On Some Strong and $\Delta$–Convergence Theorems for Total Asymptotically Quasi-Nonexpansive Mappings in CAT(0) Spaces

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Abstract

The aim of this article is to prove some strong and $\Delta$-convergence results for total asymptotically quasi-nonexpansive mappings using modified Khan et. al. iterative procedures in CAT(0) spaces. Our results are the extension and generalization of some results of Sahin and Basarir [1], Basarir and Sahin[13], Chang et. al.[24], Agarwal et. al. [15], Aggarwal and Chugh[14], Khan et. al.[19], Khan, Cho and Abbas[21] and Khan and Abbas[20].

Keywords: CAT(0) spaces; $\Delta$ – convergence; strong convergence; total asymptotically quasi nonexpansive mappings; common fixed point; Iterative procedures.

Mathematics Subject Classifications: 47H09, 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Kirk [27, 28] initiated the study of fixed point theory in CAT(0) spaces. He showed that every nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. In 2008, Kirk and Panyanak[29] generalized Lim’s[25] concept of $\Delta$-convergence in CAT(0) spaces to prove the CAT(0) space analogs of some Banach space results which involve weak convergence. Dhompongsa and Panyanak[17] obtained $\Delta$-convergence theorems for the Picard, Mann and Ishikawa iterative procedures in the CAT(0) space setting. Since
then many authors have studied the existence and convergence theorems of fixed points (see [2], [7], [13], [26], [31]). In 2011, Khan and Abbas [20] obtained strong and $\Delta$-convergence theorems for S-iterative procedure which is both faster than and independent of the Ishikawa iterative procedure. They also obtained some convergence results for two mappings using the Ishikawa-type iterative procedure. In 2013, Sahin and Basarir [1] studied modified S-iterative procedure and proved strong convergence theorems in CAT(0) spaces which generalize some results of Khan and Abbas [20]. Recently, Basarir and Sahin [13] gave strong and $\Delta$-convergence theorems for modified S-iterative procedure and modified two step iterative procedure for total asymptotically nonexpansive mappings on a CAT(0) space. In this paper, we establish some strong and $\Delta$-convergence results for total asymptotically quasi-nonexpansive mappings using modified Khan et al. iterative procedure of total asymptotically quasi nonexpansive mappings in CAT(0) spaces. The results obtained extend and generalize some results of Sahin and Basarir [1], Basarir and Sahin [13], Chang et al.[24], Agarwal et al. [15], Aggarwal and Chugh [14], Khan et al.[19], Khan, Cho and Abbas [21] and Khan and Abbas [20].

Now, we recall some well known concepts and results.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subseteq \mathbb{R}$ to $X$ such that $c(0) = x, c(l) = y,$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

In particular, $c$ is an isometry and $d(x, y) = l$. Usually, the image $c([0, l])$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. A geodesic segment joining $x$ and $y$ is not necessarily unique in general. In particular, in the case when the geodesic segment joining $x$ and $y$ is unique, we use $[x, y]$ to denote the unique geodesic segment joining $x$ and $y$.

The space $(X, d)$ is said to be a geodesic space, if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic space, if there is exactly one geodesic joining $x$ and $y$, for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex, if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points $x_1, x_2, x_3 \in X$ (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(x_1, x_2, x_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d_{(X, d)}(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

The point $\overline{p} \in [\overline{x}, y]$ is called a comparison point in $\overline{\Delta}$ for $p \in [x, y]$, if $d(x, p) = d_{\mathbb{E}^2}(\overline{x}, \overline{p})$. 


A geodesic space is said to be a CAT(0) space, if all geodesic triangles satisfy the following comparison axiom.

**CAT(0):** Let $\Delta$ be a geodesic triangle in $X$ and let $\overline{\Delta}$ be comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\text{cat}}(\overline{x}, \overline{y}).$$

If $x, y_1, y_2$ are points of a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$(\text{CN}) \quad d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits[3]. In fact, (c.f. [12], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies (CN) inequality.

**Remark 1.1.** For $\kappa < 0$, a CAT($\kappa$) space is defined in terms of comparison triangles in the hyperbolic plane (see [12] for details). Here, for sake of simplicity, we omit definition, since it is known (see [12, page 165]) that any CAT($\kappa_1$) space is also CAT($\kappa_2$) space for any pair $(\kappa_1, \kappa_2)$ with $\kappa_2 \geq \kappa_1$. This means that the results in CAT(0) spaces can be applied to CAT($\kappa$) spaces with $\kappa \leq 0$.

We now give some definitions and results which will be required in the sequel.

**Lemma 1.2[17]** Let $(X, d)$ be a CAT(0) space. Then

(i) $(X, d)$ is uniquely geodesic.

(ii) Let $p, x, y$ be points of $X$, let $\alpha \in [0,1]$, and let $m_1$ and $m_2$ denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then

$$d(m_1, m_2) \leq \alpha d(x, y). \quad (1.1.1)$$

(iii) Let $x, y \in X$, $x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$.

(iv) Let $x, y \in X$. For each $t \in [0,1]$, there exists unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y). \quad (1.1.2)$$

For convenience, from now onwards we will use the notation $(1-t)x \oplus ty$ for the unique point $z$ satisfying (1.1.2).

**Lemma 1.3.[17]** Let $(X, d)$ be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z) \quad \text{for all } x, y, z \in X \text{ and } t \in [0,1].$$

**Lemma 1.4. [28]** Let $p, x, y$ be points of a CAT(0) space $X$, let $\alpha \in [0,1]$. Then

$$d((1-\alpha)p \oplus \alpha x, (1-\alpha)p \oplus \alpha y) \leq \alpha d(x, y).$$
The following Lemma is a generalization of (CN) inequality.

**Lemma 1.5.** Let \((X, d)\) be a CAT(0) space. Then
\[
d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2
\]
for all \(x, y, z \in X\) and \(t \in [0, 1]\).

Let \(\{x_n\}\) be a bounded sequence in a CAT(0) space \(X\). For each \(x \in X\), we set
\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]
The **asymptotic radius** \(r(\{x_n\})\) of \(\{x_n\}\) is given by
\[
r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},
\]
And the **asymptotic center** \(A(\{x_n\})\) of \(\{x_n\}\) is the set
\[
A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.
\]
Therefore, the following equivalence holds for any point \(u \in X\):
\[
u \in A(\{x_n\}) \iff \limsup_{n \to \infty} d(u, x_n) \leq \limsup_{n \to \infty} d(x, x_n), \quad \text{for all } x \in X. \quad (1.1.3)
\]
It is known (see, e.g., [18], Proposition 7) that in a CAT(0) space, \(A(\{x_n\})\) consists of exactly one point.

We now give the definition of \(\Delta\)-convergence in a CAT(0) space.

**Definition 1.6.** A sequence \(\{x_n\}\) in \(X\) is said to be **\(\Delta\)-convergent** to \(x \in X\) if \(x\) is the unique asymptotic center of \(\{u_n\}\) for every subsequence \(\{u_n\}\) of \(\{x_n\}\). In this case we write
\[
\Delta \lim_{n \to \infty} x_n = x \quad \text{and call } x \text{ the } \Delta\text{-limit of } \{x_n\}.
\]
We denote \(\omega_{\Delta}(x_n) = \bigcup\{ A(\{u_n\}) \}, \) where the union is taken over all subsequence \(\{u_n\}\) of \(\{x_n\}\).

**Definition 1.7.** Let \(C\) be a nonempty subset of a CAT(0) space \(X\). A mapping \(I - T : C \to C\) is said to be **demiclosed** at zero, if for each sequence \(\{x_n\}\) in \(C\) that \(\Delta\)-converges to a point \(x \in C\) and \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\), imply \(Tx = x\).

**Lemma 1.8.** Let \((X, d)\) be a CAT(0) space. Then
(i) Every bounded sequence in \(X\) has a \(\Delta\)-convergent subsequence.
(ii) If \(C\) is a closed convex subset of \(X\) and if \(\{x_n\}\) is a bounded sequence in \(C\), then the asymptotic center of \(\{x_n\}\) is in \(C\).

**Definition 1.9.** Let \(C\) be a nonempty subset of a CAT(0) space \(X\). Then \(T : C \to C\) is called
a) **Uniformly L-Lipschitzian** if there exists a constant \(L > 0\) such that
\[
d(T^n x, T^n y) \leq L d(x, y)
\]
for all \(x, y \in C, n \in \mathbb{N}\).
b) **nonexpansive** if \( d(Tx, Ty) \leq d(x, y) \) for all \( x, y \in C \).

c) **quasi-nonexpansive**\(^{[16]}\) if \( d(Tx, p) \leq d(x, p) \) for all \( x \in C, p \in F(T) \).

d) **asymptotically nonexpansive**\(^{[10]}\) if for a sequence \( \{k_n\} \subset [1, \infty) \) with 
\[
\lim_{n \to \infty} k_n = 1, \quad \text{we have} \quad d(T^n x, T^n y) \leq k_n d(x, y) \quad \text{for all} \quad x, y \in C, n \in N.
\]
e) **asymptotically quasi-nonexpansive**\(^{[11]}\) if for a sequence \( \{k_n\} \subset [1, \infty) \) with 
\[
\lim_{n \to \infty} k_n = 1, \quad \text{we have} \quad d(T^n x, p) \leq k_n d(x, p) \quad \text{for all} \quad x \in C, p \in F(T), n \in N.
\]
f) **nearly asymptotically nonexpansive** if there exists constants \( a_n \in [0, 1), k_n \geq 0 \) with 
\[
\lim_{n \to \infty} a_n = 0, \quad \eta(T^n) \geq 1, \quad \lim_{n \to \infty} \eta(T^n) = 0 \quad (\text{where} \ \eta(T^n) \ \text{denotes the infimum of constants} \ k_n) \quad \text{such that}
\]
\[
d(T^n x, T^n y) \leq k_n \left(d(x, y) + a_n\right) \quad \text{for all} \quad x, y \in C, n \in N.
\]
g) **total asymptotically nonexpansive**\(^{[24]}\) if there exist non-negative real sequences 
\( \{\mu_n\}, \{\nu_n\} \) with \( \mu_n \to 0, \nu_n \to 0 \), and a strictly increasing continuous function 
\( \zeta : [0, \infty) \to [0, \infty) \) with \( \zeta(0) = 0 \) such that 
\[
d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n \quad \text{for all} \quad x, y \in C, n \in N.
\]
h) **total asymptotically quasi-nonexpansive mapping** if there exist non-negative real sequences 
\( \{\mu_n\}, \{\nu_n\} \) with \( \mu_n \to 0, \nu_n \to 0 \), and a strictly increasing continuous function 
\( \zeta : [0, \infty) \to [0, \infty) \) with \( \zeta(0) = 0 \) such that 
\[
d(T^n x, p) \leq d(x, p) + \nu_n \zeta(d(x, p)) + \mu_n \quad \text{for all} \quad x \in C, p \in F(T), n \in N.
\]
i) **semi-compact** if for a sequence \( \{x_n\} \) in \( C \) with \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \), exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to a point \( p \in C \).

**Remark 1.10** It is clear from the above definition that the class of total asymptotically quasi-nonexpansive mappings includes the classes of total asymptotically nonexpansive, nearly asymptotically nonexpansive, asymptotically quasi-nonexpansive mappings. But the converse of each may not be true (see\([11],[24]\) and \([32]\)).

**Lemma 1.11.**\(^{[11]}\) Let \( \{a_n\}, \{b_n\} \) and \( \{\delta_n\} \) be sequences of nonnegative real numbers such that 
\[
a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \text{for all} \quad n \in N.
\]

If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \) then \( \lim a_n \) exists.

**Lemma 1.12.**\(^{[24]}\): Let \( X \) be a CAT(0) space, \( x \in X \) be a given point and let \( \{t_n\} \) be a sequence in \([b, c]\) with \( b, c \in (0, 1) \) and \( 0 < b(1-c) \leq \frac{1}{2} \). Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences in \( X \) such that \( \limsup_{n \to \infty} d(x_n, x) \leq r, \limsup_{n \to \infty} d(y_n, x) \leq r \) and 
\[
\lim_{n \to \infty} d\big((1-t_n)x_n \oplus t_n y_n, x\big) = r \quad \text{for some} \quad r \geq 0, \quad \text{then} \lim_{n \to \infty} d(x_n, y_n) = 0.
\]
Let $C$ be a nonempty subset of a Banach space $X$ and $T, S : C \to C$ be two mappings. In the sequel $F$ denotes the set of common fixed points of the mappings $T$ and $S$.

Schu [8] defined the modified Mann iterative procedure which is a generalization of Mann iterative procedure,

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)x_n + a_nT^n x_n, n \in N,
\end{align*}
\]

where $\{a_n\}$ is in $(0,1)$. If $a_n=1$ for all $n \in N$, then it reduces to modified Picard iteration defined as

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= T^n x_n, n \in N.
\end{align*}
\]

Tan and Xu [9] generalized Ishikawa iteration procedure and studied modified Ishikawa iteration procedure,

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)x_n + a_nT^n y_n, \\
    y_n &= (1-b_n)x_n + b_nT^n y_n, n \in N,
\end{align*}
\]

where $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. By taking $b_n=0$ for all $n \in N$ in (1.1.6), we obtain modified Mann iterative procedure (1.1.4).

Khan and Takahashi [22] constructed and studied the following Ishikawa type iterative procedure which modify the iterative procedure defined by Das and Debata [4]:

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)x_n + a_nT^n y_n, \\
    y_n &= (1-b_n)x_n + b_nS^n x_n, n \in N,
\end{align*}
\]

where $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. If we take $S=T$, then we get modified Ishikawa iteration procedure (1.1.6).

In 2007, Agarwal et. al. [15] defined the modified S-iterative procedure as follows:
Recently, Khan, Cho and Abbas [21] introduced modified Khan et.al. iterative procedure as follows:

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)T^s x_n + a_n T^s y_n, \\
    y_n &= (1-b_n)x_n + b_n T^s x_n, n \in \mathbb{N},
\end{align*}
\]

where \( \{a_n\} \) and \( \{b_n\} \) are in \([0,1]\). We note that (1.1.8) is independent of (1.1.6) (and hence of (1.1.4)).

In 2013, Sahin et al. [1] modified S-iteration (1.1.8) in CAT(0) spaces as follows:

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)T^s x_n + a_n S^s y_n, \\
    y_n &= (1-b_n)x_n + b_n T^s x_n, n \in \mathbb{N},
\end{align*}
\]

where \( \{a_n\} \) and \( \{b_n\} \) are in \((0, 1)\).

We now modify (1.1.9) in CAT(0) spaces as follows:

Let \( C \) be a nonempty subset of a CAT(0) space \( X \) and \( T, S : C \to C \) be two mappings with \( F \neq \emptyset \). Then the sequence \( \{x_n\} \) generated by

\[
\begin{align*}
    x_1 &= x \in C, \\
    x_{n+1} &= (1-a_n)T x_n + a_n T y_n, \\
    y_n &= (1-b_n)x_n + b_n T x_n, n \in \mathbb{N},
\end{align*}
\]
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where \( \{a_n\} \) and \( \{b_n\} \) are in \((0, 1)\) called modified Khan et. al. iterative procedure. It reduces to the modified S-iteration (1.1.10) for \( S=I \).

If we take \( n=1 \) in (1.1.12) then the following Khan et. al. iterative procedure defined by Khan et. al.[19] will be obtained as a special case of iterative procedure (1.1.12):

\[
\begin{align*}
 x_1 &= x \in C, \\
 x_{n+1} &= (1-a_n)Tx_n \oplus a_nSy_n, \\
 y_n &= (1-b_n)x_n \oplus b_nTx_n, \quad n \in \mathbb{N},
\end{align*}
\]

(1.1.13)

where \( \{a_n\} \) and \( \{b_n\} \) are in \((0, 1)\).

2. MAIN RESULTS

In this section, we prove strong and \( \Delta \)-convergence of the modified khan et. al. iterative procedure (1.1.12) to a common fixed point of two total asymptotically quasi-nonexpansive mappings \( T \) and \( S \) in CAT(0) spaces.

Let \( T, S : C \to C \) be two total asymptotically quasi-nonexpansive mappings satisfying

\[
d(T^n x, p) \leq d(x, p) + \nu_n \mathcal{C}(d(x, p)) + \mu_n \quad \text{and} \quad d(S^n x, p) \leq d(x, p) + \nu_n \mathcal{C}(d(x, p)) + \mu_n
\]

for all \( x \in C, p \in F(T), n \in \mathbb{N}, \) with non-negative real sequences \( \{\mu_n\}, \{\nu_n\} \) with \( \mu_n \to 0, \nu_n \to 0, \) and a strictly increasing continuous function \( \mathcal{C} : [0, \infty) \to [0, \infty) \) with \( \mathcal{C}(0) = 0 \).

From now onwards, we will denote the set of common fixed points of \( T \) and \( S \) by \( F = \{ p \in C : Tp = p = Sp \} \).

**Lemma 2.1** Let \( C \) be a nonempty closed convex subset of a CAT(0) space \( X \) and \( T, S : C \to C \) be two uniformly \( L_1 \) and \( L_2 \)-Lipschitzian and total asymptotically quasi-nonexpansive mappings and \( L = \max \{L_1, L_2\} \). Let \( \{x_n\} \) be defined by iterative procedure (1.1.12) with \( F \neq \emptyset \). If the following conditions are satisfied:

a) \( \sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} a_n < \infty; \)

b) there exists a constant \( M^* > 0 \) such that \( \mathcal{C}(r) \leq M^*r, \ r \geq 0; \)

c) \( \{b_n\} \) is the sequence in \([0,1]\):

d) \( \sum_{n=1}^{\infty} \sup \{d(z, T^n z) : z \in B \} < \infty \) for each bounded subset \( B \) of \( C \).
e) there exists constants $b, c \in (0, 1)$ with $0 < b(1 - c) \leq \frac{1}{2}$ such that $\{a_n\} \subset [b, c]$.

then

(i) $\lim_{n \to \infty} d(x_n, q)$ exists for all $q \in F$.

(ii) $\lim_{n \to \infty} d(x_n, T^n x_n) = 0 = \lim_{n \to \infty} d(x_n, S^n x_n)$.

**Proof.** Let $q \in F$. Then by Lemma 1.3,

$$d(y_n, q) = d((1 - b_n)x_n \oplus b_n T^n x_n), q)$$

$$\leq (1 - b_n)d(x_n, q) + b_n d(T^n x_n, q)$$

$$\leq (1 - b_n)d(x_n, q) + b_n \left\{d(x_n, q) + \nu_n \zeta \left(d(x_n, q)\right) + \mu_n\right\}$$

$$\leq (1 + b_n \nu_n M^*)d(x_n, q) + b_n \mu_n$$

$$\leq (1 + \nu_n M^*)d(x_n, q) + \mu_n$$

Now, using (2.1.1), we get

$$d(x_{n+1}, q) = d((1 - a_n)T^n x_n \oplus a_n S^n y_n), q)$$

$$\leq (1 - a_n)d(T^n x_n, q) + a_n d(S^n y_n, q)$$

$$\leq (1 - a_n) \left\{d(x_n, q) + \nu_n \zeta \left(d(x_n, q)\right) + \mu_n\right\} + a_n Ld(y_n, q)$$

$$\leq (1 - a_n) \left\{d(x_n, q) + \nu_n \zeta \left(d(x_n, q)\right) + \mu_n\right\} + a_n L \left\{(1 + \nu_n M^*)d(x_n, q) + \mu_n\right\}$$

$$\leq (1 - a_n) \left\{(1 + \nu_n M^*)d(x_n, q) + \mu_n\right\} + a_n L \left\{(1 + \nu_n M^*)d(x_n, q) + \mu_n\right\}$$

$$\leq \left\{(1 - a_n)(1 + \nu_n M^*) + a_n L(1 + \nu_n M^*)\right\}d(x_n, q) +\left(1 + a_n(L-1)\right) \mu_n$$

$$\leq \left\{(1 - a_n)(L-1) + \nu_n M^*(1 + a_n(L-1))\right\}d(x_n, q) +\left(1 + a_n(L-1)\right) \mu_n.$$

Thus, by Lemma 1.12 and condition(a), $\lim_{n \to \infty} d(x_n, q)$ exists for all $q \in F$.

Let $\lim_{n \to \infty} d(x_n, q) = c$. (2.1.1)

Since,

$$d\left(S^n y_n, q\right) \leq d(y_n, q) + \nu_n \zeta \left(d(y_n, q)\right) + \mu_n$$

$$\leq (1 + \nu_n M^*)d(y_n, q) + \mu_n$$

$$\leq (1 + \nu_n M^*) \left\{(1 + \nu_n M^*)d(x_n, q) + \mu_n\right\} + \mu_n$$

$$\leq (1 + \nu_n M^*)(1 + \nu_n M^*)d(x_n, q) + (2 + \nu_n M^*) \mu_n.$$


Using, (2.1.1), we have
\[ \limsup_{n \to \infty} d(S^n y_n, q) \leq c. \]  
(2.1.2)

Similarly, we obtain
\[ \limsup_{n \to \infty} d(T^n x_n, q) \leq c. \]  
(2.1.3)

In addition,
\[ c = \lim_{n \to \infty} d(x_{n+1}, q) = \lim_{n \to \infty} d((1 - a_n)T^n x_n \oplus a_n S^n y_n), q). \]

With the help of (2.1.2), (2.1.3) and Lemma 1.13, we get
\[ \lim_{n \to \infty} d(T^n x_n, S^n y_n) = 0. \]  
(2.1.4)

On the other hand, since
\[ d(x_{n+1}, T^n x_n) = d((1 - a_n)T^n x_n \oplus a_n S^n y_n), T^n x_n) \leq a_n d(S^n y_n, T^n x_n). \]

From (2.1.4),
\[ \lim_{n \to \infty} d(x_{n+1}, T^n x_n) = 0. \]  
(2.1.5)

Thus, using condition (d), we have
\[ \lim_{n \to \infty} d(x_n, T^n x_n) = 0. \]  
(2.1.6)

Hence, from (2.1.5) and (2.1.6), we get
\[ \lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \]  
(2.1.7)

Now, using Lemma 1.3,
\[ d(y_n, x_n) = d((1 - b_n)x_n \oplus b_n T^n x_n, x_n) \leq (1 - b_n)d(x_n, x_n) + b_n d(T^n x_n, x_n). \]

Using (2.1.6), we have
\[ \lim_{n \to \infty} d(y_n, x_n) = 0. \]  
(2.1.8)

Also,
\[ d(x_{n+1}, y_n) \leq d(x_{n+1}, x_n) + d(y_n, x_n) \]

which using (2.1.7) and (2.1.8) gives
\[ \lim_{n \to \infty} d(x_{n+1}, y_n) = 0. \]  
(2.1.9)

Now,
\[ d(x_{n+1}, S^n y_n) \leq d(x_{n+1}, x_n) + d(x_n, T^n x_n) + d(T^n x_n, S^n y_n). \]
By (2.1.4), (2.1.6) and (2.1.7), we obtain
\[ \lim_{n \to \infty} d(x_n, S^n x_n) = 0. \] (2.1.10)
Thus,
\[ d(x_n, S^n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, S^n x_{n+1}) + d(S^n y_n, S^n x_n) \]
\[ \leq d(x_n, x_{n+1}) + d(x_{n+1}, S^n y_n) + Ld(y_n, x_n) \]
gives by (2.1.7), (2.1.8) and (2.1.10) that
\[ \lim_{n \to \infty} d(x_n, S^n x_n) = 0. \] (2.1.11)
Then,
\[ d(x_{n+1}, T x_{n+1}) \leq d(x_{n+1}, T^a x_{n+1}) + d(T^a x_{n+1}, x_{n+1}) + d(T^a x_{n+1}, Tx_{n+1}) \]
\[ \leq d(x_{n+1}, T^a x_{n+1}) + d(x_{n+1}, x_{n+1}) + Ld(T^n x_n, x_{n+1}) \]
\[ = d(x_{n+1}, T^a x_{n+1}) + Ld(x_{n+1}, x_{n+1}) + La d(T^n x_n, S^n y_n). \]
It follows from (2.1.4), (2.1.6) and (2.1.7) that
\[ \lim_{n \to \infty} d(x_n, T x_n) = 0. \]
Finally,
\[ d(x_{n+1}, S x_{n+1}) \leq d(x_{n+1}, S^{a+1} x_{n+1}) + d(S^{a+1} x_{n+1}, x_{n+1}) \]
\[ \leq d(x_{n+1}, S^{a+1} x_{n+1}) + Ld(S^{a+1} x_{n+1}, x_{n+1}) \]
\[ \leq d(x_{n+1}, S^{a+1} x_{n+1}) + L \left[ d(S^{a+1} x_{n+1}, S^n y_n) + d(S^n y_n, x_{n+1}) \right] \]
\[ \leq d(x_{n+1}, S^{a+1} x_{n+1}) + L \left[ d(x_{n+1}, y_n) + d(S^n y_n, y_{n+1}) \right] \]
implies by using (2.1.9), (2.1.10) and (2.1.11) that
\[ \lim_{n \to \infty} d(x_n, S x_n) = 0. \]

**Theorem 2.2.** Let \( X, C, T, S, F, \{a_n\}, \{b_n\} \) and \( \{x_n\} \) be as in Lemma 2.1. If \( I - T \) and \( I - S \) are demiclosed with respect to zero, then \( \{x_n\} \) \( \Delta \)-converges to a point of \( F \).

**Proof.** Let \( q \in F \). Then by Lemma 2.1, \( \lim d(x_n, q) \) exists for all \( q \in F \). Thus \( \{x_n\} \) is bounded. From Lemma 2.1, we have
\[ \lim_{n \to \infty} d(x_n, T x_n) = 0 = \lim_{n \to \infty} d(x_n, S x_n). \]
Firstly, we show that \( \omega_\Delta(x_n) \subseteq F \). Let \( u \in \omega_\Delta(x_n) \), then there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A(\{u_n\}) = \{u\} \). By Lemma 1.8, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta-\lim_{n \to \infty} v_n = v \) for some \( v \in C \). Also \( I - T \) and \( I - S \) are demiclosed with respect to zero, therefore we obtain \( T v = v = S v \), which means that \( v \in F \). By Lemma 2.1, \( \lim d(x_n, v) \) exists. Now, we claim that \( u = v \). Assume on the contrary that \( u \neq v \). Then by the uniqueness of asymptotic centers, we have
\[
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, u) \leq \limsup_{n \to \infty} d(u_n, u) < \limsup_{n \to \infty} d(u_n, v) = \limsup_{n \to \infty} d(x_n, v) = \limsup_{n \to \infty} d(v_n, v),
\]
a contradiction. Thus \( u = v \in F \) and hence \( \omega_\Delta(x_n) \subset F \).

Now, we show that the sequence \( \{x_n\} \) \( \Delta \)-converges to a point of \( F \), we show that \( \omega_\Delta(x_n) \) consists of exactly one point. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \). By Lemma 1.8, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta \)-lim \( x_n = v \) for some \( v \in C \).

Let \( A(\{u_n\}) = \{u\} \) and \( A(\{x_n\}) = \{x\} \). We have already proved that \( u = v \in F \). Finally, we claim that \( x = v \). If is not true, then existence of \( \lim d(x_n, v) \) and uniqueness of asymptotic center imply that
\[
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, x) \leq \limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, v) = \limsup_{n \to \infty} d(v_n, v),
\]
a contradiction. Thus \( x = v \in F \) and hence \( \omega_\Delta(x_n) = \{x\} \). Thus, \( \{x_n\} \) \( \Delta \)-converges to a point of \( F \).

Now, we obtain our strong convergence results for iterative procedure (1.1.19).

**Theorem 2.3.** Let \( X \) be a complete CAT(0) space and \( C, T, S, F, \{a_n\}, \{b_n\}, \{x_n\} \) be as in Lemma 2.1. If \( F \neq \emptyset \), then \( \{x_n\} \) converges strongly to a point of \( F \) if and only if
\[
\liminf_{n \to \infty} d(x_n, F) = 0,
\]
where \( d(x, F) = \inf \{d(x, p) : p \in F\} \).

**Proof.** Necessity is obvious. Conversely, suppose that \( \liminf_{n \to \infty} d(x_n, F) = 0 \). As proved in Lemma 2.1, we have
\[
d(x_{n+1}, p) \leq d(x_n, p)
\]
for all \( p \in F \).

This implies that \( d(x_{n+1}, F) \leq d(x_n, F) \), so that \( \lim d(x_n, F) \) exists. But by hypothesis \( \liminf_{n \to \infty} d(x_n, F) = 0 \). Therefore \( \lim d(x_n, F) = 0 \).

Next, we show that \( \{x_n\} \) is a cauchy sequence in \( C \). Let \( \varepsilon > 0 \) be arbitrarily chosen. Since
\[
\lim_{n \to \infty} d(x_n, F) = 0,
\]
there exists a positive integer \( n_0 \) such that \( d(x_n, F) < \frac{\varepsilon}{4} \) for all \( n \geq n_0 \).

In particular, \( \inf \{d(x_n, p) : p \in F\} < \frac{\varepsilon}{4} \). Thus there must exist \( p^* \in F \) such that
\[
d(x_n, p^*) < \frac{\varepsilon}{2}.
\]

Now for all \( m, n \geq n_0 \), we have
\[
d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(p^*, x_n)
\]
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$$\leq 2d(x_n, p^*) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$ 

Hence $\{x_n\}$ is a cauchy sequence in a closed subset $C$ of a complete CAT(0) space and so it must converge to a point $q$ in $C$. Now, $\lim_{n \to \infty} d(x_n, F) = 0$, gives that $d(q, F) = 0$. Since $F$ is closed, so we have $q \in F$.

Fukhar-ud-din and Khan [6] introduced the concept of condition $(A')$ as follows:

Two mappings $T, S : C \to C$ are said to satisfy the condition $(A')$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either

$$f(d(x, F)) \leq d(x, Tx), \text{ or } f(d(x, F)) \leq d(x, Sx) \quad \text{for all } x \in C,$$

where $d(x, F) = \inf \{d(x, p) : p \in F\}$.

If we take $S = T$ in this condition, then it reduces to condition(A) of Senter and Doston[5].

**Theorem 2.4.** Let $X$ be a complete CAT(0) space and $C, T, S, F, \{a_n\}, \{b_n\}, \{x_n\}$ be as in Lemma 2.1. Let $S, T$ satisfy the condition $(A')$ and $F \neq \phi$. Then $\{x_n\}$ converges strongly to a point of $F$.

**Proof.** We have proved in Lemma 2.1 that $\lim_{n \to \infty} d(x_n, p)$ exists for all $p \in F$. Let this limit be $c$. As proved in Lemma 2.1, we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F$.

This gives that $\inf_{p \in F} d(x_{n+1}, p) \leq \inf_{p \in F} d(x_n, p)$, which means that $d(x_{n+1}, F) \leq d(x_n, F)$, so that $\lim_{n \to \infty} d(x_n, F)$ exists. Again using Lemma 2.1, we have $\lim_{n \to \infty} d(x_n, Tx_n) = 0 = \lim_{n \to \infty} d(x_n, Sx_n)$.

From the condition $(A')$, either

$$\lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} d(x_n, Tx_n) = 0, \text{ or } \lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} d(x_n, Sx_n) = 0.$$ 

Hence $\lim_{n \to \infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, therefore we have $\lim_{n \to \infty} d(x_n, F) = 0$. Now, all the conditions of Theorem 2.3 are satisfied, therefore $\{x_n\}$ converges strongly to a point of $F$.

**Theorem 2.5.** Let $X$ be a complete CAT(0) space and $C, T, S, F, \{a_n\}, \{b_n\}, \{x_n\}$ be as in Lemma 2.1. Suppose that one of $S$ and $T$ is semi-compact. If $F \neq \phi$, $\{x_n\}$ converges strongly to a point of $F$. 
Proof. From Lemma 2.1, we have \( \lim_{n \to \infty} d(x_n, T x_{n_k}) = 0 = \lim_{n \to \infty} d(x_n, S x_{n_k}) \). Let one of \( S \) and \( T \) is semi-compact, then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to some point \( p \in C \). Moreover, using continuity of \( S \) and \( T \), we have

\[
d(p, Tp) = \lim d(x_{n_k}, T x_{n_k}) = 0, \quad d(p, Sp) = \lim d(x_{n_k}, S x_{n_k}) = 0.
\]

Thus, \( p \in F \). Again, from Lemma 2.1, \( \lim d(x_n, p) \) exists for all \( p \in F \). Since \( \lim_{k \to \infty} d(x_{n_k}, p) = 0 \).

Therefore, \( \lim d(x_n, p) = 0 \) and therefore \( \{x_n\} \) converges strongly to a point of \( F \).

Remark. 2.6 In the light of Remark 1.1 it is also noted that our results in CAT(0) spaces can be applied to CAT(\( \kappa \)) spaces with \( \kappa \leq 0 \).

REFERENCES


