On Intuitionistic Fuzzy $\beta$ Generalized $\alpha$ Closed Sets

Gomathi M$_1$ and Jayanthi D$_2$

Department of Mathematics, Avinashilingam University,
Coimbatore, Tamil Nadu, India

Abstract

In this paper, we have introduced the notion of intuitionistic fuzzy $\beta$ generalized $\alpha$ closed sets, intuitionistic fuzzy $\beta$ generalized $\alpha$ open sets and investigated their properties and obtained some interesting characterizations.

Keywords: Intuitionistic fuzzy topology, intuitionistic fuzzy $\beta$ closed sets, intuitionistic fuzzy $\beta$ generalized $\alpha$ closed sets, intuitionistic fuzzy $\beta$ generalized $\alpha$ open sets, intuitionistic fuzzy point.

I. INTRODUCTION

The notion of intuitionistic fuzzy sets was introduced by Atanassov [1] in 1986 as a generalization of fuzzy sets by Zadeh[16]. In 1997, Coker[2] introduced the concept of intuitionistic fuzzy topological spaces and he produced many interesting results and theorems. In 2014 Jayanthi[6] introduced the generalized $\beta$ closed set in intuitionistic fuzzy topological spaces and intuitionistic fuzzy $\beta$ generalized closed sets are introduced by Saranya and Jayanthi[12] in 2016. In this paper, we have introduced the notion of intuitionistic fuzzy $\beta$ generalized $\alpha$ closed sets, intuitionistic fuzzy $\beta$ generalized $\alpha$ open sets and investigated their properties and obtained some interesting characterizations.
II. PRELIMINARIES

Definition 2.1: [1] An intuitionistic fuzzy set (IFS for short) \( A \) is an object having the form

\[ A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \]

where the functions \( \mu_A : X \to [0,1] \) and \( \nu_A : X \to [0,1] \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of non-membership (namely \( \nu_A(x) \)) of each element \( x \in X \) to the set \( A \) respectively, and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for each \( x \in X \). Denote by IFS (X), the set of all intuitionistic fuzzy sets in \( X \). An intuitionistic fuzzy set \( A \) in \( X \) is simply denoted by \( A = (x, \mu_A, \nu_A) \) instead of denoting \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \).

Definition 2.2: [1] Let \( A \) and \( B \) be two IFSs of the form \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \) and \( B = \{(x, \mu_B(x), \nu_B(x)) : x \in X\} \). Then,

(a) \( A \subseteq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in X \),

(b) \( A = B \) if and only if \( A \subseteq B \) and \( A \supseteq B \),

(c) \( A^c = \{(x, \nu_A(x), \mu_A(x)) : x \in X\} \),

(d) \( A \cup B = \{(x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)) : x \in X\} \),

(e) \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x)) : x \in X\} \).

The intuitionistic fuzzy sets \( 0^- = (x, 0, 1) \) and \( 1^- = (x, 1, 0) \) are respectively the empty set and the whole set of \( X \).

Definition 2.3: [2] An intuitionistic fuzzy topology (IFT in short) on \( X \) is a family \( \tau \) of IFSs in \( X \) satisfying the following axioms:

(i) \( 0^-, 1^- \in \tau \),

(ii) \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),

(iii) \( \cup G_i \in \tau \) for any family \( \{G_i : i \in J\} \subseteq \tau \).

In this case the pair \( (X, \tau) \) is called the intuitionistic fuzzy topological space (IFTS in short) and any IFS in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS in short) in \( X \). The complement \( A^c \) of an IFOS \( A \) in an IFTS \( (X, \tau) \) is called an intuitionistic fuzzy closed set (IFCS in short) in \( X \).

Definition 2.4: [4] An IFS \( A = (x, \mu_A, \nu_A) \) in an IFTS \( (X, \tau) \) is said to be an

(i) intuitionistic fuzzy \( \beta \) closed set (IF\( \beta \)CS in short) if \( \text{int(cl(int(A)))) \subseteq A \)

(ii) intuitionistic fuzzy \( \beta \) open set (IF\( \beta \)OS in short) if \( A \subseteq \text{cl(int(cl(A)))) \)
**Definition 2.5:** [15] An IFS $A = (x, \mu_A, \nu_A)$ in an IFTS $(X, \tau)$ is said to be an

1. intuitionistic fuzzy semi pre closed set (IFSPCS in short) if $\text{int}(B) \subseteq A \subseteq B$
2. intuitionistic fuzzy semi pre closed set (IFSPOS in short) if $B \subseteq A \subseteq \text{cl}(B)$

**Remark 2.6:** [7] Every IFSPCS is an IF$\beta$CS in $(X, \tau)$ but not conversely in general.

**Definition 2.7:** [6] Let $(X, \tau)$ be an IFTS and $A = (x, \mu_A, \nu_A)$ be an IFS in $X$. Then the intuitionistic fuzzy $\beta$ interior and intuitionistic fuzzy $\beta$ closure are defined by

$$\beta\text{int}(A) = \bigcup \{ G/ G \text{ is an IF$\beta$OS in } X \text{ and } G \subseteq A \},$$

$$\beta\text{cl}(A) = \bigcap \{ K/ K \text{ is an IF$\beta$CS in } X \text{ and } A \subseteq K \}.$$  

Note that for any IFS $A$ in $(X, \tau)$, we have $\beta\text{cl}(A^c) = (\beta\text{int}(A))^c$ and $\beta\text{int}(A^c) = (\beta\text{cl}(A))^c$.

**Result 2.8:** [12] Let $A$ be an IFS in $(X, \tau)$, then

1. $\beta\text{cl}(A) \supseteq A \cup \text{int}(\text{cl}(\text{int}(A)))$
2. $\beta\text{int}(A) \subseteq A \cap \text{cl}(\text{int}(\text{cl}(A)))$

**Definition 2.9:** [3] An intuitionistic fuzzy point (IFP in short), written as $p_{(\alpha, \beta)}$, is defined to be an IFS of $X$ given by

$$p_{(\alpha, \beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0, 1) & \text{otherwise}. \end{cases}$$

An intuitionistic fuzzy point $p_{(\alpha, \beta)}$ is said to belong to a set $A$ if $\alpha \leq \mu_A$ and $\beta \geq \nu_A$.

**Definition 2.10:** [13] Two IFSs are said to be q-coincident ($A_q B$ in short) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

**Definition 2.11:** [13] Two IFSs $A$ and $B$ are said to be not q-coincident ($A_{\bar{q}} B$ in short) if and only if $A \subseteq B^c$.

**Definition 2.12:** [3] Let $(X, \tau)$, be an IFTS and $A = (x, \mu_A, \nu_A)$ be an IFS in $X$. Then intuitionistic fuzzy kernel of $A$ is the intersection of all IFOSs containing $A$. 

---

*On Intuitionistic Fuzzy $\beta$ Generalized a Closed Sets*
III. INTUITIONISTIC FUZZY $\beta$ GENERALIZED $\alpha$ CLOSED SETS

In this section we have introduced intuitionistic fuzzy $\beta$ generalized $\alpha$ closed sets and studied some of their properties.

**Definition 3.1:** An IFS $A$ in an IFTS $(X, \tau)$ is said to be an intuitionistic fuzzy $\beta$ generalized $\alpha$ closed set (IF$\beta$G$\alpha$CS for short) if $\beta\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IF$\alpha$OS in $(X, \tau)$. The complement $A^c$ of an IF$\beta$G$\alpha$CS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy $\beta$ generalized $\alpha$ open set (IF$\beta$G$\alpha$OS for short) in $X$. The family of all IF$\beta$G$\alpha$CSs of an IFTS $(X, \tau)$ is denoted by IF$\beta$G$\alpha$C$(X)$.

**Example 3.2:** Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b) \rangle$ and $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $\tau = \{0, G_1, G_2, 1\}$ is an IFT on $X$. Then IF$\alpha$O$(X) = \{0, G_1, G_2, 1\}$ and IF$\beta$C$(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ 0.4 \leq \mu_a \leq 0.5$ whenever $0.3 \leq \mu_b \leq 0.5, 0.5 \leq \nu_a \leq 0.6, 0.5 \leq \nu_b \leq 0.7, 0 \leq \mu_a + \nu_a \leq 1$ and $0 \leq \mu_b + \nu_b \leq 1 \}$ and

$\text{IF}\beta\text{C}(X) = \{0., 1., \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \text{either } [\mu_a < 0.4 \text{ or } \mu_b \text{ < 0.3}] \text{ or } [\mu_a \geq 0.5 \text{ and } \mu_b \geq 0.5], 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}$.

Let $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$ then $A$ is an IF$\beta$G$\alpha$CS in $X$.

**Remark 3.3:** Every IFCS[2], IFSCS[4], IFPCS[4], IF$\alpha$CS[4], IF$\gamma$CS[5], IFRCS[4], IFSPCS[15], IF$\beta$CS[4] is an IF$\beta$G$\alpha$CS in $(X, \tau)$ but not conversely in general. It can be seen from the following examples.

**Example 3.4:** In Example 3.2, $A$ is an IF$\beta$G$\alpha$CS in $X$. But not an IFCS in $X$, as $\text{cl}(A) = G_1^c \neq A$.

**Example 3.5:** In Example 3.2, $A$ is an IF$\beta$G$\alpha$CS in $X$, but not an IFSCS in $X$, as $\text{int}(\text{cl}(A)) = \text{int}(G_1^c) = G_1 \not\subseteq A$.

**Example 3.6:** Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ and $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $\tau = \{0, G_1, G_2, 1\}$ is an IFT on $X$. Then

IF$\alpha$O$(X) = \{0., 1., \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}$.

Let $A = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $A$ is an IF$\beta$G$\alpha$CS in $X$ but not an IFPCS in $X$, as

**Example 3.7:** In Example 3.2, $A$ is an IF$\beta$G$\alpha$CS in $X$, but not an IF$\alpha$CS in $X$, since $\text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(G_1^c)) = \text{cl}(G_1) = G_1^c \not\subseteq A$. 
Example 3.8: Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.3_b), (0.5_a, 0.7_b) \rangle$ and $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $\tau = \{0-, G_1, G_2, 1-\}$ is an IFT on $X$. Then

$$\text{IF}^\alpha \text{O}(X) = \{0-, G_1, G_2, 1-\}$$

$$\text{IF}^\beta \text{C}(X) = \{0-, 1-, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \mu_a \geq 0.5 \text{ whenever } \mu_b \geq 0.3, \text{ or } \mu_a < 0.4 \text{ or } \mu_b < 0.3, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}$$

Let $A = \langle x, (0.4_a, 0.6_b), (0.4_a, 0.4_b) \rangle$. Then $A$ is an IF$^\beta \text{GaCS}$ in $X$. But not an IF$^\gamma \text{CS}$ as $\text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) = G_1 \cap G_1^c \neq A$.

Example 3.9: In Example 3.6, $A$ is an IF$^\beta \text{GaCS}$ in $X$, but not an IF$^\text{RCS}$ in $X$, as $\text{cl}(\text{int}(A)) = \text{cl}(G_2) = G_1^c \neq A$.

Example 3.10: Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.3_b), (0.5_a, 0.7_b) \rangle$ and $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $\tau = \{0-, G_1, G_2, 1-\}$ is an IFT on $X$. Then

$$\text{IF}^\alpha \text{O}(X) = \{0-, G_1, G_2, 1-\}$$

$$\text{IF}^\beta \text{C}(X) = \{0-, 1-, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \text{ either } \mu_a \geq 0.5 \text{ whenever } \mu_b \geq 0.3, \text{ or } \mu_a < 0.4 \text{ or } \mu_b < 0.3, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}$$

and

$$\text{IFPC}(X) = \{0-, 1-, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \text{ either } \mu_a < 0.4 \text{ or } \mu_b < 0.3 \text{ and } \mu_a \geq 0.5 \text{ whenever } \mu_b \geq 0.7, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}$$

Let $A = \langle x, (0.4_a, 0.6_b), (0.4_a, 0.4_b) \rangle$. Then $A$ is an IF$^\beta \text{GaCS}$ in $X$. But $A$ is not an IF$^\text{SPCS}$ in $X$, as we could not find any IF$^\text{PC}$ $B$ such that $\text{int}(B) \subseteq A \subseteq B$ in $X$.

Example 3.11: In Example 3.8, Let $A = \langle x, (0.4_a, 0.6_b), (0.4_a, 0.4_b) \rangle$. Then $A$ is an IF$^\beta \text{GaCS}$ in $X$. But $A$ is not an IF$^\text{BCS}$ in $X$, as $\text{int}(\text{cl}(\text{int}(A))) = \text{int}(\text{cl}(G_2)) = \text{int}(G_1^c) = G_1 \nleq A$.

Remark 3.12: Every IF$^\text{RGCS}$[13], IF$^\text{GCS}$[14], IF$^\text{GPCS}$[8], IF$^\alpha \text{GCS}$[9], IF$^\text{GSCS}$[11] are independent to every IF$^\beta \text{GaCS}$ in $(X, \tau)$ in general. This can be seen from the following examples and diagram.
Example 3.13: Let \( X = \{ a, b \} \) and \( G_1 = (x, (0.5_a, 0.7_b), (0.5_a, 0.3_b)) \) and \( G_2 = (x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)) \). Then \( \tau = \{0-, G_1, G_2, 1-\} \) is an IFT on \( X \). Then

\[
\text{IF} \alpha O(X) = \{0-, G_1, 1-\}
\]

\[
\text{IF} \beta C(X) = \{0-, 1-, \mu_a \in [0, 1], \mu_b \in [0, 1], v_a \in [0, 1], v_b \in [0, 1]/ \mu_a \geq 0.5 \text{ and } \mu_b \geq 0.6, 0 \leq \mu_a + v_a \leq 1 \text{ and } 0 \leq \mu_b + v_b \leq 1 \}
\]

Let \( A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b)) \). Here \( A \) is not an \( \text{IF}\beta G\alpha CS \) in \( X \). But \( A \) is an \( \text{IF} \alpha GCS \) in \( X \), as \( A \subseteq 1- \) and \( \text{cl}(A) = 1- \).

Example 3.14: Let \( X = \{ a, b \} \) and \( G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)) \). Then \( \tau = \{0-, G, 1-\} \) is an IFT on \( X \). Then

\[
\text{IF} \alpha O(X) = \{0-, G, 1-\}
\]

\[
\text{IF} \beta C(X) = \{0-, 1-, \mu_a \in [0, 1], \mu_b \in [0, 1], v_a \in [0, 1], v_b \in [0, 1]/ \mu_a < 0.5 \text{ or } \mu_b < 0.6, 0 \leq \mu_a + v_a \leq 1 \text{ and } 0 \leq \mu_b + v_b \leq 1 \}
\]

Let \( A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b)) \). We have \( A \subseteq G \). Now \( \beta \text{cl}(A) = A \subseteq G \). This implies that \( A \) is an \( \text{IF}\beta G\alpha CS \) in \( X \) but not an \( \text{IF} \alpha GCS \) in \( X \), as \( A \subseteq G \) and \( \text{cl}(A) = G^\circ \not\subseteq G \), where \( G \) is an \( \text{IFROS} \) in \( X \).

Example 3.15: In Example 3.2, the IFS \( A = (x, (0.4_a, 0.4_b), (0.6_a, 0.6_b)) \) is not an \( \text{IF}\beta G\alpha CS \) in \( X \), but \( A \) is an \( \text{IF} \alpha GCS \) in \( X \), as \( A \subseteq G_1 \) and \( \text{cl}(A) = G_1^\circ \subseteq G_1 \).
Example 3.16: In Example 3.2, $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$ is an IFβGαCS in X but A is not an IFGCS in X, as $A \subseteq G_1, G_2$ and $\text{cl}(A) = G_1^c \not\subseteq G_2$.

Example 3.17: Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$, $G_2 = \langle x, (0.2_a, 0.3_b), (0.8_a, 0.7_b) \rangle$ and $G_3 = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b) \rangle$. Then $\tau = \{0., G_1, G_2, G_3, 1.-\}$ is an IFT on X. Then

$\text{IFαO}(X) = \{0., G_2, 0.5 \leq \mu_a \leq 0.6$ and $0.6 \leq \mu_b \leq 0.7, 1.- \}$

$\text{IFβC}(X) = \{0., 1.-, \mu_a \epsilon [0, 1], \mu_b \epsilon [0, 1], \nu_a \epsilon [0, 1], \nu_b \epsilon [0, 1]/\text{either } \mu_a < 0.2 \text{ or } \mu_b < 0.3 \text{ or } \mu_a \geq 0.6 \text{ and } \mu_b \geq 0.7 \text{ or } 0.2 \leq \mu_a < 0.5 \text{ and } 0.3 \leq \mu_b < 0.6, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}.$

$\text{IFPC}(X) = \{0., 1.-, \mu_a \epsilon [0, 1], \mu_b \epsilon [0, 1], \nu_a \epsilon [0, 1], \nu_b \epsilon [0, 1]/\text{either } \mu_a < 0.2 \text{ or } \mu_b < 0.3 \text{ or } 0.4 \leq \mu_a \leq 0.5 \text{ and } 0.3 \leq \mu_b < 0.6 \text{ or } \mu_a \geq 0.8 \text{ and } \mu_b \geq 0.7, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}.$

Let $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b) \rangle$. Then A is not an IFβGαCS in X. But A is an IFGPCS in X. Since $A \subseteq G_1, G_3$ and $\text{pcl}(A) = A \subseteq G_1, G_3$.

Example 3.18: Let $X = \{a, b\}$ and $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ and $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then $\tau = \{0.-, G_1, G_2, 1.-\}$ is an IFT on X. Then

$\text{IFαO}(X) = \{0.-, G_1, G_2, 1.- \}$ and

$\text{IFβC}(X) = \{0.-, 1.-, \mu_a \epsilon [0, 1], \mu_b \epsilon [0, 1], \nu_a \epsilon [0, 1], \nu_b \epsilon [0, 1]/0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}.$

$\text{IFPC}(X) = \{0.-, 1.-, \mu_a \epsilon [0, 1], \mu_b \epsilon [0, 1], \nu_a \epsilon [0, 1], \nu_b \epsilon [0, 1]/\text{either } \mu_a < 0.4 \text{ or } \mu_b < 0.3 \text{ or } 0.4 \leq \mu_a \leq 0.5 \text{ and } 0 \leq \mu_b < 0.6 \text{ or } \mu_a \geq 0.5 \text{ and } 0 \leq \mu_b + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1 \}.$

Let $A = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$. Then A is an IFβGαCS in X but A is not an IFGPCS in X. Since $A \subseteq G_1, G_2$ and $\text{pcl}(A) = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b) \rangle \not\subseteq G_2$.

Example 3.19: In Example 3.2, let $A = \langle x, (0.4_a, 0.4_b), (0.6_a, 0.6_b) \rangle$. We have $A \subseteq G_1$, but $\beta \text{cl}(A) = 1. \not\subseteq G_1$. Hence A is not an IFβGαCS in X. But A is an IFαGCS, since $A \subseteq G_1$ and $\alpha \text{cl}(A) = A \cup \text{cl}(\text{int}(\text{cl}(A))) = A \cup \text{cl}(\text{int}(G_1^c)) = A \cup \text{cl}(G_1) = A \cup G_1^c = G_1^c \subseteq G_1$.

Example 3.20: In Example 3.2, A is an IFβGαCS in X, but not an IFαGCS in X, since $A \subseteq G_1, G_2$ and $A \cup (\text{cl}(\text{cl}(A))) = A \cup \text{cl}(\text{int}(G_1^c)) = A \cup \text{cl}(G_1) = A \cup G_1^c = G_1^c \not\subseteq G_2$.

Example 3.21: In Example 3.2, let $A = \langle x, (0.4_a, 0.4_b), (0.6_a, 0.6_b) \rangle$. We have $A \subseteq G_1$, but $\beta \text{cl}(A) = 1. \not\subseteq G_1$. Therefore A is not an IFβGαCS in X. But A is an IFGCS, since $A \subseteq G_1$ and $\text{sc}(A) = A \cup \text{int}(A) \subseteq A \cup G_1 = G_1 \subseteq G_1$. 

On Intuitionistic Fuzzy β Generalized α Closed Sets
Example 3.22: In Example 3.2, A is an IF\(\beta\)GaCS in X, but not an IFGSCS in X, as \(\text{scl}(A) = A \cup \text{int}(\text{cl}(A)) = A \cup \text{int}(G_1) = A \cup G_1 \not\subseteq G_2\).

Remark 3.23: The union of any two IF\(\beta\)GaCSs is not an IF\(\beta\)GaCS in general as seen in the following example.

Example 3.24: Let \(X = \{a, b\}\) and \(G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle, G_2 = \langle x, (0.2_a, 0.3_b), (0.8_a, 0.7_b) \rangle\) and \(G_3 = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b) \rangle\). Then \(\tau = \{0, 1\}, G_1, G_2, G_3, 1\} - \) is an IFT on X. Then
\[
\text{IF}_{\alpha}(X) = \{0, G_2, 0.5 \leq \mu_a \leq 0.6 \text{ and } 0.6 \leq \mu_b \leq 0.7, 1\} -
\]
\[
\text{IF}_{\beta}(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_e \in [0, 1], \nu_b \in [0, 1] / \text{ either } \mu_a < 0.2 \text{ or } \mu_b < 0.3 \text{ or } \mu_a \geq 0.6 \text{ and } \mu_b \geq 0.7 \text{ or } 0.2 \leq \mu_a < 0.5 \text{ and } 0.3 \leq \mu_b < 0.6, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.
\]
The IFSs \(A = \langle x, (0.1_a, 0.5_b), (0.9_a, 0.5_b) \rangle, B = \langle x, (0.5_a, 0.2_b), (0.5_a, 0.8_b) \rangle\) are IF\(\beta\)GaCSs in \((X, \tau)\). But \(A \cup B\) is not an IF\(\beta\)GaCS as \(A \cup B = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b) \rangle \subseteq G_1\) but \(\text{cl}((A \cup B)) = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b) \rangle \not\subseteq G_1\).

Remark 3.25: The intersection of two IF\(\beta\)GaCSs is not an IF\(\beta\)GaCS in general as seen in the following example.

Example 3.26: In Example 3.24, the IFSs \(A = \langle x, (0.5_a, 0.8_b), (0.5_a, 0.2_b) \rangle, B = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b) \rangle\) are IF\(\beta\)GaCSs in \((X, \tau)\). But \(A \cap B\) is not an IF\(\beta\)GaCS as \(A \cap B = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle \subseteq G_1\) but \(\text{cl}(A \cap B) = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b) \rangle \not\subseteq G_1\).

Theorem 3.27: Let \((X, \tau)\) be an IFTS. Then for every \(A \in \text{IFS}(X)\), \(A \subseteq B \subseteq \text{cl}(A) \Rightarrow B \in \text{IF}\beta\text{GaC}(X)\).

Proof: Let \(B \subseteq U\) and \(U\) be an IF\(\alpha\)OS in X. Then since \(A \subseteq B\), \(A \subseteq U\), by hypothesis \(B \subseteq \text{cl}(A)\). Therefore \(\text{cl}(B) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A) \subseteq U\), since \(A\) is an IF\(\beta\)GaCS. Hence \(B \in \text{IF}\beta\text{GaC}(X)\).

Theorem 3.28: An IFS A of an IFTS \((X, \tau)\) is an IF\(\beta\)GaCS if and only if \(A \not\subseteq F\Rightarrow \text{cl}(A) \not\subseteq \text{cl}(F)\) for every IF\(\alpha\)OS of X.

Proof: Necessity: Let \(F\) be an IF\(\alpha\)OS and \(A \not\subseteq F\), then \(A \subseteq F^c\), where \(F^c\) is an IF\(\alpha\)OS. Then \(\text{cl}(A) \subseteq F^c\), by hypothesis. Hence by definition, \(\text{cl}(A) \not\subseteq F\).

Sufficiency: Let \(U\) be an IF\(\alpha\)OS such that \(A \subseteq U\). Then \(U^c\) is an IF\(\alpha\)OS and \(A \subseteq (U^c)^c\). This implies that \(A \not\subseteq U^c\), by hypothesis, \(\text{cl}(A) \not\subseteq U^c\). Hence \(\text{cl}(A) \subseteq (U^c)^c = U\). Therefore \(\text{cl}(A) \subseteq U\). Hence \(A\) is an IF\(\beta\)GaCS in X.
Theorem 3.29: If \( A \) is an IF\( \alpha \)OS and an IF\( \beta \)G\( \alpha \)CS in \((X, \tau)\) then \( A \) is an IF\( \beta \)CS in \((X, \tau)\).

Proof: Since \( A \subseteq A \) and \( A \) is an IF\( \alpha \)OS in \( X \), by hypothesis \( \beta \text{cl}(A) \subseteq A \). But \( A \subseteq \beta \text{cl}(A) \). Therefore \( \beta \text{cl}(A) = A \). Hence \( A \) is an IF\( \beta \)CS in \((X, \tau)\).

Theorem 3.30: Let \( F \subseteq A \subseteq X \) where \( A \) is an IF\( \alpha \)OS and an IF\( \beta \)G\( \alpha \)CS in \( X \). Then \( F \) is an IF\( \beta \)G\( \alpha \)CS in \( A \) if and only if \( F \) is an IF\( \beta \)G\( \alpha \)CS in \( X \).

Proof: Necessity: Let \( U \) be an IF\( \alpha \)OS in \( X \) and \( F \subseteq U \). Also let \( F \) be an IF\( \beta \)G\( \alpha \)CS in \( A \). Then clearly \( F \subseteq A \cap U \) and \( A \cap U \) is an IF\( \alpha \)OS in \( A \). Hence beta closure of \( F \) in \( A \), \( \beta \text{cl}(A)(F) \subseteq A \cap U \) and by Theorem 3.29, \( A \) is an IF\( \beta \)CS. Therefore \( \beta \text{cl}(A) = A \). Now beta closure of \( F \) in \( X \), \( \beta \text{cl}(F) \subseteq \beta \text{cl}(F) \cap \beta \text{cl}(A) = \beta \text{cl}(F) \cap A = \beta \text{cl}(F) \subseteq A \cap U \subseteq U \) that is \( \beta \text{cl}(F) \subseteq U \), whenever \( F \subseteq U \). Hence \( F \) is an IF\( \beta \)G\( \alpha \)CS in \( X \).

Sufficiency: Let \( V \) be an IF\( \alpha \)OS in \( A \), such that \( F \subseteq V \). Since \( A \) is an IF\( \alpha \)OS in \( X \), \( V \) is an IF\( \alpha \)OS in \( X \). Therefore \( \beta \text{cl}(F) \subseteq V \), as \( F \) is an IF\( \beta \)G\( \alpha \)CS in \( X \). Thus, \( \beta \text{cl}(A)(F) = \beta \text{cl}(F) \cap A \subseteq V \cap A \subseteq V \). Hence \( F \) is an IF\( \beta \)G\( \alpha \)CS in \( A \).

Theorem 3.31: An IFS \( A \) which is both an IFOS and an IF\( \beta \)G\( \alpha \)CS if and only iff \( A \) is an IFROS in \( X \).

Proof: Necessity: Let \( A \) be an IFOS and an IF\( \beta \)G\( \alpha \)CS. Then \( \beta \text{cl}(A) \subseteq A \) as \( A \subseteq A \) and \( A \) is IF\( \alpha \)OS in \( X \) but \( A \subseteq \beta \text{cl}(A) \). This implies that \( \beta \text{cl}(A) = A \). Hence \( A \) is an IF\( \beta \)CS and \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \). Since \( A \) is an IFOS, \( \text{int}(A) = A \). Therefore \( \text{int}(\text{cl}(A)) \subseteq A \). Since \( A \) is an IFOS, it is an IFPOS. Hence \( A \subseteq \text{int}(\text{cl}(A)) \). Therefore \( A = \text{int}(\text{cl}(A)) \). Hence \( A \) is an IFROS in \( X \).

Sufficiency: Let \( A \) be an IFROS then \( A = \text{int}(\text{cl}(A)) \). Since every IFROS is an IFOS and \( A \subseteq A \), we have \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \). Therefore \( A \) is an IF\( \beta \)CS in \( X \). Hence \( A \) is IF\( \beta \)G\( \alpha \)CS in \( X \).

Theorem 3.32: For an IFOS \( A \) in \((X, \tau)\), the following conditions are equivalent.

(i) \( A \) is an IFCS
(ii) \( A \) is an IF\( \beta \)G\( \alpha \)CS and an IFQ set

Proof: (i) \( \Rightarrow \) (ii) Since \( A \) is an IFCS, it is an IF\( \beta \)G\( \alpha \)CS by Remark 3.3. Now \( \text{int}(\text{cl}(A)) = \text{int}(A) = A = \text{cl}(A) = \text{cl}(\text{int}(A)) \), by hypothesis. Hence \( A \) is an IFQ-set.

(ii) \( \Rightarrow \) (i) Since \( A \) is an IFOS and an IF\( \beta \)G\( \alpha \)CS, by Theorem 3.31, \( A \) is an IFROS. Therefore \( A = \text{int}(\text{cl}(A)) = \text{cl}(\text{int}(A)) = \text{cl}(A) \), by hypothesis. Hence \( A \) is an IFCS in \( X \).
Theorem 3.33: Let $A$ be an IF$\beta G\alpha CS$ in $(X, \tau)$ and $p_{(\alpha, \beta)}$ be an IFP in $X$ such that $p_{(\alpha, \beta)} \beta \text{cl}(A)$, then $\text{cl}(p_{(\alpha, \beta)}) q A$.

Proof: Let $A$ be an IF$\beta G\alpha CS$ and let $p_{(\alpha, \beta)} \beta \text{cl}(A)$. If $\text{cl}(p_{(\alpha, \beta)}) q A$, then $A \subseteq (\text{cl}(p_{(\alpha, \beta)}))^c$ where $(\text{cl}(p_{(\alpha, \beta)}))^c$ is an IFOS then it is an IF$\alpha OS$. By hypothesis, $\beta \text{cl}(A) \subseteq (\text{cl}(p_{(\alpha, \beta)}))^c = \int(\text{cl}(p_{(\alpha, \beta)}))^c \subseteq (p_{(\alpha, \beta)})^c$. This implies that $p_{(\alpha, \beta)} \beta \text{cl}(A)$, which is a contradiction to the hypothesis. Hence $A q \text{cl}(p_{(\alpha, \beta)})$.

Theorem 3.34: If an IFS $A$ of an IFTS $X$ is both IFOS and IFGCS, then $A$ is an IF$\beta G\alpha CS$ in $(X, \tau)$.

Proof: Suppose $A$ is both an IFOS and an IFGCS. Then as $A \subseteq A$, by hypothesis we have $\text{cl}(A) \subseteq A$, but $A \subseteq \text{cl}(A)$. Therefore $\text{cl}(A) = A$. That is $A$ is an IFCS and hence $A$ is an IF$\beta G\alpha CS$ in $X$, by Remark 3.3.

Theorem 3.35: Let $(X, \tau)$ be an IFTS. Then IF$\beta C(X) = \text{IF}\beta G\alpha C(X)$ if every IFS in $(X, \tau)$ is an IF$\alpha OS$ in $X$.

Proof: Suppose that every IFS in $(X, \tau)$ is an IF$\alpha OS$ in $X$. Let $A \in \text{IF}\beta G\alpha C(X)$. Then $A$ is also an IF$\alpha OS$ by hypothesis. Therefore by Theorem 3.29 $A$ is an IF$\beta CS$. Therefore $A \in \text{IF}\beta C(X)$. Hence $\text{IF}\beta G\alpha C(X) \subseteq \text{IF}\beta C(X)$ (i).

Let $A \in \text{IF}\beta C(X)$ by Remark 3.3 $A$ is an IF$\beta G\alpha CS$ and $A \in \text{IF}\beta G\alpha C(X)$. Hence $\text{IF}\beta C(X) \subseteq \text{IF}\beta G\alpha C(X)$ (ii). From (i) and (ii) $\text{IF}\beta C(X) = \text{IF}\beta G\alpha C(X)$.

Theorem 3.36: Let $A$ be an IF$\alpha OS$ and an IF$\beta G\alpha CS$ of $(X, \tau)$. Then $A \cap \text{F}$ is an IF$\beta G\alpha CS$ of $(X, \tau)$ where $\text{F}$ is an IFCS of $X$.

Proof: Suppose that $A$ is IF$\alpha OS$ and an IF$\beta G\alpha CS$ of $(X, \tau)$, then by Theorem 3.40, $A$ is an IF$\beta CS$. But $\text{F}$ is an IFCS in $X$. Hence $A \cap \text{F}$ is an IF$\beta CS$ as every IFCS is an IF$\beta CS$. Therefore $A \cap \text{F}$ is IF$\beta G\alpha CS$ of $X$, by Remark 3.3.

Theorem 3.37: Let $(X, \tau)$ be an IFTS, then for every $A \in \text{IFSPC}(X)$ and for every $B$ in $X$, $\text{int}(A) \subseteq B \subseteq A$ implies $B \in \text{IF}\beta G\alpha C(X)$.

Proof: Suppose that $A$ is IF$\alpha OS$ and an IF$\beta G\alpha CS$ of $(X, \tau)$, then by Theorem 3.40, $A$ is an IF$\beta CS$. But $\text{F}$ is an IFCS in $X$. Hence $A \cap \text{F}$ is an IF$\beta CS$ as every IFCS is an IF$\beta CS$. Therefore $A \cap \text{F}$ is IF$\beta G\alpha CS$ of $X$, by Remark 3.3.

Theorem 3.38: If a subset $A$ of an IFTS $(X, \tau)$ is nowhere dense, then it is an IF$\beta G\alpha CS$ in $(X, \tau)$.

Proof: If $A$ is nowhere dense, then $\text{int}(\text{cl}(A)) = 0$. Let $A \subseteq U$ where $U$ is an IF$\alpha OS$. Now $\beta \text{cl}(A) \subseteq \text{scl}(A) = A \cup \text{int}(\text{cl}(A)) = A \cup 0 = A \subseteq U$ and hence $A$ is an IF$\beta G\alpha CS$ in $(X, \tau)$. 


Theorem 3.39: Every IFS in \((X, \tau)\) is an IF\(\beta\)GaCS if \(\beta\text{cl}(A) \subseteq \text{ker}(A)\).

Proof: Let \(U\) be any IF\(\alpha\)OS such that \(A \subseteq U\). Since \(A \subseteq U, \text{ker}(A) \subseteq U\) by the Definition 2.12. Therefore \(\beta\text{cl}(A) \subseteq U\) and hence \(A\) is an IF\(\beta\)GaCS.

IV. INTUITIONISTIC FUZZY \(\beta\) GENERALIZED \(\alpha\) OPEN SETS

In this section we have introduced intuitionistic fuzzy \(\beta\) generalized \(\alpha\) open sets and studied some of their properties.

Definition 4.1: The complement \(A^c\) of an IF\(\beta\)GaCS \(A\) in an IFTS \((X, \tau)\) is called an intuitionistic fuzzy \(\beta\) generalized \(\alpha\) open set (IF\(\beta\)Ga\(\alpha\)OS for short) in \(X\). The family of all IF\(\beta\)Ga\(\alpha\)OSs of an IFTS \((X, \tau)\) is denoted by IF\(\beta\)Ga\(\alpha\)O(X).

Example 4.2: Let \(X = \{a, b\}\) and \(G_1 = \langle x, (0.5a, 0.5b), (0.5a, 0.5b) \rangle\) and \(G_2 = \langle x, (0.4a, 0.3b), (0.6a, 0.7b) \rangle\). Then \(\tau = \{0, G_1, G_2, 1\}\) is an IFT on \(X\). Then IF\(\alpha\)O(X) = \{0, \(G_1, G_2, 1\)\} is an IF\(\beta\)GaO(X).

Theorem 4.3: IF\(\alpha\)OS, IFS\(\alpha\)OS, IF\(\beta\)OS, IF\(\gamma\)OS, IFROS, IFS\(\alpha\)OS, IF\(\beta\)OS are IF\(\beta\)Ga\(\alpha\)OS but not conversely in general in \((X, \tau)\).

Proof: Obvious

Example 4.4: Obvious from examples 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11 by taking complement of \(A\) in the respective examples.

Remark 4.5: IF\(\beta\)Ga\(\alpha\)OS, IF\(\beta\)OS, IF\(\beta\)Ga\(\alpha\)OS, IF\(\beta\)Ga\(\gamma\)OS, IF\(\beta\)Ga\(\alpha\)CS are independent to IF\(\beta\)Ga\(\alpha\)OS in general.


Theorem 4.7: Let \((X, \tau)\) be an IFTS. Then for every \(A \in \text{IF}\beta\text{GaO}(X)\) and for every \(B \in \text{IFS}(X)\), \(\beta\text{int}(A) \subseteq B \subseteq A \Rightarrow B \in \text{IF}\beta\text{GaO}(X)\).

Proof: Let \(A\) be any IF\(\beta\)Ga\(\alpha\)OS of \(X\) and \(B\) be any IFS of \(X\). Let \(\beta\text{int}(A) \subseteq B \subseteq A\). Then \(A^c\) is an IF\(\beta\)Ga\(\alpha\)CS and \(A^c \subseteq B^c \subseteq \beta\text{cl}(A^c)\). Therefore \(B^c\) is an IF\(\beta\)Ga\(\alpha\)CS by Theorem 3.27, which implies \(B\) is an IF\(\beta\)Ga\(\alpha\)OS in \(X\). Hence \(B \in \text{IF}\beta\text{GaO}(X)\).
**Theorem 4.8:** An IFS $A$ of an IFTS $(X, \tau)$ is an IF$\beta$G$\alpha$OS if and only if $F \subseteq \beta \text{int}(A)$ whenever $F$ is an IF$\alpha$CS and $F \subseteq A$.

**Proof:**

**Necessity:** Suppose $A$ is an IF$\beta$G$\alpha$OS in $X$. Let $F$ be an IF$\alpha$CS such that $F \subseteq A$. Then $F^c$ is an IF$\alpha$OS and $A^c \subseteq F^c$. By hypothesis $A^c$ is an IF$\beta$G$\alpha$CS, we have $\beta \text{cl}(A^c) \subseteq F^c$. Therefore $F \subseteq \beta \text{int}(A)$.

**Sufficiency:** Let $F$ be an IF$\alpha$CS such that $F \subseteq A$ and $F \subseteq \beta \text{int}(A)$. Then $(\beta \text{int}(A))^c \subseteq F^c$ and $A^c \subseteq F^c$. This implies that $\beta \text{cl}(A^c) \subseteq F^c$, where $F^c$ is an IF$\alpha$OS. Therefore $A^c$ is an IF$\beta$G$\alpha$CS. Hence $A$ is an IF$\beta$G$\alpha$OS in $X$.

**Theorem 4.9:** Let $(X, \tau)$ be an IFTS. Then for every $A \in \text{IFS}(X)$ and for every $B \in \text{IF} \alpha \text{O}(X)$, $B \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(B))) \Rightarrow A \in \text{IF} \beta \text{G} \alpha \text{O}(X)$.

**Proof:** Let $B$ be an IF$\alpha$OS. Then $B \subseteq \text{int}(\text{cl}(\text{int}(B)))$. By hypothesis, $A \subseteq \text{cl}(\text{int}(\text{cl}(B))) \subseteq \text{cl}(\text{int}(\text{cl}(\text{int}(\text{int}(B))))) \subseteq \text{cl}(\text{int}(\text{cl}(\text{int}(B)))) = \text{cl}(\text{int}(\text{int}(B))) \subseteq \text{cl}(\text{int}(A))$ as $B \subseteq A$. Therefore $A$ is an IFSOS and by Theorem 4.3, $A$ is an IF$\beta$G$\alpha$OS. Hence $A \in \text{IF} \beta \text{G} \alpha \text{O}(X)$.

**Theorem 4.10:** Let $(X, \tau)$ be an IFTS then for every $A \in \text{IFSPO}(X)$ and for every IFS $B$ in $X$, $A \subseteq B \subseteq \text{cl}(A) \Rightarrow B \in \text{IF} \beta \text{G} \alpha \text{O}(X)$.

**Proof:** Let $A$ be an IF$\alpha$CS in $X$. Then there exists an IFPOS, (say) $C$ such that $C \subseteq A \subseteq \text{cl}(C)$. By hypothesis, $A \subseteq B$. Therefore $C \subseteq B$. Since $A \subseteq \text{cl}(C)$, $\text{cl}(A) \subseteq \text{cl}(C)$ and $B \subseteq \text{cl}(C)$, by hypothesis. Therefore $B$ is an IFPOS. As every IFPOS is an IF$\beta$G$\alpha$OS, $B \in \text{IF} \beta \text{G} \alpha \text{O}(X)$.

**Theorem 4.11:** If $A$ is an IF$\alpha$CS and an IF$\beta$G$\alpha$OS in $(X, \tau)$ then $A$ is an IF$\beta$OS in $(X, \tau)$.

**Proof:** Since $A \subseteq A$ and $A$ is an IF$\alpha$CS, by hypothesis $A \subseteq \beta \text{int}(A)$. But $\beta \text{int}(A) \subseteq A$. Therefore $\beta \text{int}(A) = A$. Hence $A$ is an IF$\beta$OS in $(X, \tau)$.

**Theorem 4.12:** If an IFS $A$ of an IFTS $X$ is both IFCS and IFGOS, then $A$ is an IF$\beta$G$\alpha$OS in $(X, \tau)$.

**Proof:** Suppose an IFS $A$ of an IFTS $X$ is both an IFCS and an IFGOS. Then as $A \subseteq A$, by hypothesis $A \subseteq \text{int}(A)$. But $\text{int}(A) \subseteq A$. Therefore $\text{int}(A) = A$. That is $A$ is an IFO and hence $A$ is an IF$\beta$G$\alpha$OS in $(X, \tau)$, as every IFO is an IF$\beta$G$\alpha$OS.

**Theorem 4.13:** Let $(X, \tau)$ be an IFTS. Then IF$\beta$O$(X) = \text{IF} \beta \text{G} \alpha \text{O}(X)$ if every IFS in $(X, \tau)$ is an IF$\alpha$CS in $X$.

**Proof:** Suppose that every IFS in $(X, \tau)$ is an IF$\alpha$CS in $X$. Let $A \in \text{IF} \beta \text{G} \alpha \text{O}(X)$. Then $A$ is also an IF$\alpha$CS by hypothesis. Therefore by Theorem 4.11 $A$ is an IF$\beta$OS. Therefore $A \in \text{IF} \beta \text{O}(X)$. Hence $\text{IF} \beta \text{G} \alpha \text{O}(X) \subseteq \text{IF} \beta \text{O}(X)$ (i)
Let \( A \in \text{IF}\beta\text{O}(X) \) then by Theorem 4.3 \( A \in \text{IF}\beta\text{GaO}(X) \). Hence \( \text{IF}\beta\text{O}(X) \subseteq \text{IF}\beta\text{GaO}(X) \) \( \rightarrow \) (ii). Therefore from (i) and (ii) \( \text{IF}\beta\text{O}(X) = \text{IF}\beta\text{GaO}(X) \).

V. APPLICATION OF INTUITIONISTIC FUZZY \( \beta \) GENERALIZED \( \alpha \) CLOSED SETS

In this section we have investigated many theoretical applications of intuitionistic fuzzy \( \beta \) generalized \( \alpha \) closed sets by defining new spaces and obtained many interesting theorems.

**Definition 5.1:** If every \( \text{IF}\beta\text{GaCS} \) is an \( \text{IFCS} \) in \((X, \tau)\), then the space can be called as an intuitionistic fuzzy \( \beta_{\text{GaT}1/2} \) space (\( \text{IF}\beta_{\text{GaT}1/2} \) space for short).

**Definition 5.2:** An \( \text{IFTS} \ (X, \tau) \) is an intuitionistic fuzzy \( \beta_{\text{GaT}1/2} \) space if every \( \text{IF}\beta\text{GaCS} \) is an \( \text{IFCS} \) in \( X \).

**Example 5.3:** Let \( X = \{a, b\} \) and \( G = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b) \rangle \). Then \( \tau = \{0, G, 1\} \) is an \( \text{IFTS} \). Then

\[
\text{IF}\alpha\text{O}(X) = \{0, G, 1\}
\]

\[
\text{IF}\beta\text{C}(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.
\]

Therefore the space \((X, \tau)\) is an intuitionistic fuzzy \( \beta_{\text{GaT}1/2} \) space, as every \( \text{IF}\beta\text{GaCS} \) is an \( \text{IFCS} \) in this \((X, \tau)\).

**Definition 5.4:** An \( \text{IFTS} \ (X, \tau) \) is an intuitionistic fuzzy \( \beta_{\text{GaP}T1/2} \) space if every \( \text{IF}\beta\text{GaCS} \) is an \( \text{IFPCS} \) in \( X \).

**Example 5.5:** Let \( X = \{a, b\} \) and \( G = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle \). Then \( \tau = \{0, G, 1\} \) is an \( \text{IFTS} \). Then

\[
\text{IF}\alpha\text{O}(X) = \{0, G, 1\}
\]

\[
\text{IF}\beta\text{C}(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \mu_a \geq 0.7 \text{ and } \mu_b \geq 0.8, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.
\]

\[
\text{IF}\beta\text{GaC}(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \mu_a < 0.7 \text{ or } \mu_b < 0.8, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.
\]

\[
\text{IFPCS}(X) = \{0, 1, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1]/ \mu_a < 0.7 \text{ or } \mu_b < 0.8, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.
\]

Therefore the space \((X, \tau)\) is an intuitionistic fuzzy \( \beta_{\text{GaP}T1/2} \) space, as every \( \text{IF}\beta\text{GaCS} \) is an \( \text{IFPCS} \) in this \((X, \tau)\).
Theorem 5.6: Every IF$_{\beta}g_{\alpha c}T_{1/2}$ space is an IF$_{\beta}g_{\alpha b}T_{1/2}$ space but not conversely in general.

Proof: Let X be an IF$_{\beta}g_{\alpha c}T_{1/2}$ space. Let A be an IF$_{\beta}G_{\alpha CS}$ in X. By hypothesis, A is an IFCS in X. Since every IFCS is an IFCS, A is an IFCS in X. Hence X is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space.

Example 5.7: In example 5.3, The IFTS (X, τ) is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space, as every IF$_{G_{\alpha CS}}$ is an IFCS in (X, τ), but (X, τ) is not an IF$_{\beta}g_{\alpha b}T_{1/2}$ space, since A = {x, (0.4, 0.3, b), (0.6a, 0.7, b)} is an IF$_{\beta}G_{\alpha CS}$ in (X, τ), but not an IFCS as cl(A) = G$^c$ ≠ A.

Theorem 5.8: Every IF$_{\beta}g_{\alpha c}T_{1/2}$ space is an IF$_{\beta}g_{\alpha b}T_{1/2}$ space but not conversely in general.

Proof: Let X be an IF$_{\beta}g_{\alpha c}T_{1/2}$ space. Let A be an IF$_{\beta}G_{\alpha CS}$ in X. By hypothesis, A is an IFCS in X. Since every IFCS is an IFCS, A is an IFCS in X. Hence X is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space.

Example 5.9: In Example 5.3, The IFTS (X, τ) is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space but not an IF$_{\beta}g_{\alpha b}T_{1/2}$ space, since the IFS A = {x, (0.5a, 0.4b), (0.5a, 0.6b)} is an IF$_{\beta}G_{\alpha CS}$ in (X, τ), but not an IFCS as cl(int(A)) = cl(G) = G$^c$ $\not\subset$ A.

Theorem 5.10: An IFTS (X, τ) is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space iff IF$_{\beta}G_{\alpha O}(X)$ = IF$_{\beta}O(X)$.

Proof: Necessity: Let A be an IF$_{\beta}G_{\alpha OS}$ in (X, τ) then A$^c$ is an IF$_{\beta}G_{\alpha CS}$ in (X, τ). By hypothesis A$^c$ is an IFCS in (X, τ). Hence A is an IFOS in (X, τ). Thus IF$_{\beta}G_{\alpha O}(X)$ = IF$_{\beta}O(X)$.

Sufficiency: Let A be an IF$_{\beta}G_{\alpha CS}$ in (X, τ) then A$^c$ is an IF$_{\beta}G_{\alpha OS}$ in (X, τ). By hypothesis A$^c$ is an IFOS in (X, τ). Therefore A is an IFCS in (X, τ). Hence (X, τ) is an IF$_{\beta}g_{\alpha c}T_{1/2}$ space.

Theorem 5.11: An IFTS (X, τ) is an IF$_{\beta}g_{\alpha b}T_{1/2}$ space iff IF$_{\beta}O(X)$ = IF$_{\beta}G_{\alpha O}(X)$.

Proof: Necessity: Let A be an IF$_{\beta}G_{\alpha OS}$ in (X, τ) then A$^c$ is an IF$_{\beta}G_{\alpha CS}$ in (X, τ). By hypothesis A$^c$ is an IFCS in (X, τ). Hence A is an IFOS in (X, τ). Thus IF$_{\beta}O(X)$ = IF$_{\beta}G_{\alpha O}(X)$.

Sufficiency: Let A be an IF$_{\beta}G_{\alpha CS}$ in (X, τ) then A$^c$ is an IF$_{\beta}G_{\alpha OS}$ in (X, τ). By hypothesis A$^c$ is an IFOS in (X, τ). Therefore A is an IFCS in (X, τ). Hence (X, τ) is an IF$_{\beta}g_{\alpha b}T_{1/2}$ space.

Theorem 5.12: For any IFS A in (X, τ) where X is an IF$_{\beta}g_{\alpha b}T_{1/2}$ space, A $\in$ IF$_{\beta}G_{\alpha O}(X)$ iff for every IFP $p_{(\alpha, \beta)}$ $\in$ A, there exist an IF$_{\beta}G_{\alpha OS}$ B in X such that $p_{(\alpha, \beta)}$ $\in$ B $\subseteq$ A.
Proof: Necessity: If \( A \in \text{IF}\beta G\alpha O(X) \), then we can take \( B = A \) so that \( p(\alpha, \beta) \subseteq B \subseteq A \) for every IFP \( p(\alpha, \beta) \in A \).

Sufficiency: Let \( A \) be an IFS in \( (X, \tau) \) and assume that there exists \( B \in \text{IF}\beta G\alpha O(X) \) such that \( p(\alpha, \beta) \subseteq B \subseteq A \). Since \( X \) is an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space, \( B \) is an IF\( \beta \)OS. Then \( A = \bigcup_{p(\alpha, \beta) \subseteq A} \{ p(\alpha, \beta) \} \subseteq \bigcup_{p(\alpha, \beta) \subseteq A} B \subseteq A \). Therefore \( A = \bigcup_{p(\alpha, \beta) \subseteq A} B \), which is an IF\( \beta \)OS. Hence \( A \) is an IF\( \beta \)GaOS in \( X \).

Theorem 5.13: Let \( (X, \tau) \) be an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space. Then

(i) Any union of IF\( \beta \)GaCS is an IF\( \beta \)GaCS in \( X \).

(ii) Any intersection of IF\( \beta \)GaOS is an IF\( \beta \)GaOS in \( X \).

Proof: (i) Let \( \{ A_i \} \) be the collection of IF\( \beta \)GaCS in \( X \). Since \( (X, \tau) \) is an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space, every IF\( \beta \)GaCS is an IFCS and hence each \( A_i \), \( i \in J \) is an IF\( \beta \)CS in \( (X, \tau) \). But any union of IF\( \beta \)CS is an IF\( \beta \)CS[2], \( \cup A_i \), for every \( i \in J \) is an IF\( \beta \)CS. Since every IF\( \beta \)CS is an IF\( \beta \)GaCS, \( \cup A_i \) is also an IF\( \beta \)GaCS in \( X \).

(ii) can be proved by taking complement in (i).

Theorem 5.14: If \( A \) is both an IFOS and an IF\( \beta \)GaCS in \( X \) and if \( X \) is an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space, then

(i) \( A \) is an IFROS

(ii) \( A \) is an IFRCS

(iii) \( A \) is an IFQ-set[10]

Proof: Let \( A \) be an IF\( \beta \)GaCS in \( X \), then by Definition 5.1, \( A \) is an IFCS in \( X \). Now (i) \( \text{int(cl}(A)) = \text{int}(A) = A \) and therefore \( A \) is IFROS in \( X \).

(ii) Consider \( \text{cl}(\text{int}(A)) = \text{cl}(A) = A \) and therefore \( A \) is IFRCS in \( X \).

(iii) From (i) and (ii) \( \text{int(cl}(A)) = \text{cl}(\text{int}(A)) \). Hence \( A \) is an IFQ-set.

Theorem 5.15: Let \( (X, \tau) \) be an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space, then the following conditions are equivalent:

i. \( A \) is an IF\( \beta \)GaOS in \( X \)

ii. \( A \subseteq \text{cl}(\text{int}(A)) \)

iii. \( \text{cl}(A) \in \text{IFRC}(X) \)

Proof: (i) \( \Rightarrow \) (ii): Let \( A \) be an IF\( \beta \)GaOS in \( X \). Then since \( X \) is an IF\( \beta \)g\( \alpha \beta \)T\( 1/2 \) space, \( A \) is an IF\( \beta \)OS in \( X \). Therefore \( A \subseteq \text{cl}(\text{int}(A)) \).

(ii) \( \Rightarrow \) (iii): Let \( A \subseteq \text{cl}(\text{int}(A)) \). Then \( \text{cl}(A) \subseteq \text{cl}(\text{cl}(\text{int}(A))) = \text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \). Therefore \( \text{cl}(A) = \text{cl}(\text{int}(A)) \). Hence \( \text{cl}(A) \in \text{IFRC}(X) \).
(iii) $\Rightarrow$ (i): Since $\text{cl}(A)$ is an IFRCS in $X$, $\text{cl}(A) = \text{cl}(\text{int}(\text{cl}(A)))$ and since $A \subseteq \text{cl}(A)$, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. Therefore $A$ is a $\text{IF}^{\beta}\text{OS}$. Hence $A$ is a $\text{IF}^{\beta}\text{G}^{\alpha}\text{OS}$ in $X$.

**Theorem 5.16:** Let $(X, \tau)$ be an $\text{IF}^{\beta}_{\text{g}^{\alpha}_{1/2}}$ space, then the following conditions are equivalent:

i. $A$ is an $\text{IF}^{\beta}\text{G}^{\alpha}\text{CS}$ in $X$

ii. $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$

iii. $\text{int}(A) \in \text{IFRO}(X)$

**Proof:** This theorem can be easily proved by taking complements in Theorem 5.15.

**REFERENCES**


