On the Coefficient of Entire Functions of Generalised Order

Manisha Sharma*

1Department of Applied Sciences, Krishna Engineering College, Ghaziabad, Uttar Pradesh-201007, India.

Abstract

In this paper, we define the generalised order \( \rho(\alpha, \beta, f) = \lim_{r \to \infty} \sup \frac{\log \log M(r)}{\log r} \)

and the generalised lower order \( \lambda(\alpha, \beta, f) = \lim_{r \to \infty} \inf \frac{\log \log M(r)}{\log r} \) and an entire function \( f \) and extend some known results on entire function, “of infinite order”. Our definition of \( \rho(\alpha, \beta, f) \) is essentially due to Seremata theorem

\[
\frac{\partial \delta(x,r)}{\partial (\log x)} = O(1) \text{as } x \to \infty
\]

For every \( c > 0 \),

Then

\[
\rho(\alpha, \beta, f) = \lim_{n \to \infty} \rho \left( \frac{n}{\log \left( \frac{1}{a_n} \right)} \right)
\]

See also G.VALIRON” Lectures on the general theory of integral functions “

Keywords: \( M(r, f) = \max_{|z|=r} |f(z)|, \mu(r, f) = \max_{n \geq 0} |a_n r^n|, \nu(r, f) = \max_{n \geq 0} \{ n \mid |a_n r^n| \} \)

As usual, we call \( M(r, f), \mu(r, f) \) and \( \nu(r, f) \), maximum modulus, maximum term and rank of the maximum term respectively for entire function \( f(z) \).

* Corresponding Author Dr. Manisha Sharma
Dept. of Maths, Institute Name: Krishna Engineering College Ghaziabad
Address: 95, loni road, near air force Hindon station , Ghaziabad
Email Id.- khushi.mani@gmail.com
1. INTRODUCTION

1) Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be the entire function of the fast growth. We set

\[
M(r, f) = \max_{|z|=r} |f(z)|, \quad \mu(r, f) = \max_{n \geq 0} \left| a_n r^n \right|, \quad \nu(r, f) = \max_{n \geq 0} \left\{ \mu \mid |a_n r^n| \right\}.
\]

As usual, we call \( M(r, f), \mu(r, f) \) and \( \nu(r, f) \), maximum modulus, maximum term and rank of the maximum term respectively for entire function \( f(z) \).

Then,

\[
\lim_{r \to \infty} \sup \frac{\log M(r)}{\log r} = \rho
\]

It is known that ([1], [2])

\[
(1.1) \quad \rho = \lim_{n \to \infty} \sup \frac{n \log n}{\log |a_n|}^{-1}
\]

\[
(1.2) \quad \rho \leq \lim_{n \to \infty} \sup \frac{\log n}{\log \left| \frac{a_n}{a_{n+1}} \right|}
\]

Where equality holds everywhere (1.2) if \( \left| \frac{a_n}{a_{n+1}} \right| \) forms a non-decreasing function of \( n \) for \( n > n_0 \) (\( n_0 \) is some fixed integer not necessarily same at each occurrence). These results have been generalised by considering the ratio \( \frac{\log M(r)}{\log r} \) \( (g \geq 2) \) [see 4,5,6,7].

Let \( L_0 \) denote the class of function L satisfying the following condition (H, i) and (H, ii).

(H, i) \( h(x) \) is defined on \([a, \infty]\) and is positive strictly increasing, differentiable and tends to \( \infty \) as \( x \to \infty \).

(H, ii) \( \lim_{x \to \infty} \frac{\log \left( \frac{L(x)}{L(cx)} \right)}{L(x)} = 1 \)

For every function \( \psi(x) \) such that \( \psi(x) \to \infty \) as \( x \to \infty \).

Let \( \Delta \) denote the class of function L satisfying the condition (H, i) and (H, iii)

\[
(\text{H, iii}) \quad \lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1 \quad \text{For every } c > 0
\]

Let \( f(z) \) be any entire function and suppose that \( \alpha(x) \in \Delta, \beta(x) \in L_0 \). Write

\[
\rho(\alpha, \beta, f) = \lim_{r \to \infty} \left\{ \sup \alpha(\log M(r, f)) \right\}
\]

\[
\lambda(\alpha, \beta, f) = \lim_{r \to \infty} \left\{ \inf \frac{\beta(\log r)}{\beta(\log r)} \right\}
\]
The $\rho(\alpha, \beta, f)$ is called the generalised order of $f$ and $\lambda(\alpha, \beta, f)$ the generalised lower order of $f$. If we take $\alpha(x) = \log x$ and $\beta(x) = x$, we get the familiar definition of order [3, p.8, 11, p32-34] and lower order [1].

Shah [8] has proved the following theorem:

**THEOREM A:** Let $f(z)$ be entire, set $F(x, c) = \beta^{-1}(c, \alpha(x)), F(x, 1) = F(x)$. If for some function $\psi(x) \to \infty$ (however slowly) as $x \to \infty$

$$\frac{\beta [x \psi(x)]}{\beta e^x} \to 0 \text{ as } x \to \infty. \quad (1.3)$$

And if

$$\frac{dF(x)}{d(\log x)} = O(1) \text{ as } x \to \infty. \quad (1.4)$$

Then,

$$\rho(\alpha, \beta, f) = \lim_{r \to \infty} \sup \frac{\alpha(\log \mu(r))}{\beta(\log r)} = \lim_{r \to \infty} \sup \frac{\alpha(\nu(r))}{\beta(\log r)}$$

$$\lambda(\alpha, \beta, f) = \lim_{r \to \infty} \inf \frac{\alpha(\log \mu(r))}{\beta(\log r)} = \lim_{r \to \infty} \inf \frac{\alpha(\nu(r))}{\beta(\log r)}$$

Seremeta [7] has shown

**Theorem B:** If

$$\frac{dF(x, c)}{d(\log x)} = O(1) \text{ as } x \to \infty \quad (1.5)$$

For every $c>0$,

Then

$$\rho(\alpha, \beta, f) = \lim_{n \to \infty} \frac{\alpha(n)}{\rho \left( \frac{1}{n \log \frac{1}{|a_n|}} \right)}$$

2. **APPROACH TO GENERALIZED THEOREM AND RESULT**

Theorem: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of generalised order $\rho(\alpha, \beta, f)$ and

$$\left| \frac{c_{n}}{a_{n+1}} \right|$$

is ultimately non-decreasing function of $n$ and (1.3), (1.4), (1.5) holds, then

$$\rho(\alpha, \beta, f) = \lim_{n \to \infty} \sup \frac{\alpha(n)}{\beta \left( \log \left| \frac{c_n}{a_{n+1}} \right| \right)}$$
Remark: (1) If we take $\alpha(x) = \log x$ and $\beta(x) = x$, we get the familiar result of Shah [2].

(2) If we take $\alpha(x) = \log^{q-1} x$ and $\beta(x) = x$, we get the result of S.K. Bajpai [12].

Proof: Let

$$\mu = \lim_{n \to \infty} \sup \frac{\alpha(n)}{\beta \left( \log \frac{\alpha_n}{\alpha_{n+1}} \right)}$$

Then for given $\varepsilon > 0$ and large $n \geq n_0(\varepsilon)$, we have

$$\beta \left( \log \left| \frac{\alpha_n}{\alpha_{n+1}} \right| \right) > \frac{\alpha(n)}{\mu + \varepsilon}$$

$$\log \left| \frac{\alpha_n}{\alpha_{n+1}} \right| > \frac{1}{\mu + \varepsilon}$$

$$\sum_{m=n_0}^{n} \log \left| \frac{\alpha_m}{\alpha_{m+1}} \right| > \sum_{m=n_0}^{n} \frac{1}{\mu + \varepsilon}$$

Now using Stieltjes integral method, we have

$$\log \left| \frac{a_{n_0}}{a_{n+1}} \right| > \int_{n_0}^{n} \beta^{-1} \left( \frac{\alpha(t)}{\mu + \varepsilon} \right) dt$$

$$\left[ t \beta^{-1} \left( \frac{\alpha(t)}{\mu + \varepsilon} \right) \right]_{n_0}^{n} - \int_{n_0}^{n} \beta^{-1} \left( \frac{\alpha(t)}{\mu + \varepsilon} \right) dt$$

Since,

$$\int_{n_0}^{n} [t] dt \left\{ \beta^{-1} \left( \frac{\alpha(t)}{\mu + \varepsilon} \right) \right\} = o \left( \frac{1}{\mu + \varepsilon} \right)$$

We have,

$$\log \left| \frac{a_{n_0}}{a_{n+1}} \right| > n \beta^{-1} \left( \frac{\alpha(n)}{\mu + \varepsilon} \right) \{1 + O(1)\}$$

$$\log \left| \frac{1}{a_{n+1}} \right| > n \beta^{-1} \left( \frac{\alpha(n)}{\mu + \varepsilon} \right) \{1 + O(1)\}$$

$$\frac{1}{n} \log \left| \frac{1}{a_{n+1}} \right| > \beta^{-1} \left( \frac{\alpha(n)}{\mu + \varepsilon} \right) \{1 + O(1)\}$$

$$\mu + \varepsilon > \frac{\alpha(n)}{\beta \left( \frac{1}{n} \log \left| \frac{1}{a_n} \right| \right)} \{1 + O(1)\}$$
On the Coefficient of Entire Functions of Generalised Order

2531

Conversely, for given \( \varepsilon > 0 \) and for all \( n \geq n(\varepsilon) \), we have,

\[
(\rho_0 + \varepsilon)\beta\left(\frac{1}{n} \log \left| \frac{1}{a_n} \right| \right) > \alpha(n)
\]

Since, \( \left| \frac{a_n}{a_{n+1}} \right| \) is non-decreasing function of \( n \), we have,

\[
\left| \frac{a_n}{a_{n+1}} \right|^{n-n_0} > \left| \frac{a_{n_0}}{a_{n+1}} \right| \left| \frac{a_{n_0+1}}{a_{n+1}} \right| \cdots \cdots \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{a_n}{a_{n+1}} \right|
\]

Hence, from (1.3) and (1.4), we obtain,

\[
(n - n_0) \log \left| \frac{a_n}{a_{n+1}} \right| > \log |a_{n_0}| + \log |a_{n+1}|^{-1} > (n + 1)\beta^{-1}\left(\frac{\alpha(n+1)}{\rho_0 + \varepsilon}\right)
\]

\[
(n - n_0) \log \left| \frac{a_n}{a_{n+1}} \right| > \beta^{-1}\left(\frac{\alpha(n+1)}{\rho_0 + \varepsilon}\right) > \beta^{-1}\left(\frac{\alpha(n)}{\rho_0 + \varepsilon}\right)
\]

\[
(1+O(1)) \log \left| \frac{a_n}{a_{n+1}} \right| > \beta^{-1}\left(\frac{\alpha(n)}{\rho_0 + \varepsilon}\right)
\]

\[
(\rho_0 + \varepsilon)\beta\left(\frac{\alpha(n)}{\beta \log \left| \frac{a_{n+1}}{a_{n+1}} \right|} \right) > \rho_0 \geq \mu
\]

This completes the proof of the theorem.

3. CONCLUSION

Maximum term of the entire function \( \mu(r, f) \) is equal to the order \( \rho \). “The author would like to express her thanks to “Dr. J.P. Singh” for his kind encouragement and helpful suggestions”.
REFERENCE