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On the Construction of Larger Singular Graphs

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Abstract

Graphs having 0 as an eigen value of the adjacency matrix are called singular graphs. Multiplicity of the eigen value 0 is the nullity, η(*G*) of the singular graph *G*. Deletion of a vertex from a singular graph either changes its nullity or leave it unaltered. The extent of change depends on the type vertex we are deleting. Vertices of singular graphs are classified as core and noncore vertices. Noncore vertices are further classified as noncore vertices of zero null spread and of null spread −1. If *G* is a graph on *n* vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its eigen values , then the energy of *G* is $E = E(G) = \sum_{j=1}^{n} \lambda_j$. A graph with energy, E(*G*) < n is said to be hypoenergetic. Graph for which E(*G*) ≥ n are called nonhypoenergetic. In this paper larger singular graphs are constructed by joining singular graphs by an edge. As singular graphs have different types of vertices, the graphs constructed in this way will differ in nullity depending on the vertex we are joining to the edge uv. An attempt was made to study various spectral properties and energy of resulting graphs. We used many properties of coalescence of graphs with respect to different types of vertices to prove the results.

Keywords: Singular graphs, Nullity, Core vertex, Null spread, Energy of graph, Nonhypoenergetic.

1 INTRODUCTION

Let G be a finite, undirected simple graph of order n , with vertex set V(G) and Edge set ε(G). The adjacency matrix A(G) of the graph G is a square matrix of order n whose entries a_{ij} denote the number of edges from the vertex v_i to the vertex v_j. The characteristic polynomial φ (G, x) = det (x I_n – A (G)) of G is a polynomial of degree n in x. The roots of φ (G, x) = 0 are called the eigen values of G. The set of eigen values together with their multiplicities constitute the spectrum of G, denoted by spec(G). If zero is an eigen values of G, then G is a singular graph. The multiplicity of zero in spec(G) is called the nullity of G, denoted by η(G). The nonzero vector X satisfying the equation AX=0 is called the kernel eigen vector of G.

A singular graph on at least two vertices, with a kernel eigen vector having nonzero entries, is said to be a core graph. The core graph of nullity one, is called nut graph. Core graphs have nullity one or more. If G is a singular graph of nullity one and if X is a kernel eigen vector such that $X = [x_1,x_2,\ldots x_m,0,\ldots 0]^t$, with $x_m \neq 0, i = 1,2,\ldots,m$, then the subgraph F of G induced by the nonzero entries $x_1,x_2,\ldots x_m$ is called the core graph of G. The set of remaining vertices called core-forbidden vertices or noncore vertices constitute the periphery of G. (See Figure 1)

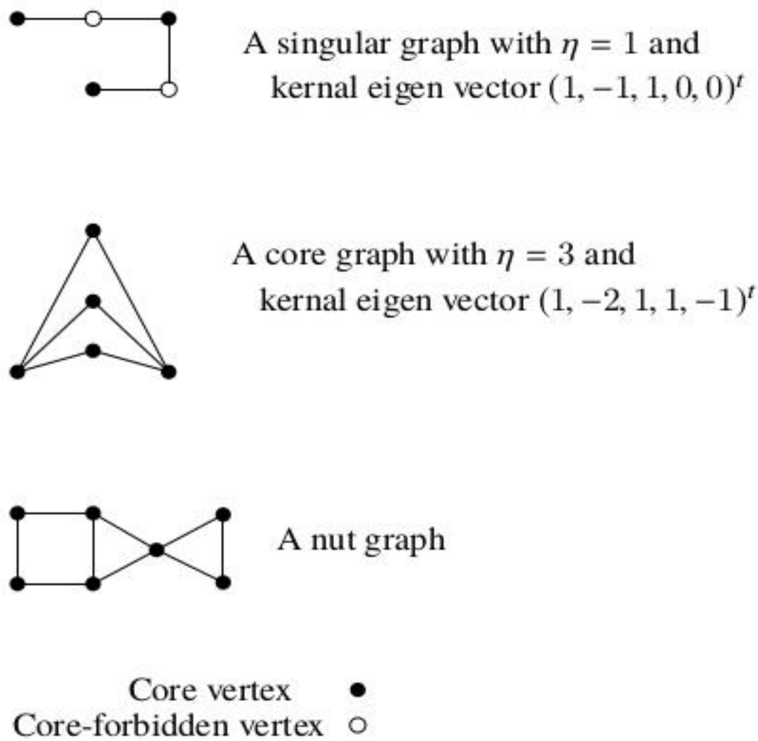

Figure 1: Three singular graphs.

Definition 1.1 [10] A graph G, |G | ≥ 3 is a singular configuration with core (F, X_F), if it is a singular graph of nullity one, with F as an induced subgraph, having |F| + η(F) −1 vertices, satisfying |F| ≥ 2 , $FX_F = 0$ and $G\begin{pmatrix} X_F \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

A singular configuration can be constructed as follows [11]: starting with a core graph (F, X_F) of nullity η(*F*) > 1, if a connected graph *S* of nullity one by adding a minimal set *P* of independent vertices, until the nullity is reduced to one can be constructed, then *S* is a singular configuration. The vertices which reduces the nullity of the core by one are called peripheral vertices. There are η(*F*) − 1 independent peripheral vertices. The set of peripheral vertices is called the periphery of *S* and is denoted by *P*. A simplest singular configuration in which there is no edges between pairs of vertices of *P* is called minimal configuration (MC).(See Figure 2).

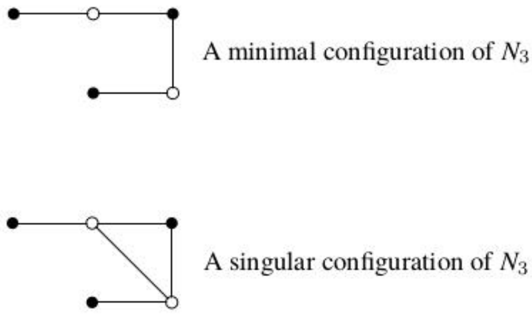

Figure 2: Minimal and singular configuration.

We have the following very important interlacing theorem about eigen values of graphs.

Theorem 1.1 [8] *If G is an n-vertex graph with eigen values* $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ *and H is a vertex deleted subgraph of G with eigen values* $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ *, then* $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ *, i= 1,2,...,n −1.*

It is clear from interlacing theorem that the multiplicity of an eigen value and hence the multiplicity of nullity can change at most one upon deleting or adding a vertex of the graph.

Definition 1.2 [3] Let G − u be the induced subgraph of the graph G obtained on deleting the vertex u. The null spread of the vertex u is : n_u (G) = η (G) – η (G - u).

Clearly null spread satisfies -1 ≤ n_u (G) ≤ 1. If *u* is a core vertex, then n_u (G) = 1. If *u* is a vertex in the periphery of a MC, then n_u (G) = -1. There are vertices with n_u (G) = 0 also. Thus noncore vertices (vertices other than core vertices) of a singular graph can be classified as noncore vertices of null spread −1 and noncore vertices of zero null spread.(See Figure 3).

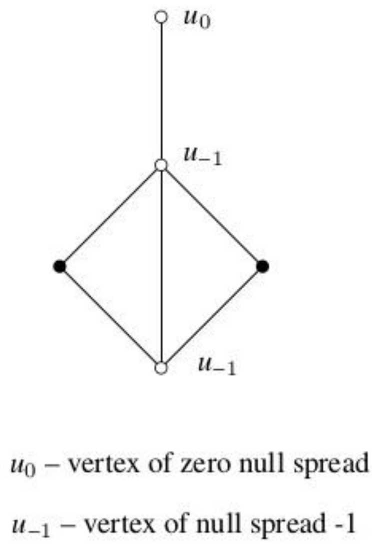

Figure 3: Graph with both types of noncore vertices.

Gutman in 1978 gave the following definition for energy of a graph

Definition 1.3 [15] If G is a graph on n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its eigen values , then the energy of *G* is

$$E = E(G) = \sum_{j=1}^{n} \lambda_j \ .$$

A graph with energy, $E(G) < n$, is said to be hypoenergetic. Graph for which $E(G) \geq n$ are called nonhypoenergetic. If $E(G) < n-1$ and *G* is connected, *G* is called strongly hypoenergetic [15].

Definition 1.4 [7] Let G_1 and G_2 be two graphs with disjoint vertex sets. If a vertex u ϵ G_1 is identified with a vertex v ϵ G_2 , then the graph G_1 o G_2 obtained of order $|G_1| + |G_2| - 1$, is said to be the coalescence of G_1 and G_2 with respect to u and v.

An expression for the characteristic polynomial $\varphi(G, x)$ of the graph G = G_1 o G_2 is given by the following theorem.

Theorem 1.2 [7] *The characteristic polynomial of the coalescence G_1 o G_2 of two rooted graphs (G_1,u) and (G_2,w) obtained by identifying the vertices u and w so that the vertex v = u = w become a cut vertex of G_1 o G_2 is given by*

$$\varphi(G_1 \, o \, G_2) = \varphi(G_1)\, \varphi(G_2 - w) + \varphi(G_1 - u)\, \varphi(G_2) - x\, \varphi(G_1 - u)\, \varphi(G_2 - w) \qquad (1.1)$$

2 CONSTRUCTION OF NEW SINGULAR GRAPHS

2.1 Joining two graphs by an edge

Definition 2.1 [9] Let (K ,u) and (H ,w) are two rooted graphs. The graph obtained by joining u and w by an edge is denoted by KH + uw. (See Figure 4).

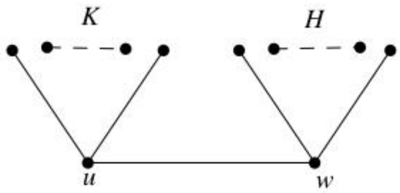

Figure 4: Joining (*K ,u*) and (*H ,w*) by an edge *uw*.

The characteristic polynomial of KH + uw [9] is

$$\varphi(KH + uw) = \varphi(K)\varphi(H) - \varphi(K - u)\varphi(H - w).$$

Example 2.3 *The characteristic polynomial of the graph in Figure* 5 *is given by* $x^8 - 8x^6 + 20x^4 - 4x^2$

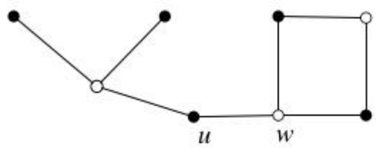

Figure 5: The graph KH + uw.

Theorem 2.4 [9] *Let the components of the graph obtained by deleting the edge uw from KH + uw be (K ,u) and (H ,w). If one of the following conditions is satisfied, then KH + uw is singular.*

1. *One component and its root-deleted subgraph are singular.*
2. *One component and the root-deleted subgraph of the other component are singular.*

We have the following theorems:

Theorem 2.5 [5] *The coalescence of two singular graphs of nullity η_1 and η_2 coalesced at a core vertex yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.*

Theorem 2.6 [6] *Let G_1 be a nonsingular graph and G_2 be a singular graph of nullity η_2. If G_1 and G_2 are coalesced at a vertex $u \in G_1$ and a core vertex $v \in G_2$, then the nullity of* $G_1 \circ G_2$ *is $\eta_2 - 1$.*

Theorem 2.7 [6] *Let G_1 and G_2 be two singular graphs of order n_1 and n_2 respectively. If* $G_1 \circ G_2$ *is the coalescence of G_1 and G_2 at a noncore vertex of null spread -1, then*

$\eta(G_1 \circ G_2) = \eta_1 + \eta_2 + 1$.

Theorem 2.8 [6] *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. The nullity of the coalescence of G_1 and G_2 at a noncore vertex of zero null spread is $\eta_1 + \eta_2$.*

Theorem 2.9 [6] *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. The coalescence of G_1 and G_2 at a core vertex of G_1 and at a noncore vertex (null spread 0 or -1) of G_2 or vice versa yield a singular graph of nullity $\eta_1 + \eta_2 + 1$.*

Theorem 2.10 [6] *Let G_1 be a non singular graph and G_2 be a singular graph of nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex of G_1 and a noncore vertex of zero null spread of G_2 is η_2.*

Theorem 2.11 [6] *Let G_1 be a nonsingular graph and G_2 be a singular graph of nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex $u \in G_1$ and a noncore vertex $w \in G_2$ of null spread -1 is*

1. *$\eta_2 + 1$, if $G_1 - u$ is singular.*
2. *η_2, if $G_1 - u$ is nonsingular.*

Theorem 2.12 *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. If a core vertex u of G_1 and a noncore vertex w (of null spread -1 or 0) of G_2 is coalesced, then in the coalesced graph, the coalesced vertex is a noncore vertex.*

Theorem 2.13 *A singular graph with noncore vertices always satisfies the following conditions.*

1. *If one ore more neighbours of a noncore vertex v is the only neighbours of another vertex v', then v' will be a noncore vertex.*
2. *the vertices having core or noncore vertex neighbours whose neighbours are noncore vertices will be noncore vertices.*

The following three examples wiil illustrate what we have stated in Theorem 2.12 and Theorem 2.13.

Example 2.14 In Figure 6, the graphs G_1 and G_2 are coalesced with respect to a core vertex u ϵ G_1 and a noncore vertex w ϵ G_2. Note that in the coalesced graph, the coalesced vertex is a noncore vertex.

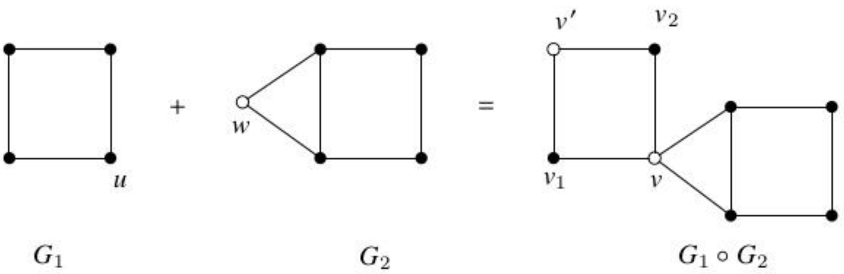

Figure 6: Coalescence of G_1 and G_2 with respect to a core vertex of G_1 and noncore vertex of G_2.

Example 2.15 In Figure 6, the vertices v_1 and v_2 are neighbours of both v and v'. Note that in the coalesced graph, the vertex v' is changed to a noncore vertex.

Example 2.16 In Figure 7, the core vertex v of G_1 is changed to a noncore vertex in G_1 o G_2 . Note that the neighbour of v's neighbour is a noncore vertex.

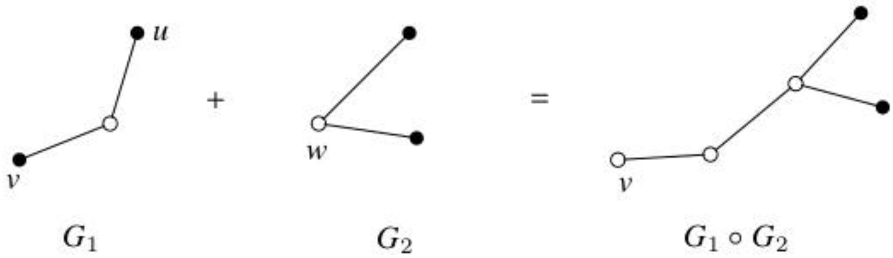

Figure 7: Coalescence of G_1 and G_2 with respect to a core vertex of G_1 and noncore vertex of G_2.

Theorem 2.17 *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively and G_1 o G_2 be the coalescence of G_1 and G_2 with respect to u ϵ G_1 and w ϵ G_2. Then, noncore vertices of G_1 and G_2 will remain as noncore vertices in G_1 o G_2.*

Theorem 2.18 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw.*

1. *Let K and H are singular graphs of nullity η_1 and η_2 respectively. If u and w are core vertices of K and H respectively, then nullity of KH + uw is $\eta_1 + \eta_2 - 2$.*

2. *Let K and H are singular graphs of nullity η_1 and η_2 respectively. If u and w are noncore vertices (of null spread 0 or −1) of K and H respectively, then the nullity of KH + uw is $\eta_1 + \eta_2$.*
3. *Let K and H are singular graphs of nullity η_1 and η_2 respectively. If u is a core vertex of K and w is a noncore vertex of null spread −1 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2$.*
4. *Let K and H be singular graphs of nullity η_1 and η_2 respectively. If u is core vertex of K and w is a noncore vertex of H of null spread 0 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2 - 1$.*
5. *Let K and H are singular graphs of nullity η_1 and η_2 respectively. If u is a noncore vertex of K of null spread 0 and w is a noncore vertex of H of null spread −1 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2$.*
6. *Let K be singular with nullity $\eta, \eta > 1$ and H nonsingular. If u is a core vertex and H − w is nonsingular, then nullity of KH + uw is $\eta - 1$.*
7. *Let K be singular with nullity η, $\eta > 1$ and H is nonsingular. If u is a core vertex and H− w is singular, then nullity of KH + uw is η.*
8. *Let K be singular with nullity η and H be nonsingular. If u is a noncore vertex (of null spread 0 or −1), then nullity of KH + uw is η.*

Proof. The graph KH + uw is in fact that coalescence of K, H and K_2 = uw. That is, KH + uw = (K o K_2) o H , where K_2 = uw. Now the proof follows from Theorem 2.5 to 2.12.

Example 2.19 See Figure 5. The nullity of the graph H is 2 and K is 2. But the nullity of KH + uw is 2 + 2 – 2 = 2. Note that u and w are core vertices of respective graphs.

Example 2.20 In Figure 8, both K and H have nullity 1. The nullity of KH + uw = 2 = 1 + 1. Here u is a noncore vertex of null spread −1 and w is a noncore vertex of zero null spread .

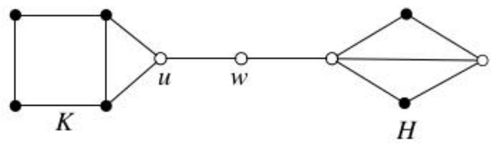

Figure 8: KH + uw with noncore u and w.

Example 2.21 In Figure 9, K is singular with nullity 4 and H is nonsingular. The root of K ,u is a core vertex. The graph KH + uw is singular with nullity 3.

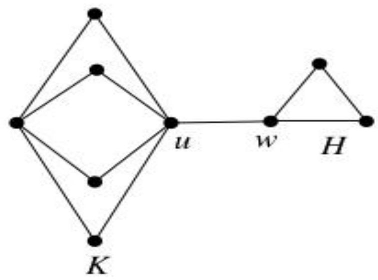

Figure 9: KH + uw with singular K and core vertex u.

Example 2.22 The graph in Figure 10 has nullity 3. Here K is singular with nullity 3 and u is a noncore vertex of null spread −1. H is obviously nonsingular.

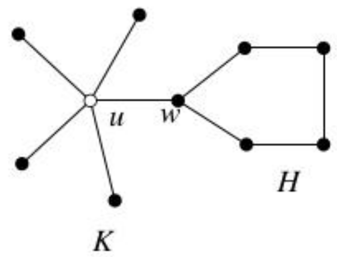

Figure 10: KH + uw with singular K and noncore vertex u.

Theorem 2.23 [15] *Let G and H be two graphs with disjoint vertex sets and G o H be the coalescence of G and H at u ϵ H and v ϵ G. Then E(G o H) ≤ E(G) + E(H). Equality is attained if and only if either u is an isolated vertex of G or v is an isolated vertex of H or both.*

Theorem 2.24 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. If both K and H are strongly hypoenergetic graphs, then KH + uw is hypoenergetic.*

Proof. We know that KH + uw = (K o K_2) o H , where K_2 = uw. By Theorem 2.23, we have

E(KH + uw) ≤ E(K) + E(uw) + E(H)

$< n_1 - 1 + 2 + n_2 - 1 = n_1 + n_2$, where $|K| = n_1$ and $|H| = n_2$.

Theorem 2.25 [15] *If the graph G is nonsingular, then G is nonhypoenergetic.*

Theorem 2.26 *Let (K ,u) and (H ,w) be the components of the graphs obtained by deleting an edge uw from KH + uw. If both K and H are singular graphs with nullity 1 and u and w are core vertices of K and H respectively, then KH + uw is nonhypoenergetic.*

Proof. By part (i) of Theorem 2.18, KH + uw is nonsingular. By Theorem 2.25, it follows that KH + uw is nonhypoenergetic.

Theorem 2.26, gives us a tool to construct nonsingular graphs using singular graphs. See the following example.

Example 2.27 See Figures 11 and 12 .

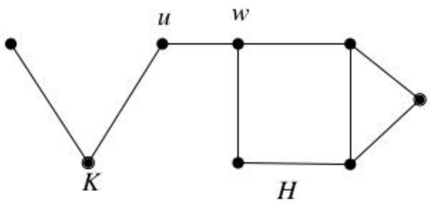

Figure 11: A nonsingular graph with hypoenergetic K and nonhypoenergetic H.

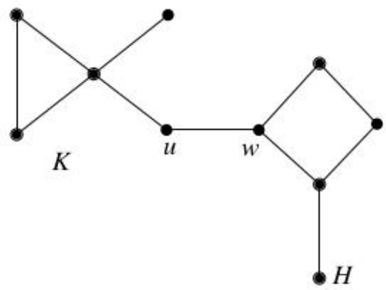

Figure 12: A nonsingular graph with both *K* and *H* are nonhypoenergetic.

Example 2.28 In Figure 13, we have a nonsingular graph, constructed using Theorem 2.18 and 2.26.

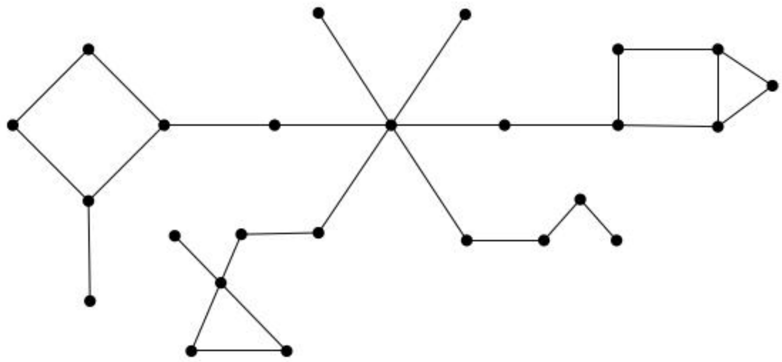

Figure 13: A nonsingular graph.

We have the following theorem about nonhypoenergetic graphs.

Theorem 2.29 [6] *Let G be a singular graph of order $n \geq 4$ with nullity η. If the absolute value of the coefficient of x^{η} in the characteristic polynomial of G is grater then or equal to e^{η}, then G is nonhypoenergetic.*

Theorem 2.30 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. Also, let K and H be singular graphs with nullity η_1 and η_2 respectively. If u and w are noncore vertices of null spread −1 of K and H respectively, then the product of the nonzero eigen values of KH + uw is the product of the nonzero eigen values of K and H.*

Proof. The characteristic polynomial of KH + uw is $\varphi(KH + uw) = \varphi(K)\varphi(H) - \varphi(K - u)\varphi(H - w)$. Since u and w are noncore vertices of null spread −1, the nullity of KH + uw is $\eta_1 + \eta_2$. There is no term in $x^{\eta_1 + \eta_2}$ in $\varphi(K - u)\varphi(H - w)$ as the removal of u and w respectively from K and H increases the nullity of K − u and H − w by one. So the product of nonzero eigen values of KH + uw is the coefficient of $x^{\eta_1 + \eta_2}$ in $\varphi(KH + uw)$, which is equal to the product of nonzero eigen values of K and H.

Corollary 2.31 *Let (K ,u) and (H ,w) be as in Theorem . If the absolute value of the coefficient of x^{η_1} in $\varphi(K)$ is greater than or equal to $e^{\eta_1 - k}$ and the absolute value of coefficient of x^{η_2} in $\varphi(H)$ is greater than or equal to $e^{\eta_2 + k}$, where $|k| \leq min(\eta_1, \eta_2)$, then KH + uw is nonhypoenergetic.*

Proof. We know that KH + uw is a singular graph with nullity $\eta_1 + \eta_2$. By the above theorem, the absolute value of the coefficient of $x^{\eta_1 + \eta_2}$ in $\varphi(KH + uw)$ = absolute value of the product of the nonzero eigen values of KH + uw = absolute value of the product of the nonzero eigen values of K and H $\geq e^{\eta_1 - k} \cdot e^{\eta_2 + k} = e^{\eta_1 + \eta_2}$. So, KH + uw is nonhypoenergetic by Theorem 2.29.

Corollary 2.32 *Let (K ,u) and (H ,w) be as in Theorem . If K and H are hypoenergetic, then the absolute value of the coefficient of* $x^{\eta_1+\eta_2}$ *in the characteristic polynomial of KH + uw is less than* $e^{\eta_1+\eta_2}$.

Proof. Since K and H are hypoenergetic, the absolute value of the coefficient of x^{η_1} and x^{η_2} in the characteristic polynomials of K and H are less than e^{η_1} and e^{η_2} respectively [see Theorem 2.29]. Now, absolute value of the coefficient of $x^{\eta_1+\eta_2}$ in φ(KH + uw) = absolute value of the nonzero eigen values of KH + uw = absolute value of the product of the nonzero eigen values of K and H $< e^{\eta_1} \cdot e^{\eta_2} = e^{\eta_1+\eta_2}$.

Example 2.33 In Figure 14, the product of nonzero eigen values of K is 4 and nullity of K is 2. Clearly $4 > e^{2-1}$. The product of nonzero eigen values of H is 11 ,which is greater than e^{1+1}.Note that the nullity of H is 1. Now, the product of nonzero eigen values of KH+uw is 44, which is the product of nonzero eigen values of K and H. Also note that $44 > e^3$.

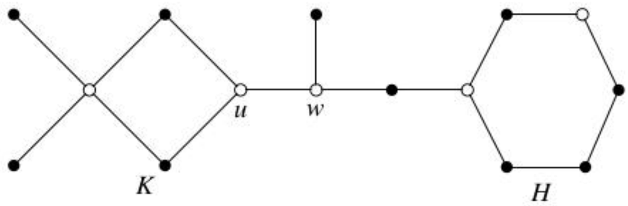

Figure 14: KH+uw with noncore vertices *u* and *w* of null spread −1.

Example 2.34 In Figure 15, both the graphs K and H are hypoenergetic. The product of the nonzero eigen values of KH + uw is 17.9923 which is less than $e^{\eta_1+\eta_2} = e^{2+3} = e^5$. Here η(K) = 2 and η(H) = 3. In this case KH + uw is also hypoenergetic.

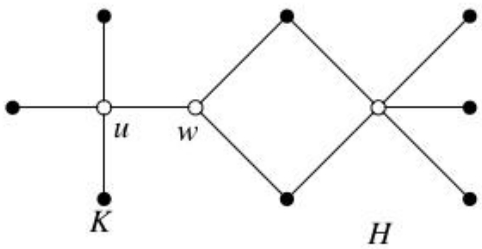

Figure 15: *KH + uw* with noncore vertices *u* and *w* of null spread −1.

Theorem 2.35 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. Also let K and H be singular graphs with nullity* η_1 *and* η_2 *respectively. If u and w are noncore vertices of null spread zero of K and H* *respectively, then the product of the nonzero eigen values of KH + uw = product of the nonzero eigen values of K× product of nonzero eigen values of H − product of nonzero eigen values of (K − u)× product of the nonzero eigen values of (H − w).*

Proof. Part (ii) of Theorem 2.18 shows that KH + uw is a singular graph of nullity $\eta_1 + \eta_2$. So the product of the nonzero eigen values of KH + uw is the coefficient of $x^{\eta_1+\eta_2}$ in the characteristic polynomial of KH + uw. The characteristic polynomial of KH + uw is

$$\varphi(KH + uw) = \varphi(K)\varphi(H) - \varphi(K - u)\varphi(H - w) \qquad (2.1)$$

Term in $x^{\eta_1+\eta_2}$ is the lowest degree term in x in φ(K)φ(H). Since u and w are noncore vertices of zero null spread, term in $x^{\eta_1+\eta_2}$ is the lowest degree term in x in φ(K − u)φ(H − w) also. Now the theorem follows from (2.1).

Corollary 2.36 *Let (K ,u) and (H ,w) be as in Theorem . If u and w are noncore vertices of zero null spread, then the coefficient of* $x^{\eta_1+\eta_2}$ *in φ(K)φ(H) cannot be equal to the coefficient of* $x^{\eta_1+\eta_2}$ *in φ(K−u)φ(H−w).*

Proof. If the coefficient of $x^{\eta_1+\eta_2}$ in φ(K)φ(H) and φ(K − u)φ(H − w) are equal, then there will be no term in $x^{\eta_1+\eta_2}$ in φ(KH + uw). This means that the nullity of KH + uw is greater than $\eta_1 + \eta_2$. As u and w are noncore vertices of zero null spread, nullity cannot be greater than $\eta_1 + \eta_2$.

Example 2.37 The product of the nonzero eigen values of the graphs KH + uw in Figure 16 is −16. The product of nonzero eigen values of the graphs K,H,K − u and H − w are respectively 2, −4, 4 and 2. Obviously −16 = 2 × −4 – 4 × 2.

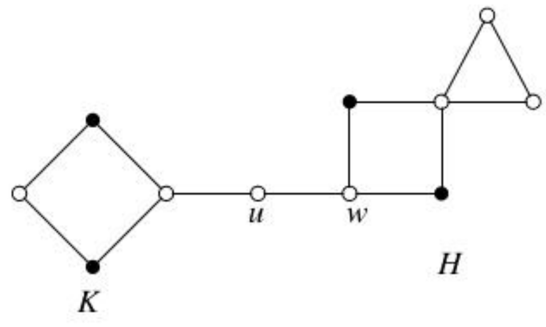

Figure 16: KH + uw with noncore vertices u and w of null spread 0.

Theorem 2.38 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. Also let K and H be singular graphs with nullity* η_1 *and* η_2 *respectively. If u and w are core vertices of K and H respectively, then the absolute value of the product of nonzero eigen values of KH + uw is the absolute value of the product of the nonzero eigen values of K − u and H − w.*

Proof. The characteristic polynomial of KH + uw is

$$\varphi(KH + uw) = \varphi(K)\varphi(H) - \varphi(K - u)\varphi(H - w).$$

Since u and w are core vertices, the nullity of KH + uw is $\eta_1 + \eta_2 - 2$. There is no term in $x^{\eta_1+\eta_2-2}$ in $\varphi(H)\varphi(K)$, as H and K are singular graphs with nullity η_1 and η_2 respectively. The removal of u and w respectively from K and H decreases their nullity by 1. So there is term in $x^{\eta_1+\eta_2-2}$ in $\varphi(K - u)\varphi(H - w)$. The coefficient of $x^{\eta_1+\eta_2-2}$ in $\varphi(KH + uw)$ is the product of the nonzero eigen values of KH + uw. Hence absolute value of the product of the nonzero eigen values of KH + uw is equal to the absolute value of the coefficient of $x^{\eta_1+\eta_2-2}$ in $\varphi(K - u)\varphi(H - w)$, which is obviously the absolute value of product of the nonzero eigen values of K − u and H − w.

Corollary 2.39 *Let (K ,u) and (H ,w) be as in Theorem . If the absolute value of the coefficient of* x^{η_1-1} *in* $\varphi(K - u)$ *is greater than or equal to* e^{η_1-k-1} *and the absolute value of the coefficient of* x^{η_2-1} *in* $\varphi(H - w)$ *is greater than or equal to* e^{η_2+k-1} *, where k is an integer such that* $|k| \leq \min(\eta_1, \eta_2)$*, then KH + uw is nonhypoenergetic.*

Proof. We know that KH + uw is a singular graph with nullity $\eta_1 + \eta_2 - 2$. By the above theorem, the absolute value of the coefficient of $x^{\eta_1+\eta_2-2}$ in $\varphi(KH + uw)$ is equal to the product of the absolute value of the coefficients of x^{η_1-1} and x^{η_2-1} in $\varphi(K - u)$ and $\varphi(H - w)$ respectively. So, absolute value of coefficient of $x^{\eta_1+\eta_2-2}$ in $\varphi(KH + uw) \geq e^{\eta_1-k-1} \cdot e^{\eta_2+k-1} = e^{\eta_1+\eta_2-2}$. Now, the corollary follows, from Theorem 2.29

Corollary 2.40 *Let (K ,u) and (H ,w) be as in Theorem . If (K − u) and (H − w) are hypoenergetic, then the absolute value of the coefficient of* $x^{\eta_1+\eta_2-2}$ *in* $\varphi(KH + uw)$ *is less than* $e^{\eta_1+\eta_2-2}$.

Theorem 2.41 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH+uw. Also let K and H are singular graphs with nullity* η_1 *and* η_2 *respectively. If u is a core vertex of K and w is a noncore vertex of null spread −1 of H or vice versa, then the product of the nonzero eigen values of KH + uw= product of the nonzero eigen values of K× product of nonzero eigen values of H − product of nonzero eigen values of (K − u)× product of the nonzero eigen values of (H − w).*

Corollary 2.42 *Let (K ,u) and (H ,w) be as in Theorem . If u is a core vertex of K and w is a noncore vertex of H of null spread −1 or vice versa, then the coefficient of* $x^{\eta_1+\eta_2}$ *in* $\varphi(K)\varphi(H)$ *cannot be equal to the coefficient of* $x^{\eta_1+\eta_2}$ *in* $\varphi(K - u)\varphi(H - w)$.

The proof of Theorem 2.41 and Corollary 2.42 are similar to the proof of Theorem 2.35 and Corollary 2.36.

Theorem 2.43 *Let (K ,u) and (H ,w) be the components of the graphs obtained by deleting an edge uw from KH+uw. Also let K and H are singular graphs with nullity* η_1 *and* η_2 *respectively. If u is a core vertex of K and w is a noncore vertex of zero null* *spread of H or vice versa, then the product of the nonzero eigen values of KH + uw = product of the nonzero eigen values of (K − u)× product of the nonzero eigen values of (H − w).*

Corollary 2.44 *Let (K,u) and (H,w) be as in Theorem . If the absolute value of the coefficient of* $x^{\eta_1 - 1}$ *in φ(K − u) is greater than or equal to* $e^{\eta_1 - 1 - k}$ *and the coefficient of* x^{η_2} *in φ(H − w) is greater than or equal to* $e^{\eta_2 + k}$*, where* $|k| \le min(\eta_1, \eta_2)$ *,then KH + uw is nonhypoenergetic.*

Corollary 2.45 *Let (K ,u) and (H ,w) be as in Theorem . If K − u and H − w are hypoenergetic, then the absolute value of the product of nonzero eigen values of KH + uw is less than* $e^{\eta_1 + \eta_2 - 1}$.

The proof of Theorem 2.43 and its corollaries are similar to the proof of Theorem 2.38 and its corollaries.

Theorem 2.46 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. Also let K and H are singular graphs with nullity* η_1 *and* η_2 *respectively. If u is a noncore vertex of K of zero null spread and w is a noncore vertex of H of null spread −1 or vice versa, then the product of the nonzero eigen values of KH + uw= product of the nonzero eigen values of K× product of nonzero eigen values of H − product of the nonzero eigen values of (K − u)× product of the nonzero eigen values of (H − w).*

Corollary 2.47 *Let (K ,u) and (H ,w) be as in Theorem . If u is a noncore vertex of K of zero null spread and w is a noncore vertex of H of null spread −1 or vice versa, then the coefficient of* $x^{\eta_1 + \eta_2}$ *in φ(K)φ(H) cannot be equal to the coefficient of* $x^{\eta_1 + \eta_2}$ *in φ(K − u)φ(H − w).*

The proof of Theorem 2.46 and Corollary 2.47 are similar to the proofs of Theorem 2.38 and Corollaries.

Theorem 2.48 *Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw. Also let K be singular with nullity η and H be nonsingular. If u is a core vertex, H − w is nonsingular and the coefficient of* $x^{\eta-1}$ *in φ(K) is greater than or equal to* $e^{\eta-1}$*, then KH + uw is nonhypoenergetic.*

Proof. By Theorem 2.18, KH + uw is a singular graph of nullity η − 1. So the absolute value of the coefficient of $x^{\eta-1}$ in φ(KH + uw) is the product of the absolute value of the coefficient of $x^{\eta-1}$ in φ(K) and the absolute value of the coefficient of x^0 in φ(H). As the absolute value of the coefficient of x^0 in φ(H) is greater than or equal to one, it clearly follows that the absolute value of the coefficient of $x^{\eta-1}$ in φ(KH + uw) is greater than or equal to $e^{\eta-1}$. Hence KH + uw is nonhypoenergetic.

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