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On Some Strong and $\Delta-$Convergence Theorems for Total Asymptotically Quasi-Nonexpansive Mappings in CAT(0) Spaces

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Abstract

The aim of this article is to prove some strong and Δ-convergence results for total asymptotically quasi-nonexpansive mappings using modified Khan et. al. iterative procedures in CAT(0) spaces. Our results are the extension and generalization of some results of Sahin and Basarir [1], Basarir and Sahin[13], Chang et. al.[24], Agarwal et. al. [15], Aggarwal and Chugh[14], Khan et. al.[19], Khan, Cho and Abbas[21] and Khan and Abbas[20].

Keywords: CAT(0) spaces; $\Delta-$ convergence; strong convergence; total asymptotically quasi nonexpansive mappings; common fixed point; Iterative procedures.

Mathematics Subject Classifications: 47H09, 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Kirk [27, 28] initiated the study of fixed point theory in CAT(0) spaces. He showed that every nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. In 2008, Kirk and Panyanak[29] generalized Lim's[25] concept of Δ-convergence in CAT(0) spaces to prove the CAT(0) space analogs of some Banach space results which involve weak convergence. Dhompongsa and Panyanak[17] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterative procedures in the CAT(0) space setting. Since then many authors have studied the existence and convergence theorems of fixed points (see [2], [7], [13], [26], [31]). In 2011, Khan and Abbas [20] obtained strong and Δ-convergence theorems for S-iterative procedure which is both faster than and independent of the Ishikawa iterative procedure. They also obtained some convergence results for two mappings using the Ishikawa-type iterative procedure. In 2013, Sahin and Basarir [1] studied modified S-iterative procedure and proved strong convergence theorems in CAT(0) spaces which generalize some results of Khan and Abbas [20]. Recently, Basarir and Sahin[13] gave strong and Δ -convergence theorems for modified S-iteartive procedure and modified two step iterative procedure for total asymptotically nonexpansive mappings on a CAT(0) space. In this paper, we establish some strong and Δ -convergence results for total asymptotically quasi-nonexpansive mappings using modified Khan et. al. iterative procedure of total asymptotically quasi nonexpansive mappings in CAT(0) spaces. The results obtained extend and generalize some results of Sahin and Basarir[1], Basarir and Sahin[13],Chang et. al.[24], Agarwal et. al. [15], Aggarwal and Chugh[14], Khan et. al.[19], Khan, Cho and Abbas[21] and Khan and Abbas[20].

Now, we recall some well known concepts and results**.**

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

Let (*X*, *d*) be a metric space. A **geodesic path** joining $x \in X$ to $y \in X$ (or, more briefly, a **geodesic** from *x* to *y*) is a map *c* from a closed interval $[0,l] \subset \mathbb{R}$ to *X* such that

$$c(0) = x, c(l) = y, \text{and } d(c(t), c(t')) = |t - t'| \text{ for all } t, t' \in [0,l].$$

In particular, *c* is an isometry and $d(x,y) = l.$ Usually, the image $c([0,l])$ of *c* is called a **geodesic** (or **metric**) **segment** joining *x* and *y*. A geodesic segment joining *x* and *y* is not necessarily unique in general. In particular, in the case when the geodesic segment joining *x* and *y* is unique, we use [*x*, *y*] to denote the **unique geodesic segment** joining *x* and *y*.

The space (*X*, *d*) is said to be a **geodesic space**, if every two points of *X* are joined by a geodesic, and *X* is said to be **uniquely geodesic space,** if there is exactly one geodesic joining *x* and *y*, for each $x, y \in X$. A subset $Y \subseteq X$ is said to be **convex,** if *Y* includes every geodesic segment joining any two of its points.

A **geodesic triangle** $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (*X*, *d*) consists of three points $x_1, x_2, x_3 \in X$ (the **vertices** of Δ) and a geodesic segment between each pair of vertices (the **edges** of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (*X*, *d*) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$.

The point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point in $\overline{\Delta}$ for $p \in [x, y]$, if $d(x, p) = d_{\mathbb{E}^2}(\overline{x}, \overline{p})$.

A geodesic space is said to be a CAT(0) space, if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let Δ be a geodesic triangle in X and Let $\overline{\Delta}$ be comparison triangle for Δ. Then Δ is said to satisfy CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$\text{(CN)} \qquad d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits[3]. In fact, (c.f. [12], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies (CN) inequality.

Remark.1.1. For $\kappa < 0$, a CAT(κ) space is defined in terms of comparison triangles in the hyperbolic plane (see [12] for details). Here, for sake of simplicity, we omit definition, since it is known (see [12, page 165]) that any CAT(κ_1) space is also CAT (κ_2) space for any pair (κ_1, κ_2) with $\kappa_2 \geq \kappa_1$. This means that the results in CAT(0) spaces can be applied to CAT(κ) spaces with $\kappa \leq 0$.

We now give some definitions and results which will be required in the sequel.

Lemma 1.2[17] Let (X, d) be a CAT(0) space. Then

(i) (X, d) is uniquely geodesic.

(ii) Let p, x, y be points of X, let $\alpha \in [0,1]$, and let m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then

$$d(m_1, m_2) \leq \alpha d(x, y). \tag{1.1.1}$$

(iii)Let $x, y \in X$, $x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$.

(iv)Let $x, y \in X$. For each $t \in [0,1]$, there exists unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1-t)d(x, y). \tag{1.1.2}$$

For convenience, from now onwards we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (1.1.2).

Lemma 1.3.[17] Let (X, d) be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z) \qquad \text{for all } x, y, z \in X \text{ and } t \in [0,1].$$

Lemma 1.4. [28]Let p, x, y be points of a CAT(0) space X, let $\alpha \in [0,1]$. Then

$$d((1-\alpha)p \oplus \alpha x, (1-\alpha)p \oplus \alpha y) \leq \alpha d(x, y).$$

The following Lemma is a generalization of (CN) inequality.

Lemma 1.5. [17]Let (X, d) be a CAT(0) space. Then

$$d((1-t)x\oplus ty,z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For each $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The **asymptotic radius** $r(\{x_n\})$of $\{x_n\}$is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

And the **asymptotic center** $A(\{x_n\})$of $\{x_n\}$is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Therefore, the following equivalence holds for any point $u \in X$:

$$u \in A\left(\{x_n\}\right) \Leftrightarrow \limsup_{n\to\infty} d(u,x_n) \le \limsup_{n\to\infty} d(x,x_n), \quad \text{for all } x \in X. \tag{1.1.3}$$

It is known (see, e.g., [18], Proposition 7) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of Δ-convergence in a CAT(0) space.

Definition 1.6.[29]A sequence $\{x_n\}$ in X is said to be **Δ-convergent** to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$for every subsequence $\{u_n\}$of $\{x_n\}$. In this case we write

$\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We denote , $\omega_\Delta(x_n) = \bigcup\left\{A\left(\{u_n\}\right)\right\}$,where the union is taken over all subsequence $\{u_n\}$of $\{x_n\}$.

Definition 1.7.[24]. Let C be a nonempty subset of a CAT(0) space X. A mapping $I - T : C \to C$ is said to be **demiclosed** at zero, if for each sequence $\{x_n\}$ in C that Δ-converges to a point $x \in C$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0,$ imply $Tx = x$.

Lemma 1.8[17] Let (X, d) be a CAT(0) space. Then

(i) Every bounded sequence in X has a Δ-convergent subsequence.
(ii) If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Definition 1.9. Let C be nonempty subset of a CAT(0) space X. Then $T : C \to C$ is called

a) **Uniformly L-Lipschitzian** if there exists a constant L>0 such that

$d(T^n x, T^n y) \le Ld(x,y)$ for all $x, y \in C, n \in N.$

b) **nonexpansive** if $d(Tx,Ty)\le d(x,y)$ for all $x,y\in C.$

c) **quasi-nonexpansive[16]** if $d(Tx,p)\le d(x,p)$ for all $x\in C, p\in F(T).$

d) **asymptotically nonexpansive[10]** if for a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1,$ we have $d(T^n x,T^n y)\le k_n d(x,y)$ for all $x,y\in C, n\in N.$

e) **asymptotically quasi-nonexpansive[11]** if for a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1,$ we have $d(T^n x,p)\le k_n d(x,p)$ for all $x\in C, p\in F(T), n\in N.$

f) **nearly asymptotically nonexpansive** if there exists constants $a_n\in[0,1), k_n\ge 0$ with $\lim_{n\to\infty}a_n=0, \eta(T^n)\ge 1, \lim_{n\to\infty}\eta(T^n)=0$ (where $\eta(T^n)$ denotes the infimum of constants k_n) such that

$d(T^n x,T^n y)\le k_n\left(d(x,y)+a_n\right)$ for all $x,y\in C, n\in N.$

g) **total asymptotically nonexpansive[24]** if there exist non-negative real sequences $\{\mu_n\},\{\upsilon_n\}$ with $\mu_n\to 0, \upsilon_n\to 0,$ and a strictly increasing continuous function $\zeta:[0,\infty)\to[0,\infty)$ with $\zeta(0)=0$ such that

$$d(T^n x,T^n y)\le d(x,y)+\upsilon_n\zeta(d(x,y))+\mu_n \quad \text{for all } x,y\in C, n\in N.$$

h) **total asymptotically quasi-nonexpansive mapping** if if there exist non-negative real sequences $\{\mu_n\},\{\upsilon_n\}$ with $\mu_n\to 0, \upsilon_n\to 0,$ and a strictly increasing continuous function $\zeta:[0,\infty)\to[0,\infty)$ with $\zeta(0)=0$ such that

$$d(T^n x,p)\le d(x,p)+\upsilon_n\zeta(d(x,p))+\mu_n \quad \text{for all } x\in C, p\in F(T), n\in N.$$

i) **semi-compact** if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty}d(x_n,Tx_n)=0,$ exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p\in C$.

Remark 1.10 It is clear from the above definition that the class of total asymptotically quasi-nonexpansive mappings includes the classes of total asymptotically nonexpansive, nearly asymptotically nonexpansive, asymptotically quasi-nonexpansive, asymptotically nonexpansive, quasi-nonexpansive and nonexpansive mappings. But the converse of each may not be true (see[11],[24] and [32]).

Lemma 1.11.[11] Let $\{a_n\},\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that $a_{n+1}\le(1+\delta_n)a_n+b_n,$ for all $n\in N$.

If $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$ then $\lim_{n\to\infty}a_n$ exists.

Lemma 1.12.[24]: Let X be a CAT(0) space, $x\in X$ be a given point and let $\{t_n\}$ be a sequence in $[b,c]$ with $b,c\in(0,1)$ and $0<b(1-c)\le\frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty}d(x_n,x)\le r,$ $\limsup_{n\to\infty}d(y_n,x)\le r$ and $\lim_{n\to\infty}d\left((1-t_n)x_n\oplus t_n y_n,x\right)=r$ for some $r\ge 0$, then $\lim_{n\to\infty}d(x_n,y_n)=0.$

Let C be a nonempty subset of a Banach space X and T, $S : C \to C$ be two mappings. In the sequel F denotes the set of common fixed points of the mappings T and S.

Schu [8] defined the modified Mann iterative procedure which is a generalization of Mann iterative procedure,

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T^n x_n, n \in N, \end{cases} \tag{1.1.4}$$

where $\{a_n\}$is in $(0,1)$. If a_n =1 for all $n \in N$, then it reduces to modified Picard iteration defined as

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T^n x_n, n \in N. \end{cases} \tag{1.1.5}$$

Tan and Xu [9] generalized Ishikawa iteration procedure and studied modified Ishikawa iteration procedure,

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, n \in N, \end{cases} \tag{1.1.6}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0,1)$. By taking b_n =0 for all $n \in N$ in (1.1.6), we obtain modified Mann iterative procedure (1.1.4).

Khan and Takahashi [22] constructed and studied the following Ishikawa type iterative procedure which modify the iterative procedure defined by Das and Debata [4]:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n S^n x_n, n \in N, \end{cases} \tag{1.1.7}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$. If we take S=T, then we get modified Ishikawa iteration procedure (1.1.6).

In 2007, Agarwal et. al. [15] defined the modified S-iterative procedure as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)T^n x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, n \in N, \end{cases} \tag{1.1.8}$$

where $\{a_n\}$ and $\{b_n\}$are in $[0,1]$. We note that (1.1.8) is independent of (1.1.6) (and hence of (1.1.4)).

Recently, Khan, Cho and Abbas [21] introduced modified Khan et.al. iterative procedure as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)T^n x_n + a_n S^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, n \in N, \end{cases} \tag{1.1.9}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$.

In 2013, Sahin et. al. [1] modified S-iteration (1.1.8) in CAT(0) spaces as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)T^n x_n \oplus a_n T^n y_n, \\ y_n = (1-b_n)x_n \oplus b_n T^n x_n, n \in N, \end{cases} \tag{1.1.10}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$.

If we take n=1, then we obtain the following S-iterative procedure defined by Khan et. al. [20] as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)Tx_n \oplus a_n Ty_n, \\ y_n = (1-b_n)x_n \oplus b_n Tx_n, n \in N, \end{cases} \tag{1.1.11}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$.

We now modify (1.1.9) in CAT(0) spaces as follows:

Let C be a nonempty subset of a CAT(0) space X and $T, S : C \to C$ be two mappings with $F \neq \phi$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)T^n x_n \oplus a_n S^n y_n, \\ y_n = (1-b_n)x_n \oplus b_n T^n x_n, n \in N, \end{cases} \tag{1.1.12}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$ called modified Khan et. al. iterative procedure. It reduces to the

modified S-iteration (1.1.10) for $S=I$.

If we take n=1 in (1.1.12) then the following Khan et. al. iterative procedure defined by Khan et. al.[19] will be obtained as a special case of iterative procedure (1.1.12):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)Tx_n \oplus a_n S y_n, \\ y_n = (1-b_n)x_n \oplus b_n T x_n, n \in N, \end{cases} \tag{1.1.13}$$

where $\{a_n\}$ and $\{b_n\}$are in $(0, 1)$.

2. MAIN RESULTS

In this section, we prove strong and Δ-convergence of the modified khan et. al. iterative procedure (1.1.12) to a common fixed point of two total asymptotically quasi-nonexpansive mappings T and S in CAT(0) spaces.

Let T, S: $C \to C$ be two total asymptotically quasi-nonexpansive mappings satisfying

$d(T^n x, p) \le d(x,p) + \upsilon_n \zeta(d(x,p)) + \mu_n$ and $d(S^n x, p) \le d(x,p) + \upsilon_n \zeta(d(x,p)) + \mu_n$ for all $x \in C, p \in F(T), n \in N,$ with non-negative real sequences $\{\mu_n\}, \{\upsilon_n\}$ with $\mu_n \to 0, \upsilon_n \to 0,$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$.

From now onwards, we will denote the set of common fixed points of T and S by $F = \{p \in C : Tp = p = Sp\}$.

Lemma 2.1 Let C be a nonempty closed convex subset of a CAT(0) space X and T, S : $C \to C$ be two uniformly L_1 and L_2-Lipschitizian and total asymptotically quasi-nonexpansive mappings and $L = \max\{L_1, L_2\}$. Let $\{x_n\}$ be defined by iterative procedure (1.1.12) with $F \ne \phi$. If the following conditions are satisfied:

a) $\sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} a_n < \infty;$

b) there exists a constant $M^* > 0$ such that $\zeta(r) \le M^* r, r \ge 0;$

c) $\{b_n\}$ is the sequence in [0,1]:

d) $\sum_{n=1}^{\infty} \sup\{d(z, T^n z) : z \in B\} < \infty$ for each bounded subset B of C.

e) there exists constants $b, c \in (0,1)$ with $0 < b(1-c) \le \frac{1}{2}$ such that $\{a_n\} \subset [b,c]$,

then

(i) $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F$.

(ii) $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$.

Proof. Let $q \in F$. Then by Lemma 1.3,

$$\begin{aligned}
d(y_n, q) &= d((1-b_n)x_n \oplus b_n T^n x_n), q) \\
&\le (1-b_n)d(x_n, q) + b_n d(T^n x_n, q) \\
&\le (1-b_n)d(x_n, q) + b_n \left\{d(x_n, q) + \upsilon_n \zeta\left(d(x_n, q)\right) + \mu_n\right\} \\
&\le (1+b_n \upsilon_n M^*)d(x_n, q) + b_n \mu_n \\
&\le (1+\upsilon_n M^*)d(x_n, q) + \mu_n
\end{aligned}$$

Now, using (2.1.1), we get

$$\begin{aligned}
d(x_{n+1}, q) &= d((1-a_n)T^n x_n \oplus a_n S^n y_n), q) \\
&\le (1-a_n)d(T^n x_n, q) + a_n d(S^n y_n, q) \\
&\le (1-a_n)\left\{d(x_n, q) + \upsilon_n \zeta\left(d(x_n, q)\right) + \mu_n\right\} + a_n L d(y_n, q) \\
&\le (1-a_n)\left\{d(x_n, q) + \upsilon_n \zeta\left(d(x_n, q)\right) + \mu_n\right\} + a_n L\left\{(1+\upsilon_n M^*)d(x_n, q) + \mu_n\right\} \\
&\le (1-a_n)\left\{(1+\upsilon_n M^*)d(x_n, q) + \mu_n\right\} + a_n L\left\{(1+\upsilon_n M^*)d(x_n, q) + \mu_n\right\} \\
&\le \left\{(1-a_n)(1+\upsilon_n M^*) + a_n L(1+\upsilon_n M^*)\right\} d(x_n, q) + \left(1+a_n\left(L-1\right)\right)\mu_n \\
&\le \left\{\left(1+a_n\left(L-1\right)\right) + \upsilon_n M^*\left(1+a_n\left(L-1\right)\right)\right\} d(x_n, q) + \left(1+a_n\left(L-1\right)\right)\mu_n.
\end{aligned}$$

Thus, by Lemma 1.12 and condition(a), $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F$.

Let $\lim_{n\to\infty} d(x_n, q) = c$. (2.1.1)

Since,

$$\begin{aligned}
d\left(S^n y_n, q\right) &\le_n d(y_n, q) + \upsilon_n \zeta\left(d(y_n, q)\right) + \mu_n \\
&\le (1+\upsilon_n M^*)d(y_n, q) + \mu_n \\
&\le (1+\upsilon_n M^*)\left\{(1+\upsilon_n M^*)d(x_n, q) + \mu_n\right\} + \mu_n \\
&\le (1+\upsilon_n M^*)(1+\upsilon_n M^*)d(x_n, q) + (2+\upsilon_n M^*)\mu_n.
\end{aligned}$$

Using, (2.1.1), we have

$$\limsup_{n\to\infty} d(S^n y_n, q) \le c. \tag{2.1.2}$$

Similarly, we obtain

$$\limsup_{n\to\infty} d(T^n x_n, q) \le c. \tag{2.1.3}$$

In addition,

$$c = \lim_{n\to\infty} d(x_{n+1}, \mathrm{q}) = \lim_{n\to\infty} d((1-a_n)T^n x_n \oplus a_n S^n y_n), \mathrm{q}).$$

With the help of (2.1.2), (2.1.3) and Lemma 1.13, we get

$$\lim_{n\to\infty} d(T^n x_n, S^n y_n) = 0\,. \tag{2.1.4}$$

On the other hand, since

$$d(x_{n+1}, T^n x_n) = d((1-a_n)T^n x_n \oplus a_n S^n y_n), T^n x_n) \le a_n d(S^n y_n, T^n x_n).$$

From (2.1.4),

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_n) = 0.. \tag{2.1.5}$$

Thus, using condition (d), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0.. \tag{2.1.6}$$

Hence, from (2.1.5) and (2.1.6), we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.. \tag{2.1.7}$$

Now, using Lemma 1.3,

$$d(y_n, x_n) = d((1-b_n)x_n \oplus b_n T^n x_n, x_n)$$

$$\le (1-b_n) d(x_n, x_n) + b_n d(T^n x_n, x_n).$$

Using (2.1.6), we have

$$\lim_{n\to\infty} d(y_n, x_n) = 0\,. \tag{2.1.8}$$

Also, $d(x_{n+1}, y_n) \le d(x_{n+1}, x_n) + d(y_n, x_n)$

which using (2.1.7) and (2.1.8) gives

$$\lim_{n\to\infty} d(x_{n+1}, y_n) = 0\,. \tag{2.1.9}$$

Now, $d(x_{n+1}, S^n y_n) \le d(x_{n+1}, x_n) + d(x_n, T^n x_n) + d(T^n x_n, S^n y_n)$.

By (2.1.4), (2.1.6) and (2.1.7), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, S^n y_n) = 0. \tag{2.1.10}$$

Thus,

$$d(x_n, S^n x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, S^n y_n) + d(S^n y_n, S^n x_n)$$
$$\le d(x_n, x_{n+1}) + d(x_{n+1}, S^n y_n) + Ld(y_n, x_n)$$

gives by (2.1.7), (2.1.8) and (2.1.10) that

$$\lim_{n\to\infty} d(x_n, S^n x_n) = 0. \tag{2.1.11}$$

Then,

$$d(x_{n+1}, Tx_{n+1}) \le d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_{n+1})$$
$$\le d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(x_{n+1}, x_n) + Ld(T^n x_n, x_{n+1})$$
$$= d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(x_{n+1}, x_n) + La_n d(T^n x_n, S^n y_n).$$

It follows from (2.1.4), (2.1.6) and (2.1.7) that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Finally,

$$d(x_{n+1}, Sx_{n+1}) \le d(x_{n+1}, S^{n+1}x_{n+1}) + d(S^{n+1}x_{n+1}, Sx_{n+1})$$
$$\le d(x_{n+1}, S^{n+1}x_{n+1}) + Ld(S^n x_{n+1}, x_{n+1})$$
$$\le d(x_{n+1}, S^{n+1}x_{n+1}) + L\left[d(S^n x_{n+1}, S^n y_n) + d(S^n y_n, x_{n+1})\right]$$
$$\le d(x_{n+1}, S^{n+1}x_{n+1}) + L\left[Ld(x_{n+1}, y_n) + d(S^n y_n, x_{n+1})\right]$$

implies by using (2.1.9), (2.1.10) and (2.1.11) that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$.

Theorem 2.2. Let X, C, T, S, F, $\{a_n\}$, $\{b_n\}$ and $\{x_n\}$ be as in Lemma 2.1. If $I-T$ and $I-S$ are demiclosed with respect to zero, then $\{x_n\}$ Δ-converges to a point of F.

Proof. Let $q \in F$. Then by Lemma 2.1, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F$. Thus $\{x_n\}$ is bounded. From Lemma 2.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$.

Firstly, we show that $\omega_\Delta(x_n) \subset F$. Let $u \in \omega_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 1.8 , there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v$ for some $v \in C$. Also $I-T$ and $I-S$ are demiclosed with respect to zero,, therefore we obtain Tv = v= Sv, which means that $v \in F$. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, v)$ exists. Now, we claim that $u = v$. Assume on the contrary that $u \neq v$. Then by the uniqueness of asymptotic centers, we have

$$\limsup_{n\to\infty} d(v_n, v) < \limsup_{n\to\infty} d(v_n, u) \le \limsup_{n\to\infty} d(u_n, u) < \limsup_{n\to\infty} d(u_n, v)$$
$$= \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),$$

a contradiction. Thus $u = v \in F$ and hence $\omega_\Delta(x_n) \subset F$.

Now, we show that the sequence $\{x_n\}$ Δ-converges to a point of F, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.8, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v$ for some $v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already proved that $u = v \in F$. Finally, we claim that $x = v$. If is not true, then existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic center imply that

$$\limsup_{n\to\infty} d(v_n, v) < \limsup_{n\to\infty} d(v_n, x) \le \limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),$$

a contradiction. Thus $x = v \in F$ and hence $\omega_\Delta(x_n) = \{x\}$. Thus, $\{x_n\}$ Δ-converges to a point of F.

Now, we obtain our strong convergence results for iterative procedure (1.1.19).

Theorem 2.3. Let X be a complete CAT(0) space and C, T, S, F, $\{a_n\}$, $\{b_n\}$, $\{x_n\}$ be as in Lemma 2.1. If $F \neq \phi$, then $\{x_n\}$ converges strongly to a point of F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 2.1, we have $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in F$.

This implies that $d(x_{n+1}, F) \le d(x_n, F)$, so that $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since

$\lim_{n\to\infty} d(x_n, F) = 0$, there exists a positive integer n_0 such that $d(x_n, F) < \frac{\varepsilon}{4}$ for all $n \ge n_0$.

In particular, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{2}.$$

Now for all m, $n \ge n_0$, we have

$$d(x_{n+m}, x_n) \le d(x_{n+m}, p^*) + d(p^*, x_n)$$

$$\leq 2d(x_{n_0}, p^*) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence $\{x_n\}$ is a cauchy sequence in a closed subset C of a complete CAT(0) space and so it must converge to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F) = 0$, gives that $d(q, F) = 0$. Since F is closed, so we have $q \in F$.

Fukhar-ud-din and Khan [6] introduced the concept of **condition** (A') as follows:

Two mappings T, $S : C \to C$ are said to satisfy the condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either

$$f(d(x, F)) \leq d(x, Tx), \text{ or } \quad f(d(x, F)) \leq d(x, Sx) \quad \text{for all } x \in C,$$

where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

If we take $S = T$ in this condition, then it reduces to condition(A) of Senter and Doston[5].

Theorem 2.4. Let X be a complete CAT(0) space and C, T, S, F, $\{a_n\}$, $\{b_n\}$, $\{x_n\}$ be as in Lemma 2.1. Let S, T satisfy the condition (A') and $F \neq \phi$. Then $\{x_n\}$ converges strongly to a point of F.

Proof. We have proved in Lemma 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. Let this limit be c. As proved in Lemma 2.1, we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F$.

This gives that $\inf_{p\in F} d(x_{n+1}, p) \leq \inf_{p\in F} d(x_n, p)$, which means that $d(x_{n+1}, F) \leq d(x_n, F)$, so that $\lim_{n\to\infty} d(x_n, F)$ exists. Again using Lemma 2.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$.

From the condition (A'), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0, \text{ or } \lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, therefore we have $\lim_{n\to\infty} d(x_n, F) = 0$. Now, all the conditions of Theorem 2.3 are satisfied, therefore $\{x_n\}$ converges strongly to a point of F.

Theorem 2.5. Let X be a complete CAT(0) space and C, T, S, F, $\{a_n\}$, $\{b_n\}$, $\{x_n\}$ be as in Lemma 2.1. Suppose that one of S and T is semi-compact. If $F \neq \phi$, $\{x_n\}$ converges strongly to a point of F.

Proof. From Lemma 2.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$. Let one of S and T is semi-compact, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p \in C$. Moreover, using continuity of S and T, we have

$$d(p, Tp) = \lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0, \;\; d(p, Sp) = \lim_{n\to\infty} d(x_{n_k}, Sx_{n_k}) = 0.$$

Thus, $p \in F$. Again, from Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. Since $\lim_{k\to\infty} d(x_{n_k}, p) = 0.$

Therefore, $\lim_{n\to\infty} d(x_n, p) = 0$ and therefore $\{x_n\}$ converges strongly to a point of F.

Remark. 2.6 In the light of Remark.1.1 it is also noted that our results in CAT(0) spaces can be applied to CAT(κ) spaces with $\kappa \le 0$.

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