Forgotten Polynomial and Forgotten Index for the Line Graphs of Banana Tree Graph, Firecracker Graph and Subdivision Graphs

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Abstract
A chemical graph can be recognized by a numerical number (topological index), algebraic polynomial or any matrix. These numbers and polynomials help to predict many physico-chemical properties of underline chemical compound. In this report

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we compute forgotten polynomial and forgotten index of line graphs of Banana tree graph, Firecracker graph and subdivision graphs.

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1. Introduction

In chemical graph theory, a molecular graph is a simple graph (having no loops and multiple edges) in which atoms and chemical bonds between them are represented by vertices and edges respectively. A graph \( G(V, E) \) with vertex set \( V(G) \) and edge set \( E(G) \) is connected if there exist a connection between any pair of vertices in \( G \). A network is simply a connected graph having no multiple edges and loops. The degree of a vertex is the number of vertices which are connected to that fixed vertex by the edges. In a chemical graph, the degree of any vertex is at most 4. The distance between two vertices \( u \) and \( v \) is denoted as \( d(u, v) = d_G(u, v) \) and is the length of shortest path between \( u \) and \( v \) in graph \( G \). The number of vertices of \( G \), adjacent to a given vertex \( v \), is the “degree” of this vertex, and will be denoted by \( d_v(G) \) or, if misunderstanding is not possible, simply by \( d_v \). The concept of degree is somewhat closely related to the concept of valence in chemistry. For details on bases of graph theory, any standard text such as [17] can be of great help.

Cheminformatics is another emerging field in which quantitative structure-activity and structure-property relationships predict the biological activities and properties of nanomaterial. In these studies, some Physico-chemical properties and topological indices are used to predict bioactivity of the chemical compounds see [3, 5, 11, 16, 18]. Algebraic polynomials have also useful applications in chemistry such as Hosoya polynomial (also called Wiener polynomial) [9] which play a vital role in determining distance-based topological indices. The \( M \)-polynomial [6] introduced in 2015, plays the same role in determining the closed form of many degree-based topological indices [1, 12, 13, 14, 15]. Recently in 2015 Furtula and Gutman [8] introduced another topological index called forgotten index or \( F \)-index

\[
F(G) = \sum_{uv \in E(G)} [(d_u)^2 + (d_v)^2].
\]

For more detail on the “\( F \)-index”, we refer to the articles [2, 4, 7].

The Forgotten polynomial of a graph \( G \) is defined as

\[
F(G, x) = \sum_{uv \in E(G)} x[(d_u)^2 + (d_v)^2].
\]

The line graph \( L(G) \) of a graph \( G \) is the graph each of whose vertex represents an edge of \( G \) and two of its vertices are adjacent if their corresponding edges are adjacent
in $G$. In this article, we compute closed form of forgotten polynomials of the line graphs of Banana tree graph, Firecracker graph and subdivision graphs. The following two lemmas [10] help us in our main results.

**Lemma 1.1.** Let $G$ be a graph with $u, v \in V(G)$ and $e = uv \in E(G)$. Then $de = du + dv - 2$.

**Lemma 1.2.** Let $G$ be a graph of order $p$ and size $q$. Then the line graph $L(G)$ of $G$ is a graph of order $p$ and size $\frac{1}{2}M_1(G) - q$.

## 2. Results and discussions

In this part, we give our main computational results.

### 2.1. Forgotten polynomial and forgotten index for the line graph of Firecracker graph

The Firecracker graph $F_{n,k}$ is the graph obtained by the concatenation of $nk$-strees by linking one leaf from each. The $F_{n,k}$ has order $nk$ and size $nk - 1$. $F_{4,7}$ is shown in the Figure 1.

![Figure 1: The Firecracker graph $F_{4,7}$](image)

**Theorem 2.1.** Let $G$ be the line graph of Firecracker graph. Then the forgotten polynomial and forgotten index of $G$ are

$$
F(G, x) = 2x^{25} + 2x^{9+k^2} + 2x^{9+(k-1)^2} + (n - 4)x^{32} + (2n - 6)x^{16+k^2} + 2(k - 2)x^{(k-1)^2+(k-2)^2} + \frac{nk^2 - 5nk + 6n}{2}x^{2(k-2)^2} + (n - 2)(k - 2)x^{k^2+(k-2)^2},
$$

$$
F(G) = k(n(k^2 - 5k + 6))^2 - 4k(n(k^2 - 5k + 6)) + 2k^3n + (-6n - 6)k^2 + (12n + 6)k + 56n - 136.
$$

**Proof.** The graph $G$ for $n = 4$ and $k = 7$ is shown in Figure 2. By using Lemma 1.1, it is easy to see that the order of $G$ is $nk - 1$ out of which 2 vertices are of degree 3, 2
vertices are of degree $k - 1$, $n - 3$ vertices are of degree 4, $n(k - 2)$ vertices are of degree $k - 2$, and $n - 2$ vertices are of degree $k$. Therefore by using Lemma 1.2, $G$ has size $\frac{nk^2 - 3nk + 8n - 8}{2}$. We partition the size of $G$ into edges of the type $E(du, dv)$, where $uv$ is an edge.

Figure 2: The line graph of Firecracker graph $F_{4, 7}$

In $G$, there are eight types of edges in $G$ based on degrees of end vertices of each edge. The first edge partition $E_1(G)$ contains 2 edges $uv$, where $d_u = 3$, $d_v = 4$. The second edge partition $E_2(G)$ contains 2 edges $uv$, where $d_u = 3$, $d_v = k$. The third edge partition $E_3(G)$ contains 2 edges $uv$, where $d_u = 3$, $d_v = k - 1$. The forth edge partition $E_4(G)$ contains $n-4$ edges $uv$, where $d_u = d_v = 4$ The fifth edge partition $E_5(G)$ contains $2n - 6$ edges $uv$, where $d_u = 4$, $d_v = k$. The sixth edge partition $E_6(G)$ contains $2(k - 2)$ edges $uv$, where $d_u = k - 1$, $d_v = k - 2$. The seventh edge partition $E_7(G)$ contains $\frac{nk^2 - 5nk + 6n}{2}$ edges $uv$, where $d_u = d_v = k - 2$ and the eighth edge partition $E_8(G)$ contains $(n - 2)(k - 2)$ edges $uv$, where $d_u = k - 2$, $d_v = k$.

The Forgotten polynomial of $G$ is

$$F(G, x) = \sum_{uv \in E(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$= \sum_{uv \in E_1(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_2(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_3(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$+ \sum_{uv \in E_4(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_5(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_6(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$+ \sum_{uv \in E_7(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_8(G)} x^{[(d_u)^2 + (d_v)^2]}$$
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\[ E_1(G)x^{25} + |E_2(G)|x^{9+k^2} + |E_3(G)|x^{9+(k-1)^2} + |E_4(G)|x^{32} \]
\[ + |E_5(G)|x^{16+k^2} + |E_6(G)|x^{(k-1)^2+(k-2)^2} + |E_7(G)|x^{2(k-2)^2} \]
\[ + |E_8(G)|x^{k^2+(k-2)^2} = 2x^{25} + 2x^{9+k^2} + 2x^{9+(k-1)^2} + (n-4)x^{32} + (2n-6)x^{16+k^2} \]
\[ + (2(k-2))x^{(k-1)^2+(k-2)^2} + \frac{nk^2-5nk+6n}{2}x^{2(k-2)^2} + (n-2)(k-2)x^{k^2+(k-2)^2}. \]

The forgotten index is

\[ F(G) = \sum_{uv \in E(G)} \left[ (d_u)^2 + (d_v)^2 \right] \]
\[ = \sum_{uv \in E_1(G)} \left[ (d_u)^2 + (d_v)^2 \right] + \sum_{uv \in E_2(G)} \left[ (d_u)^2 + (d_v)^2 \right] \]
\[ + \sum_{uv \in E_3(G)} \left[ (d_u)^2 + (d_v)^2 \right] + \sum_{uv \in E_4(G)} \left[ (d_u)^2 + (d_v)^2 \right] \]
\[ + \sum_{uv \in E_5(G)} \left[ (d_u)^2 + (d_v)^2 \right] + \sum_{uv \in E_6(G)} \left[ (d_u)^2 + (d_v)^2 \right] \]
\[ + \sum_{uv \in E_7(G)} \left[ (d_u)^2 + (d_v)^2 \right] + \sum_{uv \in E_8(G)} \left[ (d_u)^2 + (d_v)^2 \right] \]
\[ = 25|E_1(G)| + (9+k^2)|E_2(G)| + (9+(k-1)^2)|E_3(G)| \]
\[ + 32|E_4(G)| + (16+k^2)|E_5(G)| + ((k-1)^2+(k-2)^2)|E_6(G)| \]
\[ + (2(k-2)^2)|E_7(G)| + (k^2+(k-2)^2)|E_8(G)| \]
\[ = k(n(k^2-5k+6))^2 - 4k(n(k^2-5k+6)) + 2k^3n \]
\[ + (-6n-6)k^2 + (12n+6)k + 56n - 136. \]

\[ \blacksquare \]

2.2. Forgotten polynomial and forgotten index for the line graph of Banana tree graph

The Banana tree graph \( B_{n,k} \) is the graph obtained by connecting one leaf of each of \( n \) copies of a \( k \)-star graph with a single root vertex that is distinct from all the stars. The \( B_{n,k} \) has order \( nk+1 \) and size \( nk \). \( B_{3,5} \) is shown in the Figure 3.

**Theorem 2.2.** Let \( G \) be the line graph of Banana graph. Then the forgotten polynomial
and forgotten index of $G$ are

$$F(G, x) = \frac{n(n-1)}{2}x^{2n^2} + nx^{(k-1)^2 + n^2} + n(k-2)x^{(k-1)^2 + (k-2)^2}$$

$$+ \frac{nk^2 + 6n - 5nk}{2}x^{2(k-2)^2}.$$ 

$$F(G) = n(k^4 - 7k^3 + n^3 + 21k^2 - 29k + 15).$$

**Proof.** The graph $G$ for $n = 3$ and $k = 5$ is shown in Figure 4. By using Lemma 1.1, it is easy to see that the order of $G$ is $nk$ out of which $(k-2)n$ vertices are of degree $k-2$, $n$ vertices are of degree $k-1$ and $n$ vertices are of degree $n$. Therefore by using Lemma 1.2, $G$ has size $\frac{n^2 + 3n + nk^2 - 3nk}{2}$.

There are four types of edges in $G$ based on degrees of end vertices of each edge. The first edge partition $E_1(G)$ contains $\frac{n(n-1)}{2}$ edges $uv$, where $d_u = d_v = n$. The second edge partition $E_2(G)$ contains $n$ edges $uv$, where $d_u = k-1$, $d_v = n$. The third edge partition $E_3(G)$ contains $(k-2)n$ edges $uv$, where $d_u = k-1$, $d_v = k-2$ and the forth edge partition $E_4(G)$ contains $\frac{nk^2 + 6n - 5kn}{2}$ edges $uv$, where $d_u = d_v = k-2$. 
The Forgotten polynomial of $G$ is

$$F(G, x) = \sum_{uv \in E(G)} x^{(d_u)^2 + (d_v)^2}$$

$$= \sum_{uv \in E_1(G)} x^{(d_u)^2 + (d_v)^2} + \sum_{uv \in E_2(G)} x^{(d_u)^2 + (d_v)^2}$$
$$+ \sum_{uv \in E_3(G)} x^{(d_u)^2 + (d_v)^2} + \sum_{uv \in E_4(G)} x^{(d_u)^2 + (d_v)^2}$$

$$= |E_1(G)| x^{2n^2} + |E_2(G)| x^{(k-1)^2 + n^2} + |E_3(G)| x^{(k-1)^2 + (k-2)^2}$$
$$+ |E_4(G)| x^{2(k-2)^2}$$

$$= n(n-1) \frac{x^{2n^2} + nx^{(k-1)^2 + n^2} + n(k-2)x^{(k-1)^2 + (k-2)^2}}{2}$$
$$+ \frac{nk^2 + 6n - 5nk}{2} x^{2(k-2)^2}.$$ 

The forgotten index is

$$F(G) = \sum_{uv \in E(G)} [(d_u)^2 + (d_v)^2]$$

$$= \sum_{uv \in E_1(G)} [(d_u)^2 + (d_v)^2] + \sum_{uv \in E_2(G)} [(d_u)^2 + (d_v)^2]$$
$$+ \sum_{uv \in E_3(G)} [(d_u)^2 + (d_v)^2] + \sum_{uv \in E_4(G)} [(d_u)^2 + (d_v)^2]$$

$$= 2n^2 |E_1(G)| + ((k-1)^2 + n^2) |E_2(G)|$$
$$+ ((k-1)^2 + (k-2)^2) |E_3(G)| + 2(k-2)^2 |E_4(G)|$$

$$= n(k^4 - 7k^3 + n^3 + 21k^2 - 29k + 15).$$

### 2.3. Forgotten polynomial and forgotten index for the line graph of subdivision of friendship graph

Friendship graph $F_n$ is a planer undirected graph with $2n + 1$ vertices and $3n$ edges, can be constructed by joining $n$ copies of cycle graph $C_3$ with a common vertex. The subdivision graph $S(F_n)$ is the graph obtained from $F_n$ by replacing each of its edge by a path of length 2 or equivalently by inserting an additional vertex into each edge of $F_n$. The line graph $L[S(F_n)]$ of a subdivision graph $S(F_n)$ is the graph whose vertices are the edges of $S(F_n)$, two vertices $e$ and $f$ are incident if and only if they have common end vertex in $S(F_n)$. 

■
Theorem 2.3. Let $G$ be of line graph of subdivision graph of friendship graph. Then the forgotten polynomial and forgotten index of $G$ are

$$F(G, x) = 3nx^8 + n(2n - 1)x^{8n^2} + 2n^4x^{4n^2+4},$$

$$F(G) = 16n^4 + 32n.$$

Proof. There are two types of vertices with respect to degree in line graph of subdivision graph of friendship graph $G = L[S(F_n)]$, their degrees are $2n$ and $2$. $2n$ vertices have degree $2n$ and $4n$ vertices have degree $2$. $G = L[S(F_n)]$ contains $6n$ vertices and $2n(n + 2)$ edges. In $G$, there are three types of edges based on degrees of end vertices of each edge. The first edge partition $E_1(G)$ contains $3n$ edges $uv$, where $d_u = d_v = 2$. The second edge partition $E_2(G)$ contains $n(2n - 1)$ edges $uv$, where $d_u = d_v = 2n$. The third edge partition $E_3(G)$ contains $2n$ edges $uv$, where $d_u = 2n$, $d_v = 2$.

The Forgotten polynomial of $G$ is

$$F(G, x) = \sum_{uv \in E(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$= \sum_{uv \in E_1(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_2(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_3(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$= |E_1(G)|x^8 + |E_2(G)|x^{8n^2} + |E_3(G)|x^{4n^2+4}$$

$$= 3nx^8 + n(2n - 1)x^{8n^2} + 2n^4x^{4n^2+4}.$$

The forgotten index is

$$F(G) = \sum_{uv \in E(G)} [(d_u)^2 + (d_v)^2]$$

$$= \sum_{uv \in E_1(G)} [(d_u)^2 + (d_v)^2] + \sum_{uv \in E_2(G)} [(d_u)^2 + (d_v)^2]$$

$$+ \sum_{uv \in E_3(G)} [(d_u)^2 + (d_v)^2]$$

$$= 8|E_1(G)| + 8n^2|E_2(G)| + (4n^2 + 4)|E_3(G)|$$

$$= 16n^4 + 32n.$$

\[\blacksquare\]

2.4. Forgotten polynomial and forgotten index for the line graph of subdivision of star graph

Star graph $S_n$ is the complete bipartite graph $K_{1,n}$, a tree with one internal node and $n$ leaves with $n + 1$ vertices and $n$ edges. The subdivision graph $S(S_n)$ is the graph obtained from $S_n$ by replacing each of its edge by a path of length 2 or equivalently by inserting an
additional vertex into each edge of $S_n$. The line graph $L[S(S_n)]$ of a subdivision graph $S(S_n)$ is the graph whose vertices are the edges of $S(S_n)$.

**Theorem 2.4.** Let $G$ be the Line Graph of subdivision of star graph. Then the forgotten polynomial and forgotten index of $G$ are

$$F(G, x) = (n - 1)x^{1+(n-1)^2} + \frac{n^2 - 3n + 2}{2}x^{2(n-1)^2},$$

$$F(G) = n^4 - 4n^3 + 6n^2 - 3n.$$  

**Proof.** There are two types of vertices with respect to degree in line graph of subdivision graph of star graph $G = L[S(S_n)]$ their degree are $n − 1$ and 1. $n − 1$ vertices have degree $n − 1$ and $n − 1$ vertices have degree 1. $G = L[S(S_n)]$ contains $2(n − 1)$ vertices and $\frac{n^2 − n}{2}$ edges. There are two types of edges in $G$ based on degrees of end vertices of each edge. The first edge partition $E_1(G)$ contains $n − 1$ edges $uv$, where $d_u = 1$, $d_v = n − 1$. The second edge partition $E_2(G)$ contains $\frac{n^2 − 3n + 2}{2}$ edges $uv$, where $d_u = d_v = n − 1$.

The Forgotten polynomial of $G$ is

$$F(G, x) = \sum_{uv \in E(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$= \sum_{uv \in E_1(G)} x^{[(d_u)^2 + (d_v)^2]} + \sum_{uv \in E_2(G)} x^{[(d_u)^2 + (d_v)^2]}$$

$$= |E_1(G)|x^{1+(n-1)^2} + |E_2(G)|x^{2(n-1)^2}$$

$$= (n - 1)x^{1+(n-1)^2} + \frac{n^2 - 3n + 2}{2}x^{2(n-1)^2}.$$  

The forgotten index is

$$F(G) = \sum_{uv \in E(G)} [(d_u)^2 + (d_v)^2]$$

$$= \sum_{uv \in E_1(G)} [(d_u)^2 + (d_v)^2] + \sum_{uv \in E_2(G)} [(d_u)^2 + (d_v)^2]$$

$$= (1 + (n - 1)^2)|E_1(G)| + 2(n - 1)^2|E_2(G)|$$

$$= n^4 - 4n^3 + 6n^2 - 3n.$$  

\[\square\]

**References**


