Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 2837-2848

http://www.ripublication.com

New Results on Energy of Graphs of Small Order

Sophia Shalini G. B.[a*] and Mayamma Joseph[b]

[a*]*Department of Mathematics, Mount Carmel College, Bangalore-58, India.*

[b]*Department of Mathematics, Christ University, Bangalore-29, India.*

Abstract

The concept of energy of graphs was coined by Ivan Gutman in 1978. The energy of a graph G is the sum of the absolute values of eigenvalues of adjacency matrix A(G) of the graph G. In this paper we do a detailed study on the bounds of energy of simple, connected and undirected graphs of order less than 10. The problem of finding bounds of energy of graphs was addressed by the researchers by obtaining results in terms of order and size of the given graph. Our study gives emphasis to bounds of the energy of a class of graph in terms of energy of another class of graph. Identification of Some new families of hyper-energetic graphs and the relation between energy of wheel graph and fan graph which is its sub graph also obtained. It has been observed that star graphs possess the least energy among all the families of graphs under consideration. Also the MATLAB code to generate the adjacency matrix and hence to find the energy of certain families of graphs are given.

Keywords: Adjacency matrix; Graph energy

AMS classification: 05C50

1. INTRODUCTION

A graph can be represented in terms of a matrix using the adjacency relation between its vertices. Hence it is natural to study the properties of adjacency matrices associated with graphs. In this direction, the idea of energy of graphs which is a number assigned to a graph in terms of eigenvalues of its adjacency matrix emerged as an interesting area of study in graph theory.

All graphs considered in this paper are simple, finite and undirected. Let G be the simple graph with the vertex set V(G) consisting of n vertices labeled by $v_1, v_2, \ldots v_n$ and an edge set E(G). The **adjacency matrix** A(G) of the graph G is the square matrix of order n, whose (i,j)th entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to zero, otherwise. That is

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ otherwise} \end{cases}$$

where a_{ij} is any element of A(G).

The concept of energy was introduced by Ivan Gutman in 1978 [7]. The energy of a graph is calculated from the eigen values of the corresponding adjacency matrix of the graph. For details and an overview of the results on graph energy we refer to the comprehensive work by Li, Shi and Gutman [8].

The **energy** of a graph is the sum of absolute values of the eigenvalues of its adjacency matrix A(G). In other words, if λ_i where i = 1,2,…n denote the n eigenvalues of the adjacency matrix of the graph G of order n, then the energy of the graph denoted by $\mathrm{E}(G)$ is given by

$$\mathrm{E}(G) = \sum_{i=1}^{n} |\lambda_i| \qquad (1)$$

Determining the energy of different classes of graphs is a challenging task as it requires the construction of adjacency matrix of each and every graph under consideration. This is one of the reasons why researchers tried obtaining bounds for the energy of graphs. The first bound for graph energy was by McClelland [1]. He proved that for a graph G of order n and size m, $\mathrm{E}(G) \leq a\sqrt{2mn}$, where a is empirical constant and $a \cong 0.9$ and it is generally called as McClelland inequality. He also showed that for a graph G with n vertices and m edges $2\sqrt{m} \leq \mathrm{E}(G) \leq 2m$. The McClelland inequality was improved by Koolen, Moulton and Gutman [9]. They have shown that the maximum energy of a graph of order n is $\frac{\left(n^{2/3} + n\right)}{2}$. For the graph G with large number of vertices, Nikifov

[11] proved that $E(G) \le \frac{n^{2/3}}{2-n^{11/10}}$. The McClelland lower bound inequality was improved by Kinkar Das and Gutman [10].

From the review of the above results it was observed that not all the bounds give a correct estimate of the actual value of the energy of a class of graphs. Motivated by this observation we have attempted to answer the following questions: What are the exact bounds for $E(G)$ when G is a tree of order less than 10? Is it possible to obtain a formula for the bounds of energy of a tree of order n?

It has also been observed that most of the bounds of energy of graphs are expressed in terms of the order and size of the given graph. But there are not many results where the bounds of energy of a class of graphs is obtained in relation with energy of some other classes of graphs. In this direction, we explore the following problem. Given families of graphs G, H, S does there exist any relation of the form $E(G) \le E(H) \le E(S)$?

When we consider simple graphs on n vertices, the complete graph has the maximum number of edges. Gutman [7] has proved that the energy of a complete graph on n vertices is 2(n-1) and conjectured that all simple graphs of order n has energy less than 2(n-1). However this conjecture was proved to be false [6] leading to the study of a new class of graphs called **hyper-energetic graphs.**

A graph G on n vertices is said to be hyper-energetic if $E(G) > 2(n-1)$. Walikar, Ramane and Hampiholi [6] presented a systematic construction of hyper-energetic graphs. In this paper, we classify the non-hyper-energetic graphs when $2(n-1) < m$ for $n \le 10$, also, we check are there any graphs of order less than 10 other than complete graphs, complete bipartite graphs whose line graph is hyper-energetic?

For all the basic notations and definitions we refer the books by West and Harary [2, 4].

2. RESULTS AND OBSERVATIONS

2.1 Bounds for Energy of graphs

In this section we present some results on bounds on energy of planar graphs in terms of its order and size. Further we obtain refined bounds for two families of planar graphs namely wheels and fans. First we present the following theorem which is obtained using the property that the number of edges in any triangle free planar graph cannot exceed $2n-4$ [4].

Theorem 1. If G is a planar graph of order n and size m and without triangles, then

$\sqrt{2(m+2)} \le \mathrm{E}(G) \le 2\sqrt{n(n-2)}$

Proof.

If G is a planar graph without triangles, then $m \le 2n-4$.

Consider the McClelland inequality [1], $\mathrm{E}(G) \le \sqrt{2mn}$,

Since $m \le 2n-4$, we have $\mathrm{E}(G) \le \sqrt{2(2n-4)n}$

therefore, $\mathrm{E}(G) \le 2\sqrt{n(n-2)}$

Also $\mathrm{E}(G) \ge 2\sqrt{(n-1)}$ $\Rightarrow$ $\mathrm{E}(G) \ge 2\sqrt{\frac{m+4}{2}-1}$ $\Rightarrow$ $\mathrm{E}(G) \ge 2\sqrt{\frac{m+2}{2}}$

This implies that $\mathrm{E}(G) \ge \sqrt{2(m+2)}$

Thus $\sqrt{2(m+2)} \le \mathrm{E}(G) \le 2\sqrt{n(n-2)}$ □

Among several families of graphs we consider wheels and fan graphs and study their energy for the order less than or equal to 10. The above theorem gives the relation between the energy of a particular graph with the order and size of that graph. We try to obtain a relation between energy of a graph and energy of its sub graph. One such families of graphs to be are wheel graphs and fan graphs.

We present the following MATLAB code to generate the adjacency matrix of wheel graphs and fan graphs and hence to find its energy.

2.1.1 Matlab code to generate the energy of a wheel graph:

```
n=  ;  %enter the number of vertices
A=zeros(n);
for i=1:n-1
        A(i,i+1)=1;
        A(i+1,i)=1;
         A(i,n)=1;
```

```
        A(n,i)=1;
end
A(1,n-1)=1;
A(n-1,1)=1;
A
eigenvaluesofwheel=eig(A)
energy=sum(abs(eigenvaluesofwheel))
```

2.1.2 Matlab code to generate the energy of a fan graph:

```
n=  ;  %enter the number of vertices
A=zeros(n);
for i=1:n-1
        A(i,i+1)=1;
        A(i+1,i)=1;
        A(i,n)=1;
        A(n,i)=1;
end
A
eigenvaluesoffan=eig(A)
energy=sum(abs(eigenvaluesoffan))
```

Using the above MATLAB program the energy of wheel graphs and fan graphs of order upto 10 are found shown in Table 1.

Table 1: Energy of Wheel graphs and Fan graphs of order up to 10.

W_n	$\mathrm{E}(W_n)$	F_n	$\mathrm{E}(F_n)$
W_4	6.472	F_4	5.124
W_5	9.3711	F_5	7.107
W_6	11.292	F_6	8.671
W_7	12.034	F_7	10.098
W_8	13.656	F_8	11.87
W_9	15.84	F_9	13.631
W_{10}	17.578	F_{10}	15.114

Using the values from the Table 1, we conjecture the following:

Conjecture 1. If W_n is the wheel graph with n vertices and m edges, then

$$\sqrt{2m(n-2)} \le \mathrm{E}(W_n) \le 2\sqrt{n(n-2)}$$

Conjecture 2. If F_n is a fan graph on n vertices then $\left\lfloor \frac{2n+3}{2} \right\rfloor - 1 \le \mathrm{E}(F_n) \le \left\lceil \frac{3n+1}{2} \right\rceil$

We also observe that the energy of fan graph is less than the energy of a wheel graph. But we cannot generalize the energy of a sub graph is always smaller than the energy of its original graph.

We observed that most of the bounds are obtained in terms of the order and size of the graph. Now we try to find whether there exist any relation between the energies of two different classes of graphs by analyzing the energy of all trees of order less than 10.

2.2 Bounds for energy of trees

We now discuss the results on the energy of trees of order less than 10. We have obtained an interesting result concerning with the energies of star graph S_n and the path graph P_n. Up to order 3, the results on the energy of path and the star graphs are trivial. So we consider graphs of order greater than 3.

2.2.1 Matlab code to generate the energy of path:

```
n=  ;  %enter the number of vertices
A=zeros(n);
for i=1:n-1
        A(i,i+1)=1;
        A(i+1,i)=1;
end
A
eigenvaluesofpath=eig(A)
energy=sum(abs(eigenvaluesofpath))
```

2.2.2 Matlab code to generate the energy of star:

```
n=  ;  %enter the number of vertices
A=zeros(n);
for i=1:n-1
        A(1,i+1)=1;
        A(i+1,1)=1;
end
A
eigenvaluesofstar=eig(A)
energy=sum(abs(eigenvaluesofstar))
```

Using the above MATLAB program the energy of Paths and Stars of order up to 10 are found shown in Table 2.

Table 2: Energy of path graphs and star graphs of order less than 10

No.of vertices	$\mathrm{E}(P_n)$	$\mathrm{E}(S_n)$
4	4.4721	3.4641
5	5.4641	4
6	6.9879	4.4721
7	8.0547	4.8890
8	9.5175	5.2915
9	10.6275	5.6569
10	12.0533	6

From table 2, we observe the following:

Proposition 1. If S_n is the star graph and P_n is the path graph on n vertices, $\mathrm{E}(S_n) < \mathrm{E}(P_n)$.

Harary [4] has mentioned the number of graphs on number of vertices with respect to the number of edges. Cvetkovic and Petric [3] have enlisted all the connected graphs on 6 vertices. On finding the energy of all those graphs, we observe the following:

Proposition 2. If G is a simple connected graph on n vertices with n=4,5,6 then $\mathrm{E}(S_n) \leq \mathrm{E}(G)$.

We now consider all the tree graphs of order less than 10. Harary [4] has enlisted all the tree graphs of order up to 10. The energy of all the tree graphs are listed below:

Table 3: Energy of trees of order less than 10

No. of vertices	No. of Trees	Energy
4	2	3.4641,4.4721
5	3	4, 5.4641, 6.4286
6	6	4.4721, 5.8186, 6, 6.1554,6.8990, 6.9879
7	11	4.8990, 6.3246, 6.6027,6.7206, 6.8284, 7.5959, 7.6630, 7.7274, 7.8785, 8, 8.0547
8	23	5.2915, 6.7741, 7.1153, 7.2111, 7.2111,7.3846, 8.1528, 8.2611, 8.3128, 8.3751, 8.4243,8.4721, 8.5652, 8.6471, 8.7206, 8.7626, 8.8284,9.3317, 9.4093, 9.4459, 9.5175, 9.9317, 10.0942
9	47	5.6569, 7.1830, 7.3006, 7.6507, 7.7274,7.8098, 7.9362, 8.6332, 8.7710, 8.8146, 8.9405,8.9981, 9.0210, 9.0513, 9.1279, 9.1530, 9.1900,9.2472, 9.3006, 9.3006, 9.3353, 9.3353, 9.4311,9.4311, 9.4641, 9.9425, 9.5076, 10.0362, 10.0629, 10.0943, 10.0943, 10.1505, 10.1751, 10.1751, 10.1991, 10.2517, 10.2611, 10.2908, 10.2908, 10.3128, 10.3840, 10.4243, 10.4721, 10.4721, 10.4914, 10.5652, 10.6275

From table 3, we obtain a relation between energy of star graphs and the energy of other tree graphs.

Conjecture 3. If S_n is a star graph of order n and T_n is tree graph with n vertices, then $\mathrm{E}(S_n) < \mathrm{E}(T_n)$.

2.3 Hyper energetic graphs

Theorem 2. If G is a graph with n vertices and $T(K_n)$ is a total graph of complete graph on n vertices, then for $n \geq 3$, $T(K_n)$ is hyper energetic.

Proof.

The number of vertices of $T(K_n)$ = number of vertices of K_n + number of edges of K_n

$$= n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

The total graph of K_n is isomorphic to the line graph of K_{n+1}.

If *G* is a non-empty r-regular graph with n vertices and m edges [8], then

$$\phi(L(G):\lambda) = \phi(G:\lambda+2-r)(\lambda+2)^{m-n}$$

where $L(G)$ is the line graph of *G* and $\phi(G:\lambda+2-r)$ is the polynomial obtained by replacing λ with $\lambda+2-r$ in $\phi(G:\lambda)$.

The characteristic polynomial of K_n is $\phi(K_n:\lambda) = (\lambda-n+1)(\lambda-1)^{n-1}$

Thus, $\phi(L(K_n):\lambda) = (\lambda-2n+4)(\lambda-n+4)^{n-1}(\lambda+2)^{n(n-3)/2}$

$$\phi(L(K_{n+1}):\lambda) = (\lambda-2(n+1)+4)(\lambda-(n+1)+4)^n(\lambda+2)^{(n+1)(n-2)/2}$$

$$= (\lambda-2n+2)(\lambda-n+3)^n(\lambda+2)^{(n+1)(n-2)/2}$$

Since, $T(K_n) \cong L(K_{n+1})$, we can write

$\phi(L(K_n):\lambda) = (\lambda - 2n + 2)(\lambda - n + 3)^n (\lambda + 2)^{(n+1)(n-2)/2}$

Therefore, $E(T(K_n)) = |2n-2|(1) + |n-3|(n) + |-2|(n+1)(n-2)/2$

$$= 2n - 2 + n^2 - 3n + n^2 - n - 2$$

$$= 2n^2 - 2n - 4$$

To show that $\mathrm{E}(G) > \mathrm{E}(K_n)$, it is sufficient to show that $2n^2 - 2n - 4 > 2(n-1)$

Suppose, $2n^2 - 2n - 4 < 2(n-1)$, then $n^2 - 2n - 1 < 0$. This holds only when $n < 3$, a contradiction to the fact that $n \geq 3$.

Hence, $2n^2 - 2n - 4 > 2(n-1)$ which completes the proof. □

We have also observed the following results:

Proposition 4:

1. For $n \geq 3$, the total graph of $K_{n,n}$ is hyper-energetic.

2. For $n \geq 6$, the total graph of n-cocktail party graph is hyper-energetic.

3. For $n \geq 4$, the total graph of wheel graph W_n is hyper-energetic.

4. For $6 \leq n \leq 9$, the line graph of wheel graph is hyper-energetic.

Theorem 3.

For $n \geq 4$, all the cycle graphs are non-hyper-energetic.

Proof.

If G is a k-regular graph on n vertices, then $\mathrm{E}(G) \leq k + \sqrt{k(n-1)(n-k)}$

Since, C_n is a 2-regular graph, we have $\mathrm{E}(C_n) \leq 2 + \sqrt{2(n-1)(n-2)}$

In order to show that $\mathrm{E}(C_n) \le 2(n-1)$, it suffices to show that

$$2+\sqrt{2(n-1)(n-2)} \le 2(n-1)$$

Suppose, $2+\sqrt{2(n-1)(n-2)} \ge 2(n-1)$ then $2n^2-10n+12<0$ which is not possible for $n \ge 4$

Thus, $\mathrm{E}(C_n) < 2(n-1)$ for $n \ge 4$ which completes the proof. □

CONCLUSION

Most of the results we have obtained in this paper is based on the study of the exact values of the energy of the graphs under consideration. Further studies are to be done and appropriate proof techniques are to be employed to generalize these results. However the results obtained can act as a pointer for articulating and proving the general results.

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