568\varphi(u+4v)+18564\varphi(u+3v)-31824\varphi(u+2v)+43758\varphi(u+v)-48620\varphi(u)\\ &+43758\varphi(u-v)-31824\varphi(u-2v)+18564\varphi(u-3v)-8568\varphi(u-4v)+3060\varphi(u-5v)\\ &-816\varphi(u-6v)+153\varphi(u-7v)-18\varphi(u-8v)+\varphi(u-9v)-6402373706000000\varphi(v)\end{aligned} \tag{1.1}$$

in Multi-Banach Spaces by using fixed point Technique.

It is easily verified that that the function $\varphi(u)=u^{18}$ satisfies the above functional equations. In other words, every solution of the Duodeviginti functional equation is called a Duodeviginti mapping.

Now, let us recall regarding some concepts in Multi-Banach spaces.

Definition 1.1 [5] *A Multi- norm on* $\{\mathcal{A}^k : k\in\mathbb{N}\}$ *is a sequence* $(\|.\|)=(\|.\|_k : k\in\mathbb{N})$ *such that* $\|.\|_k$ *is a norm on* $\mathcal{A}^k$ *for each* $k\in\mathbb{N}, \|x\|_1=\|x\|$ *for each* $x\in\mathcal{A}$, *and the following axioms are satisfied for each* $k\in\mathbb{N}$ *with* $k\geq 2$:

- $\left\|\left(x_{\sigma(1)},...,x_{\sigma(k)}\right)\right\|_k = \left\|\left(x_1...x_k\right)\right\|_k$, for $\sigma\in\Psi_k, x_1,...,x_k\in\mathcal{A}$;
- $\left\|\left(\alpha_1 x_1,...,\alpha_k x_k\right)\right\|_k \leq \left(\max_{i\in\mathbb{N}_k}|\alpha_i|\right)\left\|\left(x_1...x_k\right)\right\|_k$

for $\alpha_1...\alpha_k\in\mathbb{C}, x_1,...,x_k\in\mathcal{A}$;

- $\left\|\left(x_1,...,x_{k-1},0\right)\right\|_k = \left\|\left(x_1,...,x_{k-1}\right)\right\|_{k-1}$, for $x_1,...,x_{k-1}\in\mathcal{A}$;
- $\left\|\left(x_1,...,x_{k-1},x_{k-1}\right)\right\|_k = \left\|\left(x_1,...,x_{k-1}\right)\right\|_{k-1}$ for $x_1,...,x_{k-1}\in\mathcal{A}$.

In this case, we say that $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed space.

Suppose that $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed spaces, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [5].

$$(a)\left\|\left(x, ..., x\right)\right\|_k = \|x\|, \text{for} x \in \mathcal{A},$$

$$(b)\max_{i\in\mathbb{N}_k}\|x_i\| \le \left\|\left(x_1, ..., x_k\right)\right\|_k \le \sum_{i=1}^{k}\|x_i\| \le k\max_{i\in\mathbb{N}_k}\|x_i\|, \forall x_1, ..., x_k \in \mathcal{A}.$$

It follows from (b) that if $(\mathcal{A}, \|.\|)$ is a Banach space, then $(\mathcal{A}^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; In this case, $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - Banach space.

Definition 1.2 [5] 1 *Let* $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ *be a multi - normed space. A sequence* (x_n) *in* $\mathcal{A}$ *is a multi-null sequence if for each* $\delta > 0,$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{k\in\mathbb{N}}\left\|\left(x_n, ..., x_{n+k-1}\right)\right\|_k \le \delta \qquad \left(n \ge n_0\right). \tag{1.2}$$

Let $x \in \mathcal{A},$ we say that the sequence (x_n) is multi-convergent to x in $\mathcal{A}$ and write $\lim_{n\to\infty} x_n = x$ if $\left(x_n - x\right)$ is a multi - null sequence.

2. GENERAL SOLUTION OF (1.1)

Theorem 2.1 2 *If* $\varphi : \mathcal{U} \to \mathcal{V}$ *be the vector spaces. If* $\varphi : \mathcal{U} \to \mathcal{V}$ *be a function (1.1) for all* $u, v \in \mathcal{U},$ *then* φ *is Viginti Duo Mapping.*

Proof. Substituting $u = 0$ and $v = 0$ in (1.1), we obtain that $\varphi(0) = 0.$ Substituting (u, v) with (u, u) and $(u, -u)$ in (1.1), respectively, and subtracting two resulting equations, we can obtain $\varphi(-u) = \varphi(u),$ that is to say, φ is an even function.

Changing (u, v) by $(9u, u)$ and $(0, 2u),$ in (1.1) respectively, and subtracting the two resulting equations, we arrive

$$\begin{aligned}&18\varphi(17u) - 171\varphi(16u) + 816\varphi(15u) - 2907\varphi(14u) + 8568\varphi(13u) - 19380\varphi(12u)\\&+ 31824\varphi(11u) - 40698\varphi(10u) + 48620\varphi(9u) - 52326\varphi(8u) + 31824\varphi(7u)\\&+ 8568\varphi(5u) - 34884\varphi(4u) + 816\varphi(3u) - 3201186853000000\varphi(2u) + 18!\varphi(u) = 0\end{aligned} \tag{2.1}$$

$\forall u \in \mathcal{U}.$ Doing $u = 8u$ by $v = u$ in (1.1), we obtain

$$\begin{aligned}&\varphi(17u) - 18\varphi(16u) + 153\varphi(15u) - 816\varphi(14u) + 3060\varphi(13u)\\&- 8568\varphi(12u) + 18564\varphi(11u) - 31824\varphi(10u) + 43758\varphi(9u) - 48620\varphi(8u)\\&+ 43758\varphi(7u) - 31824\varphi(6u) + 18564\varphi(5u) - 8568\varphi(4u) + 3060\varphi(3u)\\&- 816\varphi(2u) - 18!\varphi(u) = 0\end{aligned} \tag{2.2}$$

$\forall u \in \mathcal{U}$. Multiplying (2.2) by 18, and then subtracting (2.1) from the resulting equation, we arrive

$$\begin{aligned}&153\varphi(16u)-1938\varphi(15u)+11781\varphi(14u)-46512\varphi(13u)+134844\varphi(12u)\\&-302328\varphi(11u)+532134\varphi(10u)-739024\varphi(9u)+822834\varphi(8u)\\&-755820\varphi(7u)+572832\varphi(6u)-325584\varphi(5u)+119340\varphi(4u)\\&-54264\varphi(3u)-3201186853000000\varphi(2u)+18!(19)\varphi(u)=0\end{aligned} \tag{2.3}$$

$\forall u \in \mathcal{U}$. Replacing $u=7u$ by $v=u$ in (1.1), we obtain

$$\begin{aligned}&\varphi(16u)-18\varphi(15u)+153\varphi(14u)-816\varphi(13u)+3060\varphi(12u)-8568\varphi(11u)\\&+18564\varphi(10u)-31824\varphi(9u)+43758\varphi(8u)-48620\varphi(7u)+43758\varphi(6u)\\&-31824\varphi(5u)+18564\varphi(4u)-8568\varphi(3u)+3061\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.4}$$

$\forall u \in \mathcal{U}$. Multiplying (2.4) by 153, and then subtracting (2.3) from the resulting equation, we arrive

$$\begin{aligned}&816\varphi(15u)-11628\varphi(14u)+78336\varphi(13u)-333336\varphi(12u)+1008576\varphi(11u)\\&-2308158\varphi(10u)+4130048\varphi(9u)-5872140\varphi(8u)+6683040\varphi(7u)\\&-6122142\varphi(6u)+4543488\varphi(5u)-2720952\varphi(4u)+1256640\varphi(3u)\\&-3201186853000000\varphi(2u)+18!(172)\varphi(u)=0\end{aligned} \tag{2.5}$$

$\forall u \in \mathcal{U}$. Doing $u=6u$ and $v=u$ in (1.1), we obtain

$$\begin{aligned}&\varphi(15u)-18\varphi(14u)+153\varphi(13u)-816\varphi(12u)+3060\varphi(11u)-8568\varphi(10u)\\&+18564\varphi(9u)-31824\varphi(8u)+43758\varphi(7u)-48620\varphi(6u)\\&+43758\varphi(5u)-31824\varphi(4u)+18565\varphi(3u)-8586\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.6}$$

$\forall u \in \mathcal{U}$. Multiplying (2.6) by 816, and then subtracting (2.5) from the resulting equation, we get

$$\begin{aligned}&3060\varphi(14u)-46512\varphi(13u)+332520\varphi(12u)-1488384\varphi(11u)\\&+4683330\varphi(10u)-11018176\varphi(9u)+20096244\varphi(8u)-29023488\varphi(7u)\\&+33551778\varphi(6u)-31163040\varphi(5u)+23247432\varphi(4u)\\&-13892400\varphi(3u)-3201186846000000\varphi(2u)+18!(988)\varphi(u)=0\end{aligned} \tag{2.7}$$

$\forall u \in \mathcal{U}$. Doing $u=5u$ and $v=u$ in (1.1), we obtains

$$\begin{aligned}&\varphi(14u)-18\varphi(13u)+153\varphi(12u)-816\varphi(11u)+3060\varphi(10u)\\&-8568\varphi(9u)+18564\varphi(8u)-31824\varphi(7u)+43758\varphi(6u)\\&-48620\varphi(5u)+43759\varphi(4u)-31842\varphi(3u)+18717\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.8}$$

$\forall u \in \mathcal{U}$. Multiplying (2.8) by 3060, and then subtracting (2.7) from the resulting equation,we arrive

$$\begin{aligned}&8568\varphi(13u)-135660\varphi(12u)+1008576\varphi(11u)-4680270\varphi(10u)\\&+15199904\varphi(9u)-36709596\varphi(8u)+68357952\varphi(7u)-100347702\varphi(6u)\\&+117614160\varphi(5u)-110655108\varphi(4u)+83544120\varphi(3u)\\&-3201186904000000\varphi(2u)+18!(4048)\varphi(u)=0\end{aligned} \tag{2.9}$$

$\forall u \in \mathcal{U}$. Doing $x=4x$ and $y=x$ in (1.1), we get

$$\begin{aligned}&\varphi(13u)-18\varphi(12u)+153\varphi(11u)-816\varphi(10u)+3060\varphi(9u)\\&-8568\varphi(8u)+18564\varphi(7u)-31824\varphi(6u)+43759\varphi(5u)\\&-48638\varphi(4u)+43911\varphi(3u)-32640\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.10}$$

$\forall u \in \mathcal{U}$. Multiplying (2.10) by 8568, and then subtracting (2.9) from the resulting equation, we arrive

$$\begin{aligned}&18564\varphi(12u)-302328\varphi(11u)+2311218\varphi(10u)-11018176\varphi(9u)+36701028\varphi(8u)\\&-90698400\varphi(7u)+172320330\varphi(6u)-257312952\varphi(5u)+306075276\varphi(4u)\\&-292685328\varphi(3u)-3201186624000000\varphi(2u)+18!(12616)\varphi(u)=0\end{aligned} \tag{2.11}$$

$\forall u \in \mathcal{U}$. Changing $u=3u$ and $v=u$ in (1.1), we obtain

$$\begin{aligned}&\varphi(12u)-18\varphi(11u)+153\varphi(10u)-816\varphi(9u)+3060\varphi(8u)-8568\varphi(7u)+18565\varphi(6u)\\&-31842\varphi(5u)+43911\varphi(4u)-49436\varphi(3u)+46818\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.12}$$

$\forall u \in \mathcal{U}$. Multiplying (2.12) by 18564, and then subtracting (2.11) from the resulting equation, we arrive

$$\begin{aligned}&31824\varphi(11u)-529074\varphi(10u)+4130048\varphi(9u)-20104812\varphi(8u)\\&+68357952\varphi(7u)-172320330\varphi(6u)+333801936\varphi(5u)-509088528\varphi(4u)\\&+625044576\varphi(3u)-3201187493000000\varphi(2u)+18!(31180)\varphi(u)=0\end{aligned} \tag{2.13}$$

$\forall u \in \mathcal{U}$. Changing $u=2u$ and $v=u$ in (1.1), we obtain

$$\begin{aligned}&\varphi(11u)-18\varphi(10u)+153\varphi(9u)-816\varphi(8u)+3061\varphi(7u)-8586\varphi(6u)\\&+18717\varphi(5u)-32640(4u)+46818\varphi(3u)-57188\varphi(2u)-18!\varphi(u)=0\end{aligned} \tag{2.14}$$

$\forall u \in \mathcal{U}$. Multiplying (2.14) by 31824, and then subtracting (2.13) from the resulting equation, we arrive

$$\begin{aligned}&43758\varphi(10u)-739024\varphi(9u)+5863572\varphi(8u)-29055312\varphi(7u)\\&+100920534\varphi(6u)-261847872\varphi(5u)+529646832\varphi(4u)-864891456\varphi(3u)\\&-3201185673000000\varphi(2u)+18!(63004)\varphi(u)=0\end{aligned} \tag{2.15}$$

$\forall u \in \mathcal{U}$. Changing $u=u$ and $v=u$ in (1.1), we obtain

$$\varphi(10u)-18\varphi(9u)+154\varphi(8u)-834\varphi(7u)+3213\varphi(6u)$$
$$-9384\varphi(5u)+21624\varphi(4u)-40392\varphi(3u)+62322\varphi(2u)-18!\varphi(u)=0 \quad (2.16)$$

$\forall u\in\mathcal{U}$. Multiplying (2.16) by 43758, and then subtracting (2.15) from the resulting equation, we arrive

$$48620\varphi(9u)-875160\varphi(8u)+7438860\varphi(7u)-39673920\varphi(6u)$$
$$+148777200\varphi(5u)-416576160\varphi(4u)+902581680\varphi(3u) \quad (2.17)$$
$$-3201188400000000\varphi(2u)+18!(106762)\varphi(u)=0$$

$\forall u\in\mathcal{U}$. Changing $u=0$ and $v=u$ in (1.1), we obtain

$$\varphi(9u)-18\varphi(8u)+153\varphi(7u)-816\varphi(6u)+3060\varphi(5u)$$
$$-8568\varphi(4u)+18564\varphi(3u)-31824\varphi(2u)-3201186853000000\varphi(u)=0 \quad (2.18)$$

$\forall u\in\mathcal{U}$. Multiplying (2.18) by 48620, and then subtracting (2.17) from the resulting equation, we arrive

$$-3201186853000000\varphi(2u)+839171926400000000000\varphi(u)=0 \quad (2.19)$$

$\forall u\in\mathcal{U}$. It follows from (2.19), we reach

$$\varphi(2u)=2^{18}\varphi(u) \quad (2.20)$$

$\forall u\in\mathcal{U}$. Hence φ is a Duodeviginti Mapping.

This completes the proof.

3. STABILITY OF THE FUNCTIONAL EQUATION (1.1) IN MULTI-BANACH SPACES

Theorem 3.1 3 *Let $\mathcal{U}$ be an linear space and let $\left((\mathcal{V}^k,\|.\|_k):K\in\mathbb{N}\right)$ be a multi-Banach space. Suppose that δ is a non-negative real number and $\varphi:\mathcal{U}\to\mathcal{V}$ be a function fulfills*

$$\sup_{k\in\mathbb{N}}\left\|\left(\mathcal{G}\varphi(u_1,v_1),...,\mathcal{G}\varphi(u_k,v_k)\right)\right\|_k\le\delta \quad (3.1)$$

$u_1,...,u_k,v_1,...,v_k\in\mathcal{U}$. Then there exists a unique Duodeviginti function $\mathbb{D}:\mathcal{U}\to\mathcal{V}$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(\varphi(u_1)-\mathbb{D}(u_1),...,\varphi(u_k)-\mathbb{D}(u_k)\right)\right\|_k\le\frac{4033}{2582057616000000000}\delta. \quad (3.2)$$

forall $u_i\in\mathcal{U}$, where $i=1,2,...,k$.

Proof. Taking $u_i=0$ and Changing v_i by $2u_i$ in (3.1), and dividing by 2 in the resulting equation, we arrive

$$\sup_{k\in\mathbb{N}}\Big\|\big(\varphi(18u_1)-18\varphi(16u_1)+153\varphi(14u_1)-816\varphi(12u_1)+3060\varphi(10u_1)$$
$$-8568\varphi(8u_1)+18564\varphi(6u_1)-31824\varphi(4u_1)-3201186853000000\varphi(2u_1),...,$$
$$\varphi(18u_k)-18\varphi(16u_k)+153\varphi(14u_k)-816\varphi(12u_k)+3060\varphi(10u_k)$$
$$-8568\varphi(8u_k)+18564\varphi(6u_k)-31824\varphi(4u_k)-3201186853000000\varphi(2u_k))\big)\Big\|_k \le \frac{\delta}{2} \qquad (3.3)$$

forall $u_1,...,u_k \in \mathcal{U}$. Plugging $u_1,...,u_k$ into $9u_1,...,9u_k$ and Changing $v_1,v_2,...,v_k$ by $u_1,...,u_k$ in (3.1), we have

$$\sup_{k\in\mathbb{N}}\Big\|\big((\varphi(18u_1)-18\varphi(17u_1)+153\varphi(16u_1)-816\varphi(15u_1)+3060\varphi(14u_1)$$
$$-8568\varphi(13u_1)+18564\varphi(12u_1)-31824\varphi(11u_1)+43758\varphi(10u_1)$$
$$-48620\varphi(9u_1)+43758\varphi(8u_1)-31824\varphi(7u_1)+18564\varphi(6u_1)-8568\varphi(5u_1)$$
$$+3060\varphi(4u_1)-816\varphi(3u_1)+153\varphi(2u_1)-18!\varphi(u_1),...,\varphi(18u_k)$$
$$-18\varphi(17u_k)+153\varphi(16u_k)-816\varphi(15u_k)+3060\varphi(14u_k)$$
$$-8568\varphi(13u_k)+18564\varphi(12u_k)-31824\varphi(11u_k)+43758\varphi(10u_k)$$
$$-48620\varphi(9u_k)+43758\varphi(8u_k)-31824\varphi(7u_k)+18564\varphi(6u_k)-8568\varphi(5u_k)$$
$$+3060\varphi(4u_k)-816\varphi(3u_k)+153\varphi(2u_k)-18!\varphi(u_k)\big)\Big\|_k \le \delta \qquad (3.4)$$

forall $u_1,...,u_k \in \mathcal{A}$. Combining (3.3) and (3.4), we have

$$\sup_{k\in\mathbb{N}}\Big\|\big(18\varphi(17u_1)-171\varphi(16u_1)+816\varphi(15u_1)-2907\varphi(14u_1)$$
$$+8568\varphi(13u_1)-19380\varphi(12u_1)+31824\varphi(11u_1)-40698\varphi(10u_1)+48620\varphi(9u_1)$$
$$-52326\varphi(8u_1)+31824\varphi(7u_1)-8568\varphi(5u_1)-34884\varphi(4u_1)+816\varphi(3u_1)$$
$$-3201186853000000\varphi(2u_1)+18!\varphi(u_1),...,18\varphi(17u_k)-171\varphi(16u_k)+816\varphi(15u_k)$$
$$-2907\varphi(14u_k)+8568\varphi(13u_k)-19380\varphi(12u_k)+31824\varphi(11u_k)-40698\varphi(10u_k)$$
$$+48620\varphi(9u_k)-52326\varphi(8u_k)+31824\varphi(7u_k)+8568\varphi(5u_k)-34884\varphi(4u_k)$$
$$+816\varphi(3u_k)-3201186853000000\varphi(2u_k)+18!\varphi(u_k)\big)\Big\|_k \le \frac{3}{2}\delta \qquad (3.5)$$

forall $u_1,...,u_k \in \mathcal{A}$. Taking $u_1,...,u_k$ by $8u_1,...,8u_k$ and Changing $y_1,y_2,...,y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\sup_{k\in\mathbb{N}}\Big\|\big(\varphi(17u_1)-18\varphi(16u_1)+153\varphi(15u_1)-816\varphi(14u_1)+3060\varphi(13u_1)-8568\varphi(12u_1)$$
$$+18564\varphi(11u_1)-31824\varphi(10u_1)+43758\varphi(9u_1)-48620\varphi(8u_1)+43758\varphi(7u_1)-31824\varphi(6u_1)$$
$$+18564\varphi(5u_1)-8568\varphi(4u_1)+3060\varphi(3u_1)-816\varphi(2u_1)-18!\varphi(u_1),...,$$
$$\varphi(17u_k)-18\varphi(16u_k)+153\varphi(15u_k)-816\varphi(14u_k)+3060\varphi(13u_k)-8568\varphi(12u_k)$$
$$+18564\varphi(11u_k)-31824\varphi(10u_k)+43758\varphi(9u_k)-48620\varphi(8u_k)+43758\varphi(7u_k)$$

$$-31824\varphi(6u_k)+18564\varphi(5u_k)-8568\varphi(4u_k)+3060\varphi(3u_k)-816\varphi(2u_k)-18!\varphi(u_k))\|_k \leq \delta \quad (3.6)$$

forall $u_1,...,u_k \in \mathcal{A}$. Multiplying by 18 on both sides of (3.6),then it follows from (3.5) and the resulting equation we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\|(153\varphi(16u_1)-1938\varphi(15u_1)+11781\varphi(14u_1)-46512\varphi(13u_1)+134844\varphi(12u_1)\\&-302328\varphi(11u_1)+532134\varphi(10u_1)-739024\varphi(9u_1)+822834\varphi(8u_1)-755820\varphi(7u_1)\\&+572832\varphi(6u_1)-325584\varphi(5u_1)+119340\varphi(4u_1)-54264\varphi(3u_1)-3201186853000000\varphi(2u_1)\\&+18!(19)\varphi(u_1),...,153\varphi(16u_k)-1938\varphi(15u_k)+11781\varphi(14u_k)-46512\varphi(13u_k)\\&+134844\varphi(12u_k)-302328\varphi(11u_k)+532134\varphi(10u_k)-739024\varphi(9u_k)+822834\varphi(8u_k)\\&-755820\varphi(7u_k)+572832\varphi(6u_k)-325584\varphi(5u_k)+119340\varphi(4u_k)\\&-54264(3u_k)-3201186853000000\varphi(2u_k)+18!(19)\varphi(u_k))\|_k \leq \frac{39}{2}\delta \quad (3.7)\end{aligned}$$

$u_1,...,u_k \in \mathcal{U}$. Taking $u_1,...,u_k$ by $7u_1,...,7u_k$ and Changing $y_1,y_2,...,y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\|(\varphi(16u_1)-18\varphi(15u_1)+153\varphi(14u_1)-816\varphi(13u_1)+3060\varphi(12u_1)-8568\varphi(11u_1)\\&+18564\varphi(10u_1)-31824\varphi(9u_1)+43758\varphi(8u_1)-48620\varphi(7u_1)+43758\varphi(6u_1)\\&-31824\varphi(5u_1)+18564\varphi(4u_1)-8568\varphi(3u_1)+3061\varphi(2u_1)-18!\varphi(u_1),...,\varphi(16u_k)-18\varphi(15u_k)\\&+153\varphi(14u_k)-816\varphi(13u_k)+3060\varphi(12u_k)-8568\varphi(11u_k)+18564\varphi(10u_k)-31824\varphi(9u_k)\\&+43758\varphi(8u_k)-48620\varphi(7u_k)+43758\varphi(6u_k)-31824\varphi(5u_k)\\&+18564\varphi(4u_k)-8568\varphi(3u_k)+3061\varphi(2u_k)-18!\varphi(u_k))\|_k \leq \delta \quad (3.8)\end{aligned}$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.8) by 153, then it follows from (3.7) and the resulting equation we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\|(816\varphi(15u_1)-11628\varphi(14u_1)+78336\varphi(13u_1)-333336\varphi(12u_1)+1008576\varphi(11u_1)\\&-2308158\varphi(10u_1)+4130048\varphi(9u_1)-5872140\varphi(8u_1)+6683040\varphi(7u_1)-6122142\varphi(6u_1)\\&+4543488\varphi(5u_1)-2720952\varphi(4u_1)+1256640\varphi(3u_1)-3201186853000000\varphi(2u_1)+18!(172)\varphi(u_1),...,\\&816\varphi(15u_k)-11628\varphi(14u_k)+78336\varphi(13u_k)-333336\varphi(12u_k)+1008576\varphi(11u_k)-2308158\varphi(10u_k)\\&+4130048\varphi(9u_k)-5872140\varphi(8u_k)+6683040\varphi(7u_k)-6122142\varphi(6u_k)+4543488\varphi(5u_k)\\&-2720952\varphi(4u_k)+1256640\varphi(3u_k)-3201186853000000\varphi(2u_k)+18!(172)\varphi(u_k))\|_k \leq \frac{345}{2}\delta \quad (3.9)\end{aligned}$$

forall $u_1,...,u_k \in \mathcal{U}$. Taking $u_1,...,u_k$ by $6u_1,...,6u_k$ and Changing $v_1,v_2,...,v_k$ by $u_1,...,u_k$ in (3.1) and using evenness of φ we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\big\|\big(\varphi(15u_1)-18\varphi(14u_1)+153\varphi(13u_1)-816\varphi(12u_1)+3060\varphi(11u_1)-8568\varphi(10u_1)\\&+18564\varphi(9u_1)-31824\varphi(8u_1)+43758\varphi(7u_1)-48620\varphi(6u_1)+43758\varphi(5u_1)-31824\varphi(4u_1)\\&+18565\varphi(3u_1)-8586\varphi(2u_1)-18!\varphi(u_1),...,\varphi(15u_k)-18\varphi(14u_k)+153\varphi(13u_k)-816\varphi(12u_k)\\&+3060\varphi(11u_k)-8568\varphi(10u_k)+18564\varphi(9u_k)-31824\varphi(8u_k)+43758\varphi(7u_k)-48620\varphi(6u_k)\\&+43758\varphi(5u_k)-31824\varphi(4u_k)+18565\varphi(3u_k)-8586\varphi(2u_k)-18!\varphi(u_k)\big)\big\|_k\le\delta \qquad (3.10)\end{aligned}$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.10) by 816, then it follows from (3.9) and the resulting equation we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\big\|\big(3060\varphi(14u_1)-46512\varphi(13u_1)+332520\varphi(12u_1)-1488384\varphi(11u_1)+4683330\varphi(10u_1)\\&-11018176\varphi(9u_1)+20096244\varphi(8u_1)-29023488\varphi(7u_1)+33551778\varphi(6u_1)\\&-31163040\varphi(5u_1)+23247432\varphi(4u_1)-13892400\varphi(3u_1)-3201186846000000\varphi(2u_1)\\&+18!(988)\varphi(u_1),...,3060\varphi(14u_k)-46512\varphi(13u_k)+332520\varphi(12u_k)\\&-1488384\varphi(11u_k)+4683330\varphi(10u_k)-11018176\varphi(9u_k)+20096244\varphi(8u_k)\\&-29023488\varphi(7u_k)+33551778\varphi(6u_k)-31163040\varphi(5u_k)+23247432\varphi(4u_k)\\&-13892400\varphi(3u_k)-3201186846000000\varphi(2u_k)+18!(988)\varphi(u_k)\big)\big\|_k\le\frac{1977}{2}\delta \qquad (3.10)\end{aligned}$$

forall $u_1,...,u_k \in \mathcal{U}$. Plugging $u_1,...,u_k$ into $5u_1,...,5u_k$ and Changing $v_1,v_2,...,v_k$ by $u_1,...,u_k$ in (3.1) and using evenness of φ, we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\big\|\big(\varphi(14u_1)-18\varphi(13u_1)+153\varphi(12u_1)-816\varphi(11u_1)+3060\varphi(10u_1)-8568\varphi(9u_1)\\&+18564\varphi(8u_1)-31824\varphi(7u_1)+43758\varphi(6u_1)-48620\varphi(5u_1)+43759\varphi(4u_1)-31842\varphi(3u_1)\\&+18717\varphi(2u_1)-18!\varphi(u_1),...,\varphi(14u_k)-18\varphi(13u_k)+153\varphi(12u_k)-816\varphi(11u_k)\\&+3060\varphi(10u_k)-8568\varphi(9u_k)+18564\varphi(8u_k)-31824\varphi(7u_k)+43758\varphi(6u_k)-48620\varphi(5u_k)\\&+43759\varphi(4u_k)-31842\varphi(3u_k)+18717\varphi(2u_k)-18!\varphi(u_k)\big)\big\|_k\le\delta \qquad (3.12)\end{aligned}$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.12) by 3060, then it follows from (3.11) and the resulting equation we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\big\|\big(8568\varphi(13u_1)-135660\varphi(12u_1)+1008576\varphi(11u_1)-4680270\varphi(10u_1)\\&+15199904\varphi(9u_1)-36709596\varphi(8u_1)+68357952\varphi(7u_1)-100347702\varphi(6u_1)+117614160\varphi(5u_1)\\&-110655108\varphi(4u_1)+83544120\varphi(3u_1)-3201186904000000\varphi(2u_1)+18!(4048)\varphi(u_1),...,\\&8568\varphi(13u_k)-135660\varphi(12u_k)+1008576\varphi(11u_k)-4680270\varphi(10u_k)+15199904\varphi(9u_k)\end{aligned}$$

$$-36709596\varphi(8u_k)+68357952\varphi(7u_k)-100347702\varphi(6u_k)+117614160\varphi(5u_k)-110655108\varphi(4u_k)$$
$$+83544120\varphi(3u_k)-3201186904000000\varphi(2u_k)+18!(4048)\varphi(u_k))\|_k \le \frac{8097}{2}\delta \quad (3.13)$$

forall $u_1,...,u_k \in \mathcal{U}$. Plugging $u_1,...,u_k$ into $4u_1,...,4u_k$ and Changing $y_1, y_2,..., y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\sup_{k\in\mathbb{N}}\|(\varphi(13u_1)-18\varphi(12u_1)+153\varphi(11u_1)-816\varphi(10u_1)+3060\varphi(9u_1)-8568\varphi(8u_1)$$
$$+18564\varphi(7u_1)-31824\varphi(6u_1)+43759\varphi(5u_1)-48638\varphi(4u_1)+43911\varphi(3u_1)-32640\varphi(2u_1)$$
$$-18!\varphi(u_1),...,\varphi(13u_k)-18\varphi(12u_k)+153\varphi(11u_k)-816\varphi(10u_k)+3060\varphi(9u_k)$$
$$-8568\varphi(8u_k)+18564\varphi(7u_k)-31824\varphi(6u_k)+43759\varphi(5u_k)-$$
$$48638\varphi(4u_k)+43911\varphi(3u_k)-32640\varphi(2u_k)-18!\varphi(u_k))\|_k \le \delta \quad (3.14)$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.14) by 8568, then it follows from (3.13) and the resulting equation we arrive

$$\sup_{k\in\mathbb{N}}\|(18564\varphi(12u_1)-302328\varphi(11u_1)+2311218\varphi(10u_1)-11018176\varphi(9u_1)$$
$$+36701028\varphi(8u_1)-90698400\varphi(7u_1)+172320330\varphi(6u_1)-257312952\varphi(5u_1)$$
$$+306075276\varphi(4u_1)-292685328\varphi(3u_1)-3201186624000000\varphi(2u_1)+18!(12616)\varphi(u_1),...,$$
$$18564\varphi(12u_k)-302328\varphi(11u_k)+2311218\varphi(10u_k)-11018176\varphi(9u_k)+36701028\varphi(8u_k)$$
$$-90698400\varphi(7u_k)+172320330\varphi(6u_k)-257312952\varphi(5u_k)+306075276\varphi(4u_k)$$
$$-292685328\varphi(3u_k)-3201186624000000\varphi(2u_k)+18!(12616)\varphi(u_k))\|_k \le \frac{25233}{2}\delta \quad (3.15)$$

forall $u_1,...,u_k \in \mathcal{U}$. Taking $u_1,...,u_k$ by $3u_1,...,u_k$ and Changing $y_1, y_2,..., y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\sup_{k\in\mathbb{N}}\|(\varphi(12u_1)-18\varphi(11u_1)+153\varphi(10u_1)-816\varphi(9u_1)+3060\varphi(8u_1)-8568\varphi(7u_1)$$
$$+18565\varphi(6u_1)-31842\varphi(5u_1)+43911\varphi(4u_1)-49436\varphi(3u_1)+46818\varphi(2u_1)-18!\varphi(u_1),...,$$
$$\varphi(12u_k)-18\varphi(11u_k)+153\varphi(10u_k)-816\varphi(9u_k)+3060\varphi(8u_k)-8568\varphi(7u_k)$$
$$+18565\varphi(6u_k)-31842\varphi(5u_k)+43911\varphi(4u_k)-49436\varphi(3u_k)+46818\varphi(2u_k)-18!\varphi(u_k))\|_k \le \delta \quad (3.16)$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.16) by 18564, then it follows from (3.15) and the resulting equation we arrive

$$\sup_{k\in\mathbb{N}}\|(31824\varphi(11u_1)-529074\varphi(10u_1)+4130048\varphi(9u_1)-20104812\varphi(8u_1)$$
$$+68357952\varphi(7u_1)-172320330\varphi(6u_1)+333801936\varphi(5u_1)-509088528\varphi(4u_1)$$
$$+625044576\varphi(3u_1)-3201187493000000\varphi(2u_1)+18!(31180)\varphi(u_1),...,$$

$$31824\varphi(11u_k)-529074\varphi(10u_k)+4130048\varphi(9u_k)-20104812\varphi(8u_k)$$
$$+68357952\varphi(7u_k)-172320330\varphi(6u_k)+333801936\varphi(5u_k)-509088528\varphi(4u_k)$$
$$+625044576\varphi(3u_k)-3201187493000000\varphi(2u_k)+18!(31180)\varphi(u_k))\|_k \le \frac{62361}{2}\delta \qquad (3.17)$$

forall $u_1,...,u_k \in \mathcal{U}$. Taking $u_1,...,u_k$ by $2u_1,...,2u_k$ and Changing $y_1, y_2,..., y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\sup_{k\in\mathbb{N}}\|(\varphi(11u_1)-18\varphi(10u_1)+153\varphi(9u_1)-816\varphi(8u_1)+3061\varphi(7u_1)-8586\varphi(6u_1)$$
$$+18717\varphi(5u_1)-32640(4u_1)+46818\varphi(3u_1)-57188\varphi(2u_1)-18!\varphi(u_1),...,$$
$$\varphi(11u_k)-18\varphi(10u_k)+153\varphi(9u_k)-816\varphi(8u_k)+3061\varphi(7u_k)-8586\varphi(6u_k)$$
$$+18717\varphi(5u_k)-32640(4u_k)+46818\varphi(3u_k)-57188\varphi(2u_k)-18!\varphi(u_k))\|_k \le \delta \qquad (3.18)$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.18) by 31824, then it follows from (3.17) and the resulting equation we arrive

$$\sup_{k\in\mathbb{N}}\|(43758\varphi(10u_1)-739024\varphi(9u_1)+5863572\varphi(8u_1)-29055312\varphi(7u_1)$$
$$+100920534\varphi(6u_1)-261847872\varphi(5u_1)+529646832\varphi(4u_1)-864891456\varphi(3u_1)$$
$$-3201185673000000\varphi(2u_1)+18!(63004)\varphi(u_1),...,43758\varphi(10u_k)-739024\varphi(9u_k)$$
$$+5863572\varphi(8u_k)-29055312\varphi(7u_k)+100920534\varphi(6u_k)-261847872\varphi(5u_k)+529646832\varphi(4u_k)$$
$$-864891456\varphi(3u_k)-3201185673000000\varphi(2u_k)+18!(63004)\varphi(u_k))\|_k \le \frac{126009}{2}\delta \qquad (3.19)$$

forall $u_1,...,u_k \in \mathcal{U}$. Changing $y_1, y_2,..., y_k$ by $u_1,...,u_k$ in (3.1) and using evenness of f, we arrive

$$\sup_{k\in\mathbb{N}}\|(\varphi(10u_1)-18\varphi(9u_1)+154\varphi(8u_1)-834\varphi(7u_1)+3213\varphi(6u_1)-9384\varphi(5u_1)$$
$$+21624\varphi(4u_1)-40392\varphi(3u_1)+62322\varphi(2u_1)-18!\varphi(u_1),...,$$
$$\varphi(10u_k)-18\varphi(9u_k)+154\varphi(8u_k)-834\varphi(7u_k)+3213\varphi(6u_k)-9384\varphi(5u_k)$$
$$+21624\varphi(4u_k)-40392\varphi(3u_k)+62322\varphi(2u_k)-18!\varphi(u_k))\|_k \le \delta \qquad (3.20)$$

forall $u_1,...,u_k \in \mathcal{U}$. Multiplying (3.20) by 43758, then it follows from (3.19) and the resulting equation we arrive

$$\sup_{k\in\mathbb{N}}\|(48620\varphi(9u_1)-875160\varphi(8u_1)+7438860\varphi(7u_1)-39673920\varphi(6u_1)$$
$$+148777200\varphi(5u_1)-416576160\varphi(4u_1)+902581680\varphi(3u_1)-3201188400000000\varphi(2u_1)$$
$$+18!(106762)\varphi(u_1),...,48620\varphi(9u_k)-875160\varphi(8u_k)+7438860\varphi(7u_k)$$
$$-39673920\varphi(6u_k)+148777200\varphi(5u_k)-416576160\varphi(4u_k)$$

$$+902581680\varphi(3u_k)-3201188400000000\varphi(2u_k)+18!(106762)\varphi(u_k))\Big\|_k \le \frac{213525}{2}\delta \quad (3.21)$$

forall $u_1,...,u_k \in \mathcal{U}.$ Taking $u_1 = u_2 =,...,= u_k = 0$ and Changing $y_1, y_2,..., y_k$ by $u_1,...,u_k$ (3.1), we obtain

$$\sup_{k\in\mathbb{N}}\Big\|\big(\varphi(9u_1)-18\varphi(8u_1)+153\varphi(7u_1)-816\varphi(6u_1)+3060\varphi(5u_1)-8568\varphi(4u_1)+18564\varphi(3u_1)$$

$$-31824\varphi(2u_1)-3201186853000000\varphi(u_1),...,\varphi(9u_k)-18\varphi(8u_k)+153\varphi(7u_k)-816\varphi(6u_k)$$

$$+3060\varphi(5u_k)-8568\varphi(4u_k)+18564\varphi(3u_k)-31824\varphi(2u_k)-3201186853000000\varphi(u_k))\Big\|_k \le \frac{\delta}{2} \quad (3.22)$$

forall $u_1,...,u_k \in \mathcal{U}.$ Multiplying (3.22) by 48620, then it follows from (3.21) and the resulting equation we arrive

$$\sup_{k\in\mathbb{N}}\Big\|\big(-3201186853000000\varphi(2u_1)+839171926400000000000\varphi(u_1),...,$$

$$-3201186853000000\varphi(2u_k)+839171926400000000000\varphi(u_k))\Big\|_k \le \frac{262145}{2}\delta \quad (3.23)$$

forall $u_1,...,u_k \in \mathcal{U}.$ Dividing on both sides by 3201186853000000 in (3.23), we arrive

$$\sup_{k\in\mathbb{N}}\Big\|\big(-\varphi(2u_k)+262144\varphi(u_k),...,-\varphi(2u_k)+262144\varphi(u_k)\big)\Big\|_k \le \frac{262145}{18!}\delta \quad (3.24)$$

forall $u_1,...,u_k \in \mathcal{U}.$ Again, dividing on both sides by 262144 in (3.24), we arrive

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{1}{262144}\varphi(2u_1)-\varphi(u_1),..,\frac{1}{262144}\varphi(2u_k)-\varphi(u_k)\right)\right\|_k \le \frac{262145}{18!(262144)}\delta \quad (3.25)$$

forall $u_1,...,u_k \in \mathcal{U}.$ Let $\Lambda = \{l:\mathcal{U}\to\mathcal{V} \mid l(0)=0\}$ and introduce the generalized metric d defined on λ by s

$$d(l,m) = \inf\left\{\lambda\in[0,\infty] \mid \sup_{k\in\mathbb{N}}\|l(u_1)-m(u_1),...,l(u_k)-m(u_k)\|_k \le \lambda \quad \forall \quad u_1,...,u_k \in \mathcal{U}\right\}$$

Then it is easy to show that Λ, d is a generalized complete metric space, See [16].

We define an operator $\mathcal{J}:\Lambda\to\Lambda$ by

$$\mathcal{J}l(u) = \frac{1}{2^{18}}l(2u) \qquad u\in\mathcal{U}$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l,m\in\Lambda,$ let $\lambda\in[0,\infty]$ be an arbitary constant with $d(l,m)\le\lambda.$ From the definition if follows that

$$\sup_{k\in\mathbb{N}}\|l(u_1)-m(u_1),...,l(u_k)-m(u_k)\|_k \le \lambda \quad u_1,...,u_k \in \mathcal{U}.$$

Therefore, $\sup_{k\in\mathbb{N}}\left\|\left(\mathcal{J}l(u_1)-\mathcal{J}m(u_1),\ldots,\mathcal{J}l(u_k)-\mathcal{J}m(u_k)\right)\right\|_k \le \frac{1}{2^{18}}\lambda$

$u_1,\ldots,u_k \in \mathcal{U}.$ Hence,it holds that

$$d(\mathcal{J}l,\mathcal{J}m)\le \frac{1}{2^{18}}\lambda d(\mathcal{J}l,\mathcal{J}m)\le \frac{1}{2^{18}}d(l,m)$$

$\forall l,m\in\Lambda.$

This Means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $L=\frac{1}{2^{18}}.$

By (3.25), we have $d(\mathcal{J}\varphi,\varphi)\le \frac{262145}{18!(262144)}\delta.$ Applying the Theorem 2.2 in [17], we deduce the existence of a fiued point of $\mathcal{J}$ that is the existence of mapping $\mathbb{D}:\mathcal{U}\to\mathcal{V}$ such that

$$\mathbb{D}(2u)=2^{18}\mathbb{D}(u)\qquad \forall u\in\mathcal{U}.$$

Moreover, we have $d\left(\mathcal{J}^n\varphi,\mathbb{D}\right)\to 0,$ which implies

$$\mathbb{D}(u)=\lim_{n\to\infty}\mathcal{J}^n\varphi(u)=\lim_{n\to\infty}\frac{\varphi(2^n u)}{2^{18n}}$$

for all $u\in\mathcal{U}.$

Also, $d(\varphi,\mathbb{D})\le \frac{1}{1-\mathcal{L}}d(\mathcal{J}\varphi,\varphi)$ implies the inequality

$$\begin{aligned} d(\varphi,\mathbb{D}) &\le \frac{1}{1-\frac{1}{2^{18}}}d(\mathcal{J}\varphi,\varphi)\\ &\le \frac{4033}{25820576160000000000}\delta. \end{aligned}$$

Taking $u_1=,\ldots,=u_k=2^n u, v_1=,\ldots,=v_k=2^n v$ in (3.1) and divide both sides by $2^{18n}.$ Now, applying the property (a) of multi-norms, we obtain

$$\begin{aligned} \|\mathcal{G}\mathbb{D}(u,v)\| &= \lim_{n\to\infty}\frac{1}{2^{18n}}\left\|\mathcal{G}f\left(2^n u,2^n v\right)\right\|\\ &\le \lim_{n\to\infty}\frac{1}{2^{18n}}=0 \end{aligned}$$

for all $u,v\in\mathcal{U}.$ The uniqueness of $\mathbb{D}$ follows from the fact that $\mathbb{D}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell\in(0,\infty)$ such that

$$\sup_{k\in\mathbb{N}} \left\| \left(\varphi(u_1) - \mathbb{D}(u_1), ..., \varphi(u_k) - \mathbb{D}(u_k) \right) \right\|_k \le \ell$$

for all $u_1, ..., u_k \in \mathcal{U}$.

Hence $\mathbb{D}$ is Duodeviginti function.

Example

We consider the function

$$\begin{cases} \xi(u) = u^{18}, & |u| < 1 \\ 1, & |u| \ge 1 \end{cases}$$

where $\xi : \mathbb{R} \to \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(u) = \sum_{n=0}^{\infty} 2^{-18n} \xi(2^n u) \tag{4.1}$$

for all $u \in \mathbb{R}$. If the function φ defined in (4.1) satisfies the functional inequality

$$|\mathcal{DG}\varphi(u,v)| \le \frac{6402373706000000.(262144)^2}{262144} \delta \tag{4.2}$$

for all $u, v \in \mathbb{R}$, then there do not exist an Duodeviginti function $\mathbb{D} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|\varphi(u) - \mathbb{D}(u)| \le \beta |u|^{18}, \qquad \forall u \in \mathbb{R}. \tag{4.3}$$

proof:

$$|\varphi(u)| = \left| \sum_{n=0}^{\infty} 2^{-18n} \xi(2^n u) \right| \le \sum_{n=0}^{\infty} \frac{1}{2^{18n}} = \frac{262144}{262143}$$

Therefore, we see that φ bounded by $\frac{262144}{262143}$ on $\mathbb{R}$. Now, suppose that $0 < \delta < \frac{1}{2^{18}}$. Then there exists a positive integer k such that

$$\frac{1}{\left(2^{18}\right)^{k+1}} \le \delta < \frac{1}{\left(2^{18}\right)^{k}} \tag{4.4}$$

and

$$2^n(u+9v), 2^n\varphi(u+8v), 2^n\varphi(u+7v), 2^n\varphi(u+6v), 2^n(u+5v), 2^n(u+4v),$$
$$2^n(u+3v), 2^n(u+2v), 2^n(u+v), 2^n(u), 2^n(u-v), 2^n(u-2v), 2^n(u-3v), 2^n(u-4v), 2^n(u-5v),$$

$2^n(u-6v), 2^n\varphi(u-7v), 2^n(u-8v), 2^n(u-9v), 2^n\varphi(v) \in (-1,1)$

forall $n = 0,1,2,...,k-1.$ Hence for $n = 0,1,2,...,k-1,$ From the definition of φ and the inequality, we obtain that

$$|\mathcal{G}\varphi(u,v)| \le \sum_{n=k}^{\infty} 2^{-18n}.6402373706000000$$

$$\le \frac{6402373706000000(262144)^2}{(262144)}\delta.$$

Therefore f satisfies (4.2) for all $u, v \in \mathbb{R}.$ we prove that the functional equation (1.1) is not stable. Suppose on the contrary that there exists an Duodeviginti function $\mathbb{D}: \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(u) - \mathbb{D}(u)| \le \beta |u|^{18},$$

for all $u \in \mathbb{R}.$ Then there exists a constant $c \in \mathbb{R}$ such that $\mathbb{D} = cu^{18}$ for all rational numbers $u.$ So we arrive that

$$|\varphi(u)| \le \beta + |c|.|u|^{18} \tag{4.5}$$

for all $u \in \mathbb{Q}.$ Take $m \in \mathbb{N}$ with $m+1 > \beta + |c|.$ If u is a rational number in $(0, 2^{-m}),$ then $2^n u \in (0,1)$ for all $n = 0,1,2...m,$ and for this $u,$ we get

$$\varphi(u) = \sum_{n=k}^{\infty} 2^{-18}\xi(2^n u)^{18} = (m+1)u^{18} > \beta + |c|.u^{18}$$

which contradicts (4.5). Hence the functional equation (1.1) is not stable.

REFERENCES

[1] **Aoki.T**, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Jpn. 2 (1950), 64-66.

[2] **Aczel.J**, *A short course on functional equations,* D. Reidel Publ. Co. Dordrecht., (1987).

[3] **Czerwik.S**, *Functional Equations and Inequalities in Several Variables,* World Scientific Publishing Co., Singapore, New Jersey, London, (2002).

[4] **Diaz.J.B and Margolis.B**, *A fixed point theorem of the alternative, for contraction on a generalized complete metric space,* Bulletin of the American Mathematical Society, vol. 74 (1968), 305-309.

[5] **Dales, H.G and Moslehian**, *Stability of mappings on multi-normed spaces,* Glasgow Mathematical Journal, 49 (2007), 321-332.

[6] **Fridoun Moradlou**, *Approximate Euler-Lagrange-Jensen type Additive mapping in Multi-Banach Spaces: A Fixed point Approach,* Commun. Korean Math. Soc. 28 (2013), 319-333.

[7] **Hyers.D.H**, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[8] **Hyers.D.H, Isac.G, Rassias.T.M**, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, (1998).

[9] **John M. Rassias, Arunkumar.M, Sathya.E and Namachivayam.T**, *Various generalized Ulam-Hyers Stabilities of a nonic functional equations,* Tbilisi Mathematical Journal 9(1) (2016), 159-196.

[10] **Jun.K, Kim.H**, *The Generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. 274 (2002), 867-878.

[11] **Liguang Wang, Bo Liu and Ran Bai**, *Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach,* Fixed Point Theory and Applications (2010), 9 pages.

[12] **M Eshaghi, S Abbaszadeh**, *On The Orthogonal Pexider Derivations In Orthogonality Banach Algebras,* Fixed Point Theory 17(2) (2016), 327-340.

[13] **M Eshaghi, S Abbaszadeh, M de la Sen, Z Farhad**, *A Fixed Point result and the stability problem in Lie superalgebras,* Fixed Point Theory and Applications (1) (2015), 1-12.

[14] **ME Gordji, S Abbaszadeh**, *Theory of Approximate Functional Equations: In Banach Algebras, Inner Product Spaces and Amenable Groups,* Academic Press.

[15] **Mohan Arunkumar, Abasalt Bodaghi, John Michael Rassias and Elumalai Sathiya**, *The general solution and approximations of a decic type functional equation in various normed spaces,* Journal of the Chungcheong Mathematical Society, 29(2) (2016).

[16] **Mihet.D and Radu.V**, *On the stability of the additive Cauchy functional equation in random normed spaces,* Journal of mathematical Analysis and Applications, 343 (2008), 567-572.

[17] **Radu.V**, *The fixed point alternative and the stability of functional equations,* Fixed Point Theory 4 (2003), 91-96.

[18] **K. Ravi, J.M. Rassias and B.V. Senthil Kumar**, *Ulam-Hyers stability of undecic functional equation in quasi-beta normed spaces fixed point method,* Tbilisi Mathematical Science 9 (2) (2016), 83-103.

[19] **K. Ravi, J.M. Rassias, S. Pinelas and S.Suresh**, *General solution and stability of Quattuordecic functional equation in quasi beta normed spaces,* Advances in pure mathematics 6 (2016), 921-941.

[20] **Sattar Alizadeh, Fridoun Moradlou**, *Approximate a quadratic mapping in multi-Banach Spaces,* A Fixed Point Approach. Int. J. Nonlinear Anal. Appl., 7. No.1 (2016), 63-75.

[21] **Tian Zhou Xu, John Michael Rassias and Wan Xin Xu**, *Generalized Ulam - Hyers Stability of a General Mixed AQCQ functional equation in Multi-Banach Spaces: A Fixed point Approach,* European Journal of Pure and Applied Mathematics 3 (2010), 1032-1047.

[22] **Ulam.S.M**, *A Collection of the Mathematical Problems,* Interscience, New York, (1960).

[23] **Xiuzhong Wang, Lidan Chang, Guofen Liu,** *Orthogonal Stability of Mixed Additive-Quadratic Jenson Type Functional Equation in Multi-Banach Spaces,* Advances in Pure Mathematics 5 (2015), 325-332.

[24] **Zhihua Wang, Xiaopei Li and Themistocles M. Rassias,** *Stability of an Additive-Cubic-Quartic Functional Equation in Multi-Banach Spaces,* Abstract and Applied Analysis (2011), 11 pages.