Abstract

In this paper, we introduce the concept of fuzzy Bi-Magic labeling in graphs. We defined Fuzzy Bi-Magic labeling for cycle and star graph. Further we investigated the Properties of such labeling on these graphs.

A fuzzy graph \( G : (\sigma, \mu) \) is known as fuzzy Bi-Magic graph if there exists two bijective functions \( \sigma : V \rightarrow [0,1] \) and \( \mu : V \times V \rightarrow [0,1] \) such that \( \mu(u,v) < \sigma(u) \land \sigma(v) \) with the property that the sum of the labels on the vertices and the labels of their incident edges is one of the constants either \( k_1 \) or \( k_2 \), independent of the choice of vertex. We investigated that fuzzy Cycle graphs and fuzzy Star graphs are fuzzy Bi-Magic graphs. Further, some properties related to fuzzy bridge and fuzzy cut node have been discussed.

Keywords: Fuzzy labeling, Fuzzy Bi-magic labeling, Fuzzy Bi-Magic cycle, Fuzzy Bi-Magic Star, Fuzzy bridge, Fuzzy Cut node.

AMS Mathematical Subject Classification: 03E72, 05C72, 05C78.

1. INTRODUCTION

Fuzzy is a newly emerging mathematical framework to exhibit the phenomenon of uncertainty in real life tribulations. It was introduced by Zadeh in 1965, and the concepts were pioneered by various independent researchers, namely Rosenfeld, Kauffmann, etc.
A Fuzzy set is defined mathematically by assigning a value to each possible individual in the universe of discourse, representing its grade of membership which corresponds to the degree, to which that individual is similar or compatible with the concept represented the fuzzy set. Based on Zadeh’s Fuzzy relation the first definition of a fuzzy graph was introduced by Kauffmann in 1973. Baskar Babujee has introduced the notion of Bi-Magic labeling in which there exists two constants $k_1$ and $k_2$ such that the sums involved in a specified type of magic labeling is $k_1$ and $k_2$.

In this paper, a new concept of fuzzy Bi-magic labeling has been introduced. Fuzzy Bi-magic labeling for the cycle graph and star graph are defined. Also, some properties of these graphs with fuzzy bridge and fuzzy cut nodes are discussed. The graphs which are considered in this paper are simple, finite, connected and undirected.

All basic definitions and Symbols are followed as in [2,4,5,6].

2. PRELIMINARIES AND OBSERVATIONS

Definition 2.1

A fuzzy labeling graph admits Bi-magic labeling if the sum of membership values of vertices and edges incident at the vertices are $k_1$ and $k_2$ where $k_1$ and $k_2$ are constants and denoted by $\bar{B}m_o(G)$

A fuzzy labeling graph which admits a Bi-magic labeling is called a Fuzzy Bi-magic labeling graph.

Example: 2.2

![Figure 1: Fuzzy Bi-magic path graph for n=5 $\bar{B}m_o(P_5)$](image)

$K_1=0.16$ and $K_2=0.18$

In Figure 1,

$\sigma(v_1) + \mu(v_1, v_2) + \sigma(v_2) = 0.07 + 0.01 + 0.08 = 0.16$

$\sigma(v_2) + \mu(v_2, v_3) + \sigma(v_3) = 0.08 + 0.02 + 0.06 = 0.16$
Fuzzy Bi-Magic labeling on Cycle Graph and Star Graph

\[ \sigma(v_3) + \mu(v_3, v_4) + \sigma(v_4) = 0.06 + 0.03 + 0.09 = 0.18 \]
\[ \sigma(v_4) + \mu(v_4, v_3) + \sigma(v_3) = 0.09 + 0.04 + 0.05 = 0.18 \]

**Note:** A Fuzzy Bi-magic labeling on Path graph is satisfied only if \( n=5 \).

**Definition 2.3**

A Cycle (or) Circular graph is a graph that consists of a single cycle (or) in otherwords, finite number of vertices connected in a closed chain. The cycle graph with \( n \) vertices is denoted by \( C_n \). The number of vertices in \( C_n \) equals the number of edges and every vertex has degree 2.

A cycle graph which admits fuzzy labeling is called a fuzzy labeling cycle graph and if Bi-magic labeling exists then it is called a fuzzy Bi-magic labeling cycle graph and denoted by \( \tilde{B}m_o(C_n) \).

**Example 2.7**

![Fuzzy Bi-magic cycle for n=5](image)

**Figure 2:** Fuzzy Bi-magic cycle for \( n=5 \) \( \tilde{B}m_o(C_5) \)

**Definition 2.4**

A fuzzy Star graph consists of two vertex sets \( V \) and \( U \) with \( |V| = 1 \) and \( |U| > 1 \) such that \( \mu(v, u_i) > 0 \) and \( \mu(u_i, u_{i+1}) = 0 \) for \( 1 \leq i \leq n \).
In a fuzzy Star graph, if a Bi-magic labeling exists then it is called a fuzzy Bi-magic labeling Star graph and it is denoted by $\tilde{B}m_o(S_{1,n})$.

**Example 2.5**

Figure 3: Fuzzy Bi-magic Star for $n=4$ $\tilde{B}m_o(S_{1,4})$

**Definition 2.6**

Let $G: (\sigma, \mu)$ be a fuzzy graph. The strong degree of a vertex $v$ is defined as the sum of membership values of all strong neighbours of $v$, then $d_s(v) = \sum_{u \in N_s(v)} \mu(u, v)$.

**Definition 2.7**

An edge $uv$ is called a fuzzy bridge of $G$, if its removal reduces the strength of connectedness between some pair of vertices in $G$. Equivalently $(u,v)$ is a fuzzy bridge iff there are nodes $x,y$ such that $(u,v)$ is a arc of every strongest $x$-$y$ path.

**Definition 2.8**

A node is a fuzzy cutnode of $G: (\sigma, \mu)$ if removal of it reduces the strength of connectedness between some pair of nodes in $G$. 
3. MAIN RESULTS

Proposition 3.1

If \( n \) is odd, then the cycle \( C_n \) admits a fuzzy Bi-magic labeling.

Proof:

Let \( C_n \) be any cycle with odd number of vertices and \( v_1, v_2, \ldots, v_n \) and \( v_1v_2, v_2v_3, \ldots, v_nv_1 \) be the vertices and edges of \( C_n \) respectively.

Let \( z \to [0,1] \) such that one can choose \( z=0.01 \) if \( n \geq 3 \). The fuzzy labeling is defined as follows:

\[
\sigma(v_{2i}) = (2n - 4 - i)z \quad \text{for} \quad 1 \leq i \leq \frac{n-3}{2}
\]

\[
\sigma(v_{2i}) = 2(2n - 4 - i)z \quad \text{for} \quad i \leq \frac{n-1}{2}
\]

\[
\sigma(v_{2i-1}) = \max\left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n-1}{2} \right\} - i(z) \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{2}
\]

\[
\mu(v, v_n) = \frac{1}{2} \max\{\sigma(v_i) / 1 \leq i \leq n\}
\]

\[
\mu(v_{n-i+1}, v_{n-i}) = \mu(v, v_n) - i(z) \quad \text{for} \quad 1 \leq i \leq n-1.
\]

Here, we investigated the results for fuzzy Bi-magic cycle \( \tilde{B}m_0(C_7) \)

Case (i): \( i \) is even

Then \( i=2x \) for any positive integer \( x \)

For each edge \( v_i, v_{i+1} \), the fuzzy Bi-magic labelings are,

\[
\tilde{B}m_0(C_7) = \sigma(v_i) + \mu(v_i, v_{i+1}) + \sigma(v_{i+1})
\]

\[
= \sigma(v_{2x}) + \mu(v_{2x}, v_{2x+1}) + \sigma(v_{2x+1})
\]

\[
= \left\{ \frac{(2n - 4 - x)z}{1 \leq i \leq \frac{n-3}{2}} \right\} + \frac{1}{2} \max\{\sigma(v_i) / 1 \leq i \leq n\} -
\]

\[
= (n - 2x)z + \max\{\sigma(v_2) / 1 \leq i \leq \frac{n-1}{2}\} - (x + 1)z
\]
\[ (n - 5)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \right\} \]

If \( i = 2x \) for any positive integer \( x \), \( \left\{ x \leq \frac{n - 1}{2} \right\} \)

For each edge \( v_i, v_{i+1} \), the fuzzy Bi-magic labelings are,

\[ \tilde{B}m_0(C_7) = \sigma(v_{2x}) + \mu(v_{2x}, v_{2x+1}) + \sigma(v_{2x+1}) \]

\[ = \left\{ 2(2n - 4 - x)z / i \leq \frac{n - 1}{2} \right\} + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} - (n - 2x)z + \max \left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \right\} - (x + 1)z \]

\[ = (3n - 12)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \right\} \]

(or)

\[ = (2n - 5)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \right\} \]

Case (ii): \( i \) is odd

Then \( i = 2x + 1 \) for any positive integer \( x \)

For each edge \( v_i, v_{i+1} \), the fuzzy Bi-magic labelings are,

\[ \tilde{B}m_0(C_7) = \sigma(v_i) + \mu(v_{i+1}, v_{i+2}) + \sigma(v_{i+2}) \]

\[ = \sigma(v_{2x+1}) + \mu(v_{2x+1}, v_{2x+2}) + \sigma(v_{2x+2}) \]

\[ = \left\{ \max \{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \} - (x + 1)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} - (n - 2x + 4)z + \left\{ (2n - x)z / 1 \leq i \leq \frac{n - 3}{2} \right\} \right\} \]

\[ = (n - 5)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \left\{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n - 1}{2} \right\} \]
If \( i = 2x+1 \) for any positive integer \( x \), \( x \leq \frac{n-1}{2} \)

For each edge \( v_i, v_{i+1} \), the fuzzy Bi-magic labelings are,

\[
\tilde{B}m_0(C_7) = \sigma(v_{2x+1}) + \mu(v_{2x+1}, v_{2x+2}) + \sigma(v_{2x+2})
\]

\[
= \left\{ \frac{1}{2} \left\lfloor 2(2n-x)z \leq \frac{n-1}{2} \right\rfloor + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} -
\]

\[
(n - 2x - 8)z + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\} - (x + 1)z
\]

\[
= (3n - 12)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

(or)

\[
= (2n - 5)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

In General, if \( n \) is odd

\[
\tilde{B}m_0(C_n) = 2z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

\[
\tilde{B}m_0(C_n) = (n+2)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

Therefore, from the above cases, we verified that \( G \) is a fuzzy Bi-magic graph if \( G \) has odd number of vertices.

**Note:** If we take,

(i) The Number of vertices \( n = 5 \), then

\[
\tilde{B}m_0(C_5) = (n - 3)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

\[
\tilde{B}m_0(C_5) = (3n - 8)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

(or)

\[
= (2n - 3)z + \frac{1}{2} \text{Max} \left\{ \sigma(v_i) \right\} / 1 \leq i \leq n \right\} + \text{Max} \left\{ \sigma(v_{2i}) \right\} / 1 \leq i \leq \frac{n-1}{2} \right\}
\]

(ii) The Number of vertices \( n = 9 \), then
\[ Bm_0(C_5) = (n - 7)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n-1}{2} \} \]

\[ \tilde{B}m_0(C_5) = (3n - 16)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n-1}{2} \} \]

(or) \[ = (2n - 7)z + \frac{1}{2} \max \{ \sigma(v_i) / 1 \leq i \leq n \} + \max \{ \sigma(v_{2i}) / 1 \leq i \leq \frac{n-1}{2} \} \]

and so on.

**Proposition: 3.2**

For any \( n \geq 4 \), the Star graph \( S_{1,n} \) admits a fuzzy Bi-magic labeling.

**Proof:**

Let \( S_{1,n} \) be the Star graph with \( v, u_1, u_2, \ldots, u_n \) as vertices and \( vu_1, vu_2, \ldots, vu_n \) as edges.

Let \( z \to [0,1] \) such that one can choose \( z=0.01 \) if \( n \geq 4 \). The fuzzy labeling is defined as follows:

\[ \sigma(u_i) = [2(n+1) - i]z \quad \text{for } i=1,2,3 \]

\[ \sigma(u_i) = [(2(n+1) - i) - 1]z \quad \text{for } 4 \leq i \leq n \]

\[ \sigma(u_i) = \sum_{i=1}^{n} \frac{\sigma(u_i)}{n} \quad \text{for } 1 \leq i \leq n \]

\[ \mu(v, u_1) = \max \{ \sigma(v), \sigma(u_1) \} - \min \{ \sigma(v), \sigma(u_1) \} - 2z, \quad \text{for } i=1 \]

\[ \mu(v, u_2) = \max \{ \sigma(v), \sigma(u_2) \} - \min \{ \sigma(v), \sigma(u_2) \}, \quad \text{for } i=2 \]

\[ \mu(v, u_3) = \max \{ \sigma(v), \sigma(u_3) \} - \min \{ \sigma(v), \sigma(u_3) \} + 2z, \quad \text{for } i=3 \]

\[ \mu(v, u_i) = \max \{ \sigma(v), \sigma(u_i) \} - \min \{ \sigma(v), \sigma(u_i) \} + 3z, \quad \text{for } 4 \leq i \leq n \]

Then the constants \( k_1 \) and \( k_2 \) of the fuzzy Bi-magic labeling are defined as follows:
To find $k_i$:

Case (i): for $i=1$

$$
\tilde{B}_{m_0}(S_{1,n}) = \sigma(v) + \mu(v, u_1) + \sigma(u_1)
$$

$$
= \left\{ \sum_{1 \leq i \leq n} \frac{\sigma(u_i)}{n} \right\} + \max\{\sigma(v), \sigma(u_1) \mid i = 1\} - 
\min\{\sigma(v), \sigma(u_1) \mid i = 1\} - 2z + (2n + 1 - 1)z
$$

Case (ii): for $i=2$

$$
\tilde{B}_{m_0}(S_{1,n}) = \sigma(v) + \mu(v, u_2) + \sigma(u_2)
$$

$$
= \left\{ \sum_{1 \leq i \leq n} \frac{\sigma(u_i)}{n} \right\} + \max\{\sigma(v), \sigma(u_2) \mid i = 2\} - 
\min\{\sigma(v), \sigma(u_2) \mid i = 2\} + (2n + 1 - 2)z
$$

Case (iii): for $i=3$

$$
\tilde{B}_{m_0}(S_{1,n}) = \sigma(v) + \mu(v, u_3) + \sigma(u_3)
$$

$$
= \left\{ \sum_{1 \leq i \leq n} \frac{\sigma(u_i)}{n} \right\} + \max\{\sigma(v), \sigma(u_3) \mid i = 3\} - 
\min\{\sigma(v), \sigma(u_3) \mid i = 3\} + 2z + (2n + 1 - 3)z
$$
To find $k_2$:

Case (iv): for $4 \leq i \leq n$

$$\tilde{B}m_0(S_{1,n}) = \sigma(v) + \mu(v, u_i) + \sigma(u_i)$$

$$= \left\{ \sum_{1 \leq i \leq n} \frac{\sigma(u_i)}{n} + \text{Max}\{\sigma(v), \sigma(u_i) / 4 \leq i \leq n\} \right\} -$$

$$\text{Min}\{\sigma(v), \sigma(u_i) / 4 \leq i \leq n\} + \{2n - (i - 4) / 4 \leq i \leq n\}$$

Hence the Star graph $S_{1,n}$ is fuzzy Bi-magic labeling graph $\tilde{B}m_0(S_{1,n})$ for $n \geq 4$.

4. PROPERTIES OF FUZZY BI-MAGIC GRAPHS

**Proposition 4.1**

For every Bi-magic graph $G$, there exists at least one fuzzy bridge.

**Proof:**

Let $G$ be a fuzzy Bi-magic graph, such that there exists only one edge $\mu(x, y)$ with maximum value, since $\mu$ is bijective.

Now, we claim that $\mu(x, y)$ is a fuzzy bridge, if we remove the edge $(x, y)$ from $G$, then its subgraph, we have $\mu^\circ(x, y) < \mu(x, y)$ which implies that $(x, y)$ is a fuzzy bridge.

**Proposition 4.2**

Removal of a fuzzy cut vertex from a fuzzy Bi-magic Star graph $G$, the resulting graph $G^*$ also admits a fuzzy Bi-magic labeling if $n \geq 4$.

**Proof:**

Since $G$ is a Star graph, there exists at least one fuzzy cut vertex. Now if we remove that fuzzy cut vertex from $G$, then it becomes a smaller Star $G^*$. However, $G^*$ remains to be admit fuzzy Bi-magic labeling if $n \geq 4$.

Hence, we conclude that removal of a fuzzy cut vertex from a fuzzy Bi-magic Star graph results in a fuzzy Bi-magic Star graph if $n \geq 4$. 
**Observation 4.3**

1. Every fuzzy Bi-magic graph is a fuzzy labelled graph, but the converse is not true.
2. If $G$ is a fuzzy Bi-magic graph then $d(u) \neq d(v)$ for any pair of vertices $u, v \in V(G)$.
3. For all fuzzy Bi-magic cycle graph $G$, there exists a subgraph $G^*$ which is a cycle with odd number of vertices and there exists at least one pair of vertices $u$ and $v$ such that $d_s(u) = d_s(v)$.

**5. CONCLUSION**

In this paper, the concept of fuzzy Bi-magic labeling has been introduced. Fuzzy Bi-magic labeling for Cycle and Star graphs have been discussed. Properties of fuzzy Bi-magic graphs are investigated.

We further extend this study on some more special classes of graphs.

**REFERENCES**

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