Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 2849-2858

http://www.ripublication.com

M-Projective Curvature Tensor on Lorentzian α-Sasakian Manifolds

Venkatesha[1] and Shanmukha B.

Department of Mathematics, Kuvempu University,
Shankaraghatta - 577 451, Shivamogga, Karnataka, India.

Abstract

The purpose the present paper is to study M-projective pseudosymmetric, ϕ-M-projectivelly flat and M-projectivelly flat Lorentzian α-Sasakian manifold. Here we also prove that a Lorentzian α-Sasakian manifolds satisfying $M^*(X,Y) \cdot R(U,V)Z = 0$ is an Einstein manifold. Moreover we presented an example of Lorentzian α-Sasakian manifold.

Keywords: Lorentzian α-Sasakian manifold, M-projective curvature tensor, pseudosymmetric, ϕ-M-projectively flat, M-projectively flat, Einstein manifold, η-Einstein manifold.

AMS Subject Classification: 53C15, 53C25, 53C50.

1. INTRODUCTION

The notion of odd dimensional manifolds with contact and almost contact structures was initiated by Boothby and Wang [4] in 1958 rather from topological point of view. Sasaki and Hatakeyama [19] reinvestigated them using tensor calculus in 1961. Tanno [20] classified connected almost contact metric manifolds whose automorphism groups possess maximum dimension. For such a manifold, the sectional curvature of a plane section containing ξ is a constant, say c. He showed that it can be divided into following three classes:

- Homogeneous normal contact Riemannian manifolds with $c > 0$.
- Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$.
- A warped product space $R \times_f c$ if c < 0.

It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [13] characterized the differential geometric properties of the

[1] *Corresponding Author*

manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [15].
In 1971, Pokhariyal and Mishra [18] have introduced M-projective curvature tensor on a Riemannian manifold which is defined by

$$M^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

Where X, Y, Z, W are vector fields

The properties of the M-projective curvature tensor in Sasakian and Kaehler manifold were developed by Ojha [16]. He showed that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor. Many Geometers studied M-projective curvature tensor on different manifolds such as Kenmostu [9], LP Sasakian [7] and generalized Sasakian space form [21]. Motivated by these ideas, we have an attempt to study properties of M-projective curvature tensor in Lorentzian α-Sasakian manifold. The present paper is organized as follows:

In section 2, we recall the notions and preliminary results of Lorentzian α-Sasakian manifolds that will be needed thereafter. In section 3, we prove that Lorentzian α-Sasakian manifold satisfying$M^*(X,Y)\cdot R(U,V)Z = 0$ is an Einstein manifold. In section 4, we study M-protectively pseudosymmetrc Lorentzian α-Sasakian manifold. Section 5 is devoted to the study of M-projectively flat Lorentzian α-Sasakian manifold and in section 6, we discus ϕ-M-protectively flat Lorentzian α-Sasakian manifold. Finally in section 7, we have constructed an Example for Lorentzian α-Sasakian manifold.

2. PRELIMINARIES

An n-dimensional differentiable manifold M is called Lorentzian α-Sasakian manifold if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g satisfying

(2.1) $\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \cdot \phi = 0$,

for all $X, Y \in TM$.

A Lorentzian α-Sasakian manifold M is satisfying [22], [23],

(2.2) $\nabla_X \xi = -\alpha\phi X, \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y),$

Where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. On Lorentzian α-Sasakian manifold M, the following relations hold true:

(2.3) $R(X,Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y),$

(2.4) $\eta(R(X,Y)Z) = \alpha^2\big(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big),$

(2.5) $R(\xi,X)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X),$

(2.6) $S(X,\xi) = \alpha^2(n-1)\eta(X), \quad QY = \alpha^2(n-1)Y,$

(2.7) $S(\phi X,\phi Y) = S(X,Y) + \alpha^2(n-1)\eta(X)\eta(Y),$

(2.8) $g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y),$

for all vector fields X, Y, Z.

Definition 2.1. An Lorentzian α-Sasakian manifold M is said to be η-Einstein, if it's Ricci tensor S is of the form,

(2.9) $S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y),$

for all vector field X, Y and 1-form η. If $B = 0$, then the manifold M is a Einstein manifold and if $A = 0$, then the manifold M is a special type of η-Einstein manifold.

3. LORENTZIAN α-SASAKIAN MANIFOLD SATISFYING $M^*(X,Y)\cdot R(U,V)Z = 0$.

Suppose $M^*(X,Y)\cdot R(U,V)Z = 0$.

Which implies

(3.1) $M^*(X,Y)R(U,V)Z - R(M^*(X,Y)U,V)Z$

$$-R(U,M^*(X,Y)V)Z - R(U,V)M^*(X,Y)Z = 0.$$

Putting $X = \xi$ in (3.1) and then taking the inner product with ξ, we obtain

(3.2) $g(M^*(\xi,Y)R(U,V)Z,\xi) - g(R(M^*(\xi,Y)U,V)Z,\xi)$

$$-g(R(U,M^*(\xi,Y)V)Z,\xi) - g(R(U,V)M^*(\xi,Y)Z,\xi) = 0.$$

By using (1.1), (2.3) and (2.6) in (3.2), we get

(3.3) $\frac{\alpha^2}{2}\,[-g(Y,R(U,V)Z) - \eta(R(U,V)Z)\eta(Y) - g(Y,U)\eta(R(\xi,V)Z)$

$$+\eta(U)\eta(R(Y,V)Z) - g(Y,V)\eta(R(U,\xi)Z) + \eta(V)\eta(R(U,Y)Z)$$

$$-g(Y,Z)\eta(R(U,V)\xi) - \eta(Z)\eta(R(U,V)Y)] - \frac{1}{2(n-1)}[-S(Y,R(U,V)Z)$$

$$-\alpha^2\big(n-1\big)\eta(R(U,V)Z)\eta(Y) + S(Y,U)\eta(R(\xi,V)Z)$$

$$-\alpha^2\big(n-1\big)\eta(U)\eta(R(Y,V)Z) + S(Y,V)\eta(R(U,\xi)Z)$$

$$-\alpha^2(n-1)\eta(V)\eta(R(U,Y)Z) + S(Y,Z)\eta(R(U,V)\xi)$$

$$-\alpha^2(n-1)\eta(Z)\eta(R(U,V)Y)] = 0.$$

Substituting $U = Z = \xi$ in the above equation and using (2.4), we get

$$S(Y,V) = \alpha^2(n - 1)g(Y,V).$$

Hence, we can state the following:

Theorem 3.1. A Lorentzian α-Sasakian manifold satisfying $(n > 1)$ $M^*(X,Y) \cdot R(U,V)Z = 0$ is an Einstein manifold.

4. $\boldsymbol{M}$-PROJECTIVELY PSEUDOSYMMETRIC LORENTZIAN $\boldsymbol{\alpha}$-SASAKIAN MANIFOLD

The concept of a pseudosymmetric manifold was introduced by Chaki [6] and Deszcz [12]. In this article we study properties of pseudosymmetric manifold according to Deszcz.

Definition 4.2. An n-dimensional Lorentzian α-Sasakian manifold M is said to be pseudosymmetric if

(4.1) $$R(X,Y) \cdot R(U,V)W = L_R[(X \wedge Y) \cdot R(U,V)W)].$$

where L_R is some smooth function on $U_R = \{x \in M : R \neq 0 \; at \; x\}$ and $(X \wedge Y)Z$ is the endomorphism and it is defined as,

(4.2) $$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$$

Definition 4.3. An n-dimensional Lorentzian α-Sasakian manifold M is said to be M-projectively pseudosymmetric if

(4.3) $$R(X,Y) \cdot M^*(U,V)Z = L_{M^*}[(X \wedge Y) \cdot M^*(U,V)Z],$$

holds on the set $U_{M^*} = \{x \in M : M \neq 0 \; at \; x\}$, where L_{M^*} is some function on U_{M^*}.

Suppose that Lorentzian α-Sasakian manifold M is M^*-protectively pseudosymmetric.

Then from (4.3), we have

(4.4) $$R(\xi,Y)M^*(U,V)Z - M^*(R(\xi,Y)U,V)Z - M^*(U,R(\xi,Y)V)Z - M^*(U,V)R(\xi,Y)\mathrm{Z}$$

$$= L_{M^*}[(\xi \wedge Y)M^*(U,V)Z - M^*((\xi \wedge Y)U,V)Z - M^*(U,(\xi \wedge Y))Z - M^*(U,V)(X \wedge Y)Z].$$

Making use of (2.3) in (4.4), we get

(4.5) $(\alpha^2 - L_{M^*})[-g(Y, M^*(U,V)Z) - \eta(M^*(U,V)Z)\eta(Y)$
$-g(Y,U)\eta(M^*(\xi,V)Z)$
$-\eta(U)\eta(M^*(Y,V)Z) - g(Y,V)\eta(M^*(U,\xi)Z) + \eta(V)\eta(M^*(U,Y)Z)$
$-g(Y,Z)\eta(M^*(U,V)\xi) + \eta(Z)\eta(M^*(U,V)Y)] = 0.$

Substituting $U = Z = \xi$ in (4.5) and then using (1.1), we obtain

$$(\alpha^2 - L_{M^*}) = 0 \; or \; S(Y,V) = \alpha^2(n-1)g(Y,V).$$

Thus we state:

Theorem 4.2. If an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$ is M projective pseudosymmetric, then either $(\alpha^2 = L_{M^*})$ or the manifold is an Einstein manifold.

5. M-PROJECTIVELY FLAT LORENTZIAN α-SASAKIAN MANIFOLD

Definition 5.1. A Lorentzian α-Sasakian manifold is M-projectively flat if

(5.1) $M^*(X,Y)Z = 0.$

Let us consider M-projectively flat, Lorentzian α-Sasakian manifold.

Then from (1.1) and (5.1) we have

(5.2) $R(X,Y)Z = \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$

Putting $Z = \xi$ in (5.2), one can get

(5.3) $R(X,Y)\xi = \frac{1}{2(n-1)}[S(Y,\xi)X - S(X,\xi)Y + \eta(Y)QX - \eta(X)QY].$

Taking $Y = \xi$ in (5.3) and by virtue (2.6), (5.3) yields

(5.4) $QX = \alpha^2(n-1)X.$

Setting $X = \xi$ in (5.2) and then using (2.5), we get

(5.5) $$S(Y,Z) = \alpha^2(n-1)g(Y,Z).$$

Using (5.4) and (5.5) in (5.2), it follows that

$$R(X,Y)Z = \alpha^2[g(Y,Z)X - g(X,Z)Y].$$

Hence we can state;

Theorem 5.1. An n-dimensional M-projectively flat Lorentzian α-Sasakian manifold M is locally isometric to the sphere S^n.

6. ϕ-M-PROJECTIVELY FLAT LORENTZIAN α-SASAKIAN MANIFOLDS

Definition 6.1. A differential manifold $(M^n, g), n > 3$, satisfying the condition

(6.1) $$\phi^2(M^*(\phi X, \phi Y)\phi Z) = 0$$

is called ϕ-M-projectively flat.

Suppose that the Lorentzian α-Sasakian manifold M is ϕ-M-projectively flat. Then it is easy to see that $\phi^2(M^*(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

(6.2) $$g(M^*(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for all vector fields X, Y, Z, and W.

Since M is ϕ-M-projectively flat, we have

(6.3) $$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2(n-1)}[S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W) + g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W)].$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using the fact that $\{\phi e_1, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (6.3) and sum up with respect to i, we get

(6.4) $$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)].$$

It can be easily verified that,

(6.5) $\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) = S(\phi Y,\phi Z) + g(\phi Y,\phi Z),$

(6.6) $\sum_{i=1}^{n-1} S(\phi e_i,\phi e_i) = (\tau - (n-1)\alpha^2),$

(6.7) $\sum_{i=1}^{n-1} g(\phi e_i,\phi Y)S(\phi e_i,\phi Z) = S(\phi Y,\phi Z),$

(6.8) $\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) = (n-1).$

Using (6.5)-(6.8) in (6.4), it follows that

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z),$$

where $A = -[\frac{2(n-1)-(\tau-(n-1))\alpha^2}{n+1}]$ and $B = -[\frac{\alpha^2(n-1)(n+1)-2(n-1)+(\tau-(n-1))\alpha^2}{n+1}]$.

Theorem 6.4. If M is a n-dimensional ϕ-M-projectively flat Lorentzian α-Sasakian manifold, then M is an η-Einstein manifold.

EXAMPLE

We consider a four-dimensional manifold $M = \{(x,y,z,u) \in R^4\}$, where (x,y,z,u) are the standard coordinates of R^4. Let $\{e_1,e_2,e_3,e_4\}$ be a linearly independent global frame on M given by

$$e_1 = e^u\frac{\partial}{\partial x}, \qquad e_2 = e^u\frac{\partial}{\partial y}, \qquad e_3 = e^u\frac{\partial}{\partial z}, \qquad e_4 = \alpha\frac{\partial}{\partial u}.$$

Let g be the Lorentzian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \quad g(e_4,e_4) = -1$$

$$g(e_1,e_2) = g(e_1,e_3) = g(e_1,e_4) = g(e_2,e_4) = g(e_3,e_4) = 0.$$

Let η be the 1-form defined by $\eta(U) = g(U,e_4)$ for any $U \in \chi(M)$ and ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_1$, $\phi(e_2) = -e_2$, $\phi(e_3) = -e_3$, $\phi(e_4) = 0$. Then using the linearity of ϕ and g, we have

$$\eta(e_4) = -1,$$

$$\phi^2(Z) = Z + \eta(U)e_4, \text{ and}$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y).$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g, then we have

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = 0, \qquad [e_1, e_3] = 0.$$
$$[e_1, e_4] = -\alpha e_1, \qquad [e_2, e_4] = -\alpha e_2, [e_3, e_4] = -\alpha e_3.$$

Now the Koszuls formula is defined

$$2g(\nabla_X Y, Z) = X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

By using the above equation we can calculate

$$\nabla_{e_i} e_i = \alpha e_\xi, \quad \nabla_{e_i} \xi = -\alpha e_i \;\; \nabla_{e_i} e_j = 0. \quad \text{Where } i = j = 1,2,3,4.$$

From the above it can be easily seen that for $e_4 = (\phi, \xi, \eta, g)$ is an Lorentzian α-Sasakian structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an Lorentzian α-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and S as follows:

$$R(e_1, e_3)e_1 = \alpha^2 e_3, \quad R(e_1, e_3)e_2 = -\alpha^2 e_1, \quad R(e_1, e_4)e_1 = -\alpha^2 e_4,$$
$$R(e_1, e_4)e_4 = -\alpha^2 e_1, \; R(e_2, e_3)e_3 = -\alpha^2 e_2, \; R(e_2, e_3)e_2 = -\alpha^2 e_3,$$
$$R(e_2, e_4)e_2 = \alpha^2 e_4, \quad R(e_3, e_4)e_3 = \alpha^2 e_3, \; R(e_3, e_4)e_4 = -\alpha^2 e_3,$$
$$\mathrm{R}(\mathrm{e}_2, \mathrm{e}_4)\mathrm{e}_4 = \alpha^2 \mathrm{e}_4, \; \mathrm{R}(\mathrm{e}_3, \mathrm{e}_4)\mathrm{e}_3 = -\alpha^2 \mathrm{e}_3, \; \mathrm{R}(\mathrm{e}_3, \mathrm{e}_4)\mathrm{e}_4 = -\alpha^2 \mathrm{e}_3.$$
$$S(e_1, e_1) = -3\alpha^2, \qquad S(e_2, e_2) = -3\alpha^2, \quad S(e_3, e_3) = -3\alpha^2,$$
$$S(e_4, e_4) = -3\alpha^2.$$

And the scalar $r = -12\,\alpha^2$. It is easy to check $S(e_i, e_i) = -3\alpha^2 g(e_i, e_i)$ Therefore M is an Einstein manifold.

ACKNOWLEDGEMENT

The second author is thankful to University Grants Commission, New Delhi, India for financial support in the form of National Fellowship for Higher Education (F1-17.1/ 2016-17/NFST-2015-17-ST-KAR-3079/(SA-III/Website))

REFERENCES

[1]. A. Bhattacharyya and C. Patra, Semi-symmetric metric connections on pseudosymmetric lorentzian α-Sasakian manifolds Acta Et Commentationes Universitatas Tarutuensis De Mathematica, 18 (2014), 197-210.

[2]. A. Barman and U. C. De, Semi Symmetric Non Metric Connections on Kenmotsu Manifolds, Romanian Journal of Mathematics and Computer Science, 5 (2015), 13-24.

[3]. D. E. Blair, Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics, Springer-Verlag, Berlin, 509 (1976), 146.

[4]. M. M. Boothby and R. C. Wang, On contact manifolds, Anna. Math., 68 (1958), 421-450.

[5]. M. C. Chaki and B. Chaki, On pseudosymmetric manifolds admitting a type of semisymmetric connection, Soochow J. Math., 13 (1987), 1-7.

[6]. M. C. Chaki, On pseudo symmetric manifolds, Ann.St.Univ.Al I Cuza Iasi, 33 (1987).

[7]. S. K. Chaubey, some properties of LP-Sasakian manifolds equipped with M-projective curvature tensor, Bulletin of Mathematical Analysis and Applications, 3 (2011), 50-58.

[8]. S. K. Chaubey, S. Prakash and R. Nivas, Some Properties of M-Projective Curvature Tensor in Kenmostu Manifolds, Bulletin of Mathematical Analysis and Applications, 3 (2012), 48-56.

[9]. S. K. Chaubey and R. H. Ojha, On the m-projective curvature tensor of Kenmotsu manifolds, Differential Geometry Dynamical Systems, 12 (2010), 52-60.

[10]. K. De and U. C. De, On a class of pseudosymmetric LP-Sasakian manifolds, Afr. Mat.26 (2015), 131-138.

[11]. U. C. De and S. Mallick, Space Time Admitting M-Projective Curvature tensor, Bulg.J.Phys.39 (2012), 331-338.

[12]. R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg. Ser. A, 44 (1992), 1-34.

[13]. K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J., 24 (1972), 93-103.

[14]. S. Lokesh, Venkatesha, C. S. Bagewadi and K. T. Pradeep Kumar, On Lorentzian α -Sasakian Manifolds, Advances in Pure Mathematics, 2 (2012), 177-182.

[15]. J. C. Marrero and D. Chinea On trans-Sasakian manifolds, Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Univ. La Laguna, La Laguna, 1(3) 1990, 655-659.

[16]. R. H. Ojha, A note on the m-projective curvature tensor, Indian J. pure Applied Math., 8 (1975), 1531- 1534.
[17]. A. Oubina, New classes of contact metric structures, Publ. Math. Debrecen, 32 (1985), 187-193.
[18]. G. P. Pokhariyal and R. S. Mishra, The Curvature Tensor and Their Relativistic Significance II, Yokohoma Mathematical Journal, 18 (1970), 105-108.
[19]. S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structure, Tohoku Math. J., 13 (1961), 281-294.
[20]. S. Tanno, The automorphism groups of almost contact Riemannian manifolds, The Tohoku Math. J., 21 (1969), 21-38.
[21]. Venkatesha and B. Sumangala, on M-projective curvature tensor of Generalised Sasakian space form, Acta Math., Univ. Comenianae 2 (2013), 209-217.
[22]. A. Yildiz and C. Murathan, On Lorentzian α-Sasakian Manifolds, Kyunpook Math. J., 45 (2005), 95-103.
[23]. A. Yildiz, and M.Turan, A Class of Lorentzian α-Sasakian Manifolds, Kyunpook Math. J., 49 (2009), 789-799.