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Approximate Solution of Muntz System

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Abstract

In this text, a method for constructing a higher degree precision rule has been proposed for numerical evaluation of Muntz system. The Muntz Legendre polynomial has been arised by orthogonalising the Muntz system and based on Lebesgue measure on $0 \le x \le 1$. In this paper we have transformed $0 \le x \le 1$ to $-1 \le x \le 1$ by using monomial transformations. The mixed quadrature rule has been constructed by taking the suitable convex combination of two individual quadrature rule and the singularities have been extracted. The efficiency and accuracy of the proposed method and error bound has been fully tested with some numerical examples .

Key words: Muntz system, Mixed quadrature, Error bound, Maclaurin's theorem.

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1. INTRODUCTION

A mixed quadrature rule has been proposed for numerical evaluation of Muntz system . This paper intends to define quadrature available for the Muntz polynomials or higher order logarithmic functions that allow to anticipate the precision of numerical integration of Muntz and Muntz logarithmic functions which has applied to smooth functions as well as the function with end point singularities that arises in the boundary value problems, Integral equations with singular kernel and in the potential theory etc.

The introduction about Muntz and Muntz Logarithmic polynomials, their extension, basic definitions and properties to represent the remainder for mixed quadrature rule has presented in section 1. The construction of mixed quadrature rule to remove the singularity at end points has reflected in section 2 which contains the quadrature rule of higher degree of precision eleven by the convex combination of two constituent rule each of same degree of precision nine. Section 3 contains several numerical examples to validate the efficiency of the proposed mixed integral rule. Finally section 4 attains conclusion.

There are four types of Muntz system which are numerically evaluated with mixed quadrature rule . The span of Muntz system λ_s which are the sequence of real numbers constitute the Muntz polynomial $P(\lambda_s)$

$$P(\lambda_s) = \sum_{i=1}^{n} c_i\, x^{m_i}, \quad i = 1, 2, 3......n \tag{1}$$

The second type is the Muntz logarithmic polynomial $\left(x^m \log(x)\right)$, with $m > -1$ and $\gamma \in N$

$$P(\lambda_{s,z}) = \sum_{i=1}^{n} c_i\, x^{m_i} (\log x)^{\gamma_i} \qquad i = 1, 2, ...n \tag{2}$$

Selecting the monomial transformation $y = 1 - 2x$ to evaluate c in $0 \le x \le 1$ to positive integral in $-1 \le x \le 1$. we present the estimation of the remainder $r_n(f)$ [5] by application of Muntz polynomial.

Referring [1] and [2] the exact solution of Muntz system is follows

We have $$I(x^m) = \int_0^1 x^m dx = 2^{-(1+m)} \int_{-1}^{1} (1-y)^m dy = \frac{1}{1+m} \tag{3}$$

$$I_n(f) = \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

$$r_n(f) = \frac{I(f) - I_n(f)}{I(f)} \tag{5}$$

The third Muntz Logarithmic Polynomials ,extended Muntz Logarithmic Polynomials and fourth case which is Muntz Polynomials with rational exponent are evaluated in the same manner as first and second type

$$I\left(x^m \log(x)\right) = \int_0^1 x^m \log(x)\, dx$$

$$= 2^{-(1+m)} \int_{-1}^{1} (1-y)^m \left(\log(1-y) - \log 2\right) dy \tag{6}$$

$$= -\frac{1}{(1+m)^2}$$

$$I\left(x^m (\log(x))^\gamma\right) = \frac{(-1)^\gamma \gamma!}{(1+m)^{\gamma+1}} \tag{7}$$

$$\int_0^1 x^t\, dt = \int_0^1 x^{\frac{u}{q}+k}\, dx = q \tag{8}$$

Where $t = \frac{u}{q} + k,\ q \in N, \frac{u}{q} + k > -1$ substituting the monomial transformations $x = v^q$

2. FORMULATION OF MIXED QUADRATURE RULES

Referring [3], [4], [6], [8] our proposed mixed quadrature rule (i.e mixing Steffensen four-point rule $(R_{ST4}(f))$ and Gauss Legendre-two point rule $(R_{GL2}(f))$, Gauss Legendre-three point rule $(R_{GL3}(f))$, Gauss Legendre-four point rule $(R_{GL4}(f))$,Gauss Legendre-five point rule $(R_{GL5}(f)))$ $R_{ST4GL2GL3GL4GL5}(f)$ of degree of precision eleven has been obtained by Theorem-1.

Theorem-1

The quadrature rule and error for the smooth function $f(x)$ which is defined on $0 \le x \le 1$ is obtained by convex combination of $R_{ST4GL2GL3GL4}(f)$ rules and Gauss Legendre-five point rules $R_{GL5}(f)$ respectively $E_{ST4GL2GL3GL4}(f)$ and $E_{GL5}(f)$

$$I = \int_{-1}^{1} f(x)\, dx \cong \frac{1}{112471}\left[10880\, R_{ST4GL2GL3GL4}(f) + 101591\, R_{GL5}(f)\right]$$
$$+ \frac{1}{112471}\left[10880\, E_{ST4GL2GL3GL4}(f) + 101591\, E_{GL5}(f)\right]$$

Where

$$R_{ST4GL2GL3GL4GL5}(f) = \frac{1}{112471}\left[10880\, R_{ST4GL2GL3GL4}(f) + 101591 R_{GL5}(f)\right]$$

and

$$E_{ST4GL2GL3GL4GL5}(f) = \frac{1}{112471}\left[10880\, E_{ST4GL2GL3GL4}(f) + 101591 E_{GL5}(f)\right]$$

Proof:

As it is well known [9]

$$I = \int_{-1}^{1} f(x)dx \cong \left[\frac{301}{51} R_{GL4}(f) - \frac{250}{51} R_{ST4GL2GL3}(f)\right] + \left[\frac{301}{51} E_{GL4}(f) - \frac{250}{51} E_{ST4GL2GL3}(f)\right] \quad (9)$$

Where Steffensen four- point rule is

$$R_{ST4}(f) = \left[\frac{11}{12}\left\{f\left(-\frac{3}{5}\right)\right\} + \frac{11}{12}\left\{f\left(\frac{3}{5}\right)\right\} + \frac{1}{12}\left\{f\left(-\frac{1}{5}\right)\right\} + \frac{1}{12}\left\{f\left(\frac{1}{5}\right)\right\}\right]$$

Gauss Legendre two-point rule is

$$R_{GL2}(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gauss Legendre three-point rule is

$$R_{GL3}(f) = \left[\frac{5}{9}\left\{f\left(-\sqrt{\frac{3}{5}}\right)\right\} + 8f(0) + \frac{5}{9}\left\{f\left(\sqrt{\frac{3}{5}}\right)\right\}\right]$$

Gauss Legendre four-point rule is

$$R_{GL4}(f) = \frac{1}{36}\left[\begin{array}{l}\left(18+\sqrt{30}\right)\{f(-\delta)+f(\delta)\}+ \\ \left(18-\sqrt{30}\right)\{f(-\sigma)+f(\sigma)\}\end{array}\right]$$

$$\delta = \sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}} \text{ and } \sigma = \sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$$

Gauss-Legendre -five point rule $R_{GL5}(f)$ is

$$I = \int_{-1}^{1} f(x)\, dx \cong R_{GL5}(f) = \frac{1}{900}\left[\begin{array}{l}\left(322+13\sqrt{70}\right)\{f(-\delta)+f(\delta)\}+ \\ +512f(0)+ \\ \left(322-13\sqrt{70}\right)\{f(-\sigma)+f(\sigma)\}\end{array}\right] \quad (10)$$

$$\text{where } \delta = \sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}} \text{ and } \sigma = \sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$$

Each of the constituent rule $R_{ST4GL2GL3GL4}(f)$ and $R_{GL5}(f)$ is of precision nine. Let $E_{ST4GL2GL3GL4}(f)$ and $E_{GL5}(f)$ denotes the errors in approximating the integral $I(f)$ by the rules $R_{ST4GL2GL3GL4}(f)$ and $R_{GL5}(f)$ respectively.

$$I(f) = R_{ST4GL2GL3GL4}(f) + E_{ST4GL2GL3GL4}(f) \tag{11}$$

$$I(f) = R_{GL5}(f) + E_{GL5}(f) \tag{12}$$

Using Maclaurin's expansion in equation (11) and (12), we get

$$E_{ST4GL2GL3GL4}(f) \cong -\frac{14513}{530145 \times 10!} f^{x}(0) \ldots\ldots$$

$$E_{GL5}(f) \cong \frac{128}{43659 \times 10!} f^{x}(0) + \frac{2706048}{338527917 \times 12!} f^{xii}(0) + \ldots\ldots\ldots$$

Multiplying $\left(\frac{128}{7}\right)$ and $\left(\frac{14513}{85}\right)$ by equation (11) and (12) respectively. Then adding the resulting equations

$$I = \int_{-1}^{1} f(x)dx \cong \frac{1}{112471}\left[10880\ R_{ST4GL2GL3GL4}(f) + 101591\ R_{GL5}(f)\right]$$

$$+ \frac{1}{112471}\left[10880\, E_{ST4GL2GL3GL4}(f) + 101591\ E_{GL5}(f)\right]$$

Then $I(f) = R_{ST4GL2GL3GL4GL5}(f) + E_{ST4GL2GL3GL4GL5}(f)$ $\quad (13)$

$$R_{ST4GL2GL3GL4GL5}(f) = \frac{1}{112471}\left[10880\, R_{ST4GL2GL3GL4}(f) + 101591 R_{GL5}(f)\right] \tag{14}$$

$$E_{ST4GL2GL3GL4GL5}(f) = \frac{1}{112471}\left[10880\, E_{ST4GL2GL3GL4}(f) + 101591 E_{GL5}(f)\right] \tag{15}$$

Equation (14) and (15) are the required mixed quadrature rule and the estimated error.

Corollary-1

If $E_{ST4GL2GL3GL4GL5}(f) = \left\lfloor \frac{10880}{112471} E_{ST4GL2GL3GL4}(f) + \frac{101591}{112471} E_{GL5}(f) \right\rfloor$

Then the absolute error bound is given by

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{274910122368}{38074573352907 \times 12!}\left|f^{xii}(0)\right|$$

Proof:

Using Maclaurin's theorem in equation (15), we have

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{274910122368}{38074573352907\times 12!}\left|f^{xii}(0)\right| \qquad (16)$$

Lemma-1

The error bound for the truncation error is given by

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734\ K}{22606671\times 10!} \quad \text{where } K = \max_{-1\le x\le 1}\left|f^{xi}(x)\right|$$

Proof:

$$E_{ST4GL2GL3GL4}(f) \cong \frac{59867}{22606671\times 10!} f^{x}(\eta_1),$$

$$E_{GL5}(f) = \frac{59867}{22606671\times 10!} f^{x}(\eta_2) \quad \textit{where } \eta_1, \eta_2 \in [-1, 1]$$

Referring $[7]$, As $f^x(x)$ is continuous and bounded in $[-1,1]$ So there exists points c and d in the interval $[-1,1]$ such that $M_1 = f^x(c)$ and $M_2 = f^x(d)$ where $M_1 = \max_{-1\le x\le 1}\left|f^x(x)\right|$ and $M_2 = \min_{-1\le x\le 1}\left|f^x(x)\right|$.

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{59867}{22606671\times 10!}\left\{f^x(c) - f^x(d)\right\}$$

$$= \frac{59867}{22606671\times 10!}\int_c^d f^{xi}(x)\,dx$$

$$= \frac{59867}{22606671\times 10!}(d-c)\ f^{xi}(\alpha)$$

$$\textit{for some } \alpha \in [-1,1]$$

By mean value theorem $|c-d| \le 2$

Then $\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734}{22606671\times 10!} f^{xi}(\alpha)$

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734\ K}{22606671\times 10!} \quad \text{where } K = \max_{-1\le x\le 1}\left|f^{xi}(x)\right|$$

4. NUMERICAL VERIFICATION

The numerical approximation for Muntz polynomials, Muntz extended logarithmic polynomials, Muntz logarithmic polynomials with rational polynomials, Muntz with Jacobi function of first kind $(J_0(x))$ are given in Table- 1.

$$I_1 = \int_0^1 x^{15}\,dx,\quad I_2 = \int_0^1 \log x\,dx,\quad I_3 = \int_0^1 x\log x\,dx,\quad I_4 = \int_0^1 x^3(\log x)^2\,dx,\quad I_5 = \int_0^1 x^3(\log x)^7\,dx,$$

$$I_6 = \int_0^1 x^5(\log x)^{10}dx,\quad I_7 = \int_0^1 x^{10}\log x\,dx,\quad I_8 = \int_0^1 x^2(\log x)^{25}dx,\quad I_9 = \int_0^1 x^{20}(\log x)^{25}dx,$$

$$I_{10} = \int_0^1 x^{\left(\frac{2}{5}+7\right)}dx,\quad I_{11} = \int_0^1 x^{\left(\frac{3}{2}+6\right)}dx,\quad I_{12} = \int_0^1 x^{\left(\frac{-1}{2}+1\right)}dx,\quad I_{13} = \int_0^1 J_0(x)(1+\log(x))dx.$$

Table-1 (Evaluation of remainder $r_n(f)$ with mixed quadrature rule)

I	$I_{EXACT} = I_{MÜNTZ}$	$R_{ST4GL2GL3GL4GL5}(f)$	$\lvert E_{ST4GL2GL3GL4GL5}(f)\rvert$	$\lvert r_n(f)\rvert$
I_1	0.062500000000000	0.062485268338495	1.473166150506994e-005	2.357065840811191e-004
I_2	-1.0000000000000	-0.980668900776510	0.019331099223490	.019331099223490
I_3	-0.25000000000000	-0.250228143946878	2.281439468775348e-004	9.125757875101392e-004
I_4	0.031250000000000	0.031253105303897	3.105303897486811e-006	9.936972471957795e-005
I_5	-0.07690429687500	-0.075827866089724	0.001076430785276	0.013997017449176
I_6	0.010002286236854	0.009725145089260	2.771411475935015e-004	0.027707780104548
I_7	-0.008264462809917	-0.008264454811730	7.998186999855217e-009	9.677806269825230e-007
I_8	-6.102292996923412e+012	-3.430600987823765e+008	6.101949936824630e+012	0.999943781772040
I_9	-6.500458885064872e-010	-2.290069138470313e-010	4.210389746594559e-010	0.647706542113225
I_{10}	0.119047619047619	0.119047618089707	9.579119997216168e-010	8.046460797661585e-009
I_{11}	0.117647058823529	0.117647057842355	9.811736978448948e-010	8.339976431681636e-009
I_{12}	0.666666666666667	0.667211463904330	5.447972376626531e-004	8.171958564939793e-004
I_{13}	-0.053108037589511	-0.033779126701509	0.019328910888002	0.363954530525141

5. CONCLUSION

The approximate results for different Muntz and Muntz logarithmic polynomials are convergent to the exact result. One of the remarkable point is that for small m and large γ $(\gamma \geq 10)$ the error increases gradually. The mixed quadrature rule $R_{ST4GL2GL3GL4GL5}(f)$ gives better approximation results than other rules for different texts which is evident from the Table-1.

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