Realization Problems in 1-movable Independent and 1-movable Doubly Connected Domination in Graphs

Renario G. Hinampas, Jr.
College of Teacher Education,
Bohol Island State University-Main Campus,
CPG North Avenue, 6300 Tagbilaran City,
Bohol, Philippines.

Abstract

This paper shows that every pair of positive integers is realizable as independent domination number and 1-movable independent domination number and realizable as doubly connected domination number and 1-movable doubly connected domination number.

AMS subject classification: 05C69.
Keywords: Domination, independent domination, doubly connected domination, 1-movable domination, 1-movable independent domination, 1-movable doubly connected domination.

1. Introduction

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighborhood of $S$ is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset $S$ of $V(G)$ is an independent set of $G$ if for every two elements $x, y \in S, xy \notin E(G)$. The independence number of $G$, denoted by $\beta(G)$, is the largest cardinality of an independent set in $G$. An independent set $S$ of $G$ is called a maximum independent set if $|S| = \beta(G)$. A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. It is an independent
dominating set of $G$ if $S$ is both an independent set and a dominating set. It is a connected dominating set of $G$ if the subgraph $\langle S \rangle$ induced by $S$ is connected and it is a doubly connected dominating set of $G$ if $S$ is a connected dominating set and the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The domination number $\gamma(G)$ (resp. independent domination number $\gamma_i(G)$, connected domination number $\gamma_c(G)$ and doubly connected domination number $\gamma_{cc}(G)$) is the smallest cardinality of a dominating (resp. independent dominating, connected dominating and doubly connected dominating) set of $G$.

A nonempty set $S \subseteq V(G)$ is a 1-movable dominating (resp. 1-movable independent dominating) set of $G$ if $S$ is a dominating (resp. independent dominating) set of $G$ and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of $G$ or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating (resp. independent dominating) set of $G$. It is a 1-movable doubly connected dominating set of $G$ if (i) $S = V(G)$ and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of $G$ and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of $G$ or (ii) $S$ is a doubly connected dominating set of $G$ and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of $G$ for some $u \in (V(G) \setminus S) \cap N_G(v)$. The 1-movable domination number $\gamma_1(G)$ (resp. 1-movable independent domination number $\gamma_{mi}(G)$ and 1-movable doubly connected domination number $\gamma_{mcc}(G)$) is the smallest cardinality of a 1-movable dominating (resp. 1-movable independent dominating and 1-movable doubly connected dominating) set of $G$.

Domination and some of its variants were investigated in [1], [3] and [4]. Moreover, the 1-movable domination and its variants were investigated in [2], [5], [6], [7], [8], [9], [10], [11] and [12].

The next section presents some realization problems in 1-movable independent domination and 1-movable doubly connected domination in graphs and that the difference of two parameters can be made arbitrarily large.

2. Results

**Theorem 2.1.** For every pair of positive integers $a$ and $b$, where $1 \leq a \leq b$, there exists a connected graph $G$ such that $\gamma_i(G) = a$ and $\gamma_{mi}(G) = b$.

**Proof.** Consider the following cases:

**Case 1:** $1 = a = b$

Then take $G = K_n$, where $n \geq 2$ so that $\gamma_i(G) = 1 = \gamma_{mi}(G)$.

**Case 2:** $1 = a < b$

Then consider the graph $G = G_1$ as shown in Figure 1.

The set $S_1 = \{x\}$ is a $\gamma_1$-set of $G$. Hence, $\gamma_i(G) = |S_1| = 1$. Now, if $S_2$ is a $\gamma_{mi}^1$-set of $G$, then either $u_i \in S_2$ or $v_i \in S_2$ for each $i = 1, 2, 3, \ldots, b$. Suppose that $u_i \in S_2$. Then, $S_2 = \{u_1, u_2, \ldots, u_{b-1}, u_b\}$ is a $\gamma_{mi}^1$-set of $G$. Hence, $\gamma_{mi}(G) = |S_2| = b$.

**Case 3:** $1 < a = b$
Realization Problems in 1-movable Independent

Then consider the graph \( G = G_2 \) as shown in Figure 2.

![Figure 2: A graph with \( 1 < \gamma_i(G) = a = \gamma_{mi}^1(G) \)](image)

The set \( S_3 = \{y_1, y_2, \ldots, y_{a-1}, y_a\} \) is a \( \gamma_i \)-set of \( G \). Hence, \( \gamma_i(G) = |S_3| = a \).

Moreover, for each \( i = 1, 2, \ldots, a \), the set \( (S_3 \setminus \{y_i\}) \cup \{x_i\} \) is an independent dominating set of \( G \). Hence, \( S_3 \) is a \( \gamma_{mi}^1 \)-set of \( G \). Thus, \( \gamma_{mi}^1(G) = |S_3| = a \).

**Case 4: 1 < a < b**

Then consider the graph \( G = G_3 \) as shown in Figure 3.

![Figure 3: A graph with \( 1 < \gamma_i(G) = a < b = \gamma_{mi}^1(G) \)](image)

If \( S_4 \) is a \( \gamma_i \)-set of \( G \), then \( x_a \in S_4 \) and \( y_i \in S_4 \) for each \( i = 1, 2, \ldots, a - 1 \). Hence, \( S_4 = \{y_1, y_2, \ldots, y_{a-1}\} \cup \{x_a\} \) and \( \gamma_i(G) = |S_4| = a \). Now, if \( S_5 \) is a \( \gamma_{mi}^1 \)-set of \( G \), then \( y_i \in S_5 \) for each \( i = 1, 2, 3, \ldots, a - 1 \) and either \( u_j \in S_5 \) or \( v_j \in S_5 \) for each \( j = 1, 2, \ldots, b - a + 1 \). Suppose that \( u_j \in S_5 \) for each \( j = 1, 2, \ldots, b - a + 1 \). Then, \( S_5 = \{y_1, y_2, \ldots, y_{a-1}\} \cup \{u_1, u_2, \ldots, u_{b-a+1}\} \) is a \( \gamma_{mi}^1 \)-set of \( G \). Hence, \( \gamma_{mi}^1(G) = |S_5| = (a - 1) + (b - a + 1) = b \). ■
Corollary 2.2. The difference $\gamma_{mi}^1 - \gamma_i$ can be made arbitrarily large.

Proof. Let $n \in \mathbb{N}$. By Theorem 2.1, there exists a connected graph $G$ such that $\gamma_i(G) = 1$ and $\gamma_{mi}^1(G) = n + 1$. Thus, $\gamma_{mi}^1(G) - \gamma_i(G) = (n + 1) - 1 = n$. \qed

Note that for every connected nontrivial graph $G$, $\gamma_{cc}(G) \leq \gamma_{mcc}^1(G)$ and $\gamma_{mcc}^1(K_n) = 1$ for all $n \geq 2$. Now, we show that every pair of positive integers is realizable as doubly connected domination number and 1-movable doubly connected domination number.

Theorem 2.3. For every pair of positive integers $a$ and $b$, where $1 \leq a \leq b$ and $b > 3a$, there exists a connected graph $G$ without cut-vertices such that $\gamma_{cc}(G) = a$ and $\gamma_{mcc}^1(G) = b$.

Proof. Consider the following cases:

Case 1: $1 = a = b$
Then take $G = K_n$, where $n \geq 2$ so that $\gamma_{cc}(G) = 1 = \gamma_{mcc}^1(G)$.

Case 2: $1 = a < b$
Then consider the graph $G = G_4$ as shown in Figure 4.

\[ \gamma_{cc}(G) = 1 < b = \gamma_{mcc}^1(G) \]

The set $S_1 = \{x\}$ is a $\gamma_{cc}$-set of $G$. Hence, $\gamma_{cc}(G) = |S_1| = 1$. Note that $G_4 \cong K_1 + P_b$. Hence, it follows that $\gamma_{mcc}^1(G) = \gamma_{mcc}^1(K_1 + P_b) = |P_b| = b$.

Case 3: $1 < a = b$
Then consider the graph $G = G_5$ as shown in Figure 5.

\[ \gamma_{cc}(G) = a = \gamma_{mcc}^1(G) \]

The set $S_3 = \{x_1, x_2, \ldots, x_{a-1}, x_a\}$ is a $\gamma_{cc}$-set of $G$. Hence, $\gamma_{cc}(G) = |S_3| = a$. The vertex $y_i$ dominates $z_i$ for each $i = 1, 2, \ldots, a$. Moreover, $y_1x_2 \in E(G)$, $x_{i-1}y_i \in E(G)$ and $y_ix_{i+1} \in E(G)$ for each $i = 2, 3, \ldots, a - 1$. Hence, it follows that
Realization Problems in 1-movable Independent

$(S_3 \setminus \{x_i\}) \cup \{y_i\}$ is a dominating set in $G$ and $(V(G) \setminus [(S_3 \setminus \{x_i\}) \cup \{y_i\}])$ is connected for each $i = 1, 2, \ldots, a$. Hence, $S_3$ is a $\gamma_{mcc}^1$-set of $G$. Therefore, $\gamma_{mcc}^1(G) = |S_3| = a$.

**Case 4:** $1 < a < b$

Then consider the graph $G = G_6$ as shown in Figure 6.

![Figure 6](image_url)

The set $S_4 = \{x_1, x_2, \ldots, x_a\}$ is a $\gamma_{cc}$-set of $G$. Hence, $\gamma_{cc}(G) = |S_4| = a$.

Suppose that $S_5$ is a $\gamma_{mcc}^1$-set of $G$. Since the graph $G_6$ has a subgraph isomorphic to the graph $G_5$, it follows that $u_a \in S_5$, $v_a \in S_5$ and $z_i \in S_5$ for each $i = 1, 2, \ldots, b - 3a$. Also, since the graph $G_6$ has a subgraph isomorphic to the graph $G_4$, it follows that $x_i \in S_5$ for each $i = 1, 2, \ldots, a - 1$. Suppose further that $u_i \notin S_5$ for each $i = 1, 2, \ldots, a - 1$. Then $u_iz_1 \notin E((V(G) \setminus (S_5 \setminus \{z_1\})))$. Thus, $(V(G) \setminus (S_5 \setminus \{z_1\}))$ is disconnected. Hence, $S_5 \setminus \{z_1\}$ is not a doubly connected dominating set of $G$. Moreover, the same conclusion is obtained whenever $v_i \notin S_5$. A contradiction to the assumption. Thus, $u_i \in S_5$ and $v_i \in S_5$ for each $i = 1, 2, \ldots, a - 1$. Therefore, $S_5 = \{u_1, u_2, \ldots, u_a\} \cup \{v_1, v_2, \ldots, v_a\} \cup \{x_1, x_2, \ldots, x_a\} \cup \{z_1, z_2, \ldots, z_{b-3a}\}$ is a $\gamma_{mcc}$-set of $G$ and $\gamma_{mcc}^1(G) = |S_5| = 3a + (b - 3a) = b$.

**Corollary 2.4.** The difference $\gamma_{mcc}^1 - \gamma_{cc}$ can be made arbitrarily large.

**Proof.** Let $n \in \mathbb{N}$. By Theorem 2.3, there exists a connected graph $G$ such that $\gamma_{cc}(G) = 1$ and $\gamma_{mcc}^1(G) = n + 1$. Thus, $\gamma_{mcc}^1(G) - \gamma_{cc}(G) = (n + 1) - 1 = n$.

**References**


