A Study on Fuzzy AHP method and its applications in a “tie-breaking procedure”

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Abstract

The situations in which two or more participants in a competition are equally placed, known as tie-situations. To break the tie situations, the tie break procedures or tiebreakers are developed for finding the ordering relation or ranking among the participants. In this paper, a new methodology or approach is proposed for dealing with the tie-situation, which is based on fuzzy analytical hierarchy process (Fuzzy AHP) with use of triangular fuzzy numbers for the pairwise comparison matrices. Then the extent analysis method (EAM) [7] is used for determining the fuzzy synthetic extent values and applying the method of comparison of fuzzy numbers for calculating the normalized weight vectors. Finally, the final score for each student can obtained. The working of proposed approach is illustrated with the help of a numerical example.

Keywords: Fuzzy AHP; Triangular fuzzy numbers; Extent analysis method; Synthetic extent values; Pairwise comparison matrices.

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1. INTRODUCTION

The Analytical Hierarchy Process (AHP) is one of the methods of multi-criteria decision making (MCDM) developed by Saaty (1980). AHP is a structured technique for organizing and analyzing complex decisions or issues which involves subjective judgments. In other words, an AHP is a traditional powerful decision making technique in order to determining priorities among different criteria, comparing the decision alternatives for each criterion and determining an overall ranking of the decision alternatives. The main advantages of AHP are handling multiple criteria, easy to understand and effectively dealing with both qualitative and quantitative data. In the real world, most of the information or data obtained from experts included uncertainty and vagueness because of the incomplete information, impreciseness of human judgments and uncertainty of decision environment. The combine effect of fuzzy set theory and analytical hierarchy process gives fuzzy analytical hierarchy process (Fuzzy AHP) as a more powerful methodology for multi-criteria decision making (MCDM). Hence, it can be concluded that Fuzzy AHP will find more applications than conventional AHP in the near future. There are many scientific approaches for deriving the weights (crisp or fuzzy) from fuzzy pairwise comparison matrices. Since fuzzy weights are not as easy to compute as crisp weights, then the majority of Fuzzy AHP applications use a simple extent analysis method proposed by Chang [7]. Likewise an AHP, fuzzy AHP provides a hierarchical structure, facilitates the decompositions and pairwise comparisons, reduces the inconsistency and generates the priority vectors. Also, a fuzzy AHP can solve and support spatial reasoning problems in a number of different context such as: locating convenience stores and other facilities (Kuo et al., 1999, 2002; Partovi, 2006), hospital site selection (Chi and Kuo, 2001; Witlox, 2003; H. Vahidnia and A. Alesheikh, 2009), screening potential landfill sites (Charnpratheep et al., 1997), supplier selection (Kahraman et al., 2003) and local park planning (Zucca et al., 2008). In the present work, the Fuzzy AHP method will be employed for breaking the tie situation and deciding the rank among the students, when they have obtained the same marks in a competitive examination. This paper is organized as follows: The basic concepts or preliminaries of fuzzy set theory and Fuzzy AHP method are presented in Section 2. Section 3 deals with the method of fuzzy numbers for pairwise comparisons. In Section 4, an idea is proposed for determining the priority vectors. In section 5 a numerical example in solved for illustrating the working process of proposed methodology. The results and conclusions are stated in Section 6.
2. PRELIMINARIES

This section contains some basic definitions of fuzzy set theory, classical AHP and Fuzzy AHP.

2.1. Fuzzy numbers

Definition 1. Let $M \in F(R)$ be called a fuzzy number, if the following two conditions are satisfied

1. There exists $x_0 \in R$ such that $\mu_M(x_0) = 1$.
2. For any $0 \leq \alpha \leq 1$, $A_\alpha = \{x, \mu_{A_\alpha}(x) \geq \alpha\}$ is a closed interval.

where $F(R)$ represents a family of all fuzzy sets and $R$ is the set of real numbers.

Definition 2. A fuzzy number $M$ on $R$ is said to be a triangular fuzzy number if its membership function $\mu_M(x): R \rightarrow [0,1]$ is defined as follows:

$$\mu_M(x) = \begin{cases} \frac{x-l}{m-l}, & l \leq x \leq m \\ \frac{-x+u}{u-m}, & m \leq x \leq u \\ 0, & \text{otherwise} \end{cases}$$

where $l$ and $u$ stand for the lower and upper value of the support of $M$ respectively, and $m$ represent the modal value. The triangular fuzzy numbers can be denoted by order triplet $(l, m, u)$ of real numbers or regular numbers. The support of $M$ is the set of elements $\{x \in R \mid l < x < u\}$.

Definition 3. Consider any two triangular fuzzy numbers $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$ then the following arithmetic operations can be defined as follows:

1. $(l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$.
2. $(l_1, m_1, u_1) \otimes (l_2, m_2, u_2) = (l_1 l_2, m_1 m_2, u_1 u_2)$.
3. $k(l_1, m_1, u_1) = (kl_1, km_1, ku_1), k > 0, k \in R$.
4. $(l_1, m_1, u_1)^{-1} = \left(\frac{1}{u_1}, \frac{1}{m_1}, \frac{1}{l_1}\right)$, provided $l_1 \neq 0, m_1 \neq 0, u_1 \neq 0$. 

2.2. Classical AHP method

**Definition 4.** AHP is a multi-criteria decision making tool in order to determine the priorities among different decision criteria, comparing decision alternatives for each criterion and obtaining an overall ranking of the decision alternatives. The final outcomes of an AHP are to decide best among the decision alternatives. The method for AHP consists of the following four steps (See Zahedi, 1986 [19]).

1. Decomposing the decision problem into a hierarchy.
2. Obtaining the judgmental matrix by making pairwise comparisons.
3. Evaluating the local weights and consistency of the comparisons.
4. Aggregation of local weights to obtain scores and ranking the alternatives.

2.3. Fuzzy AHP method

**Definition 5.** The classical AHP is insufficient for dealing with fuzziness and uncertainty in multi-criteria decision making (MCDM), because of incomplete information, impreciseness of human judgments and fuzzy environment. Hence, the fuzzy AHP technique can be viewed as an advanced analytical method developed from the classical AHP. The method for fuzzy AHP consists of the following six steps:

1. Development of the problem hierarchy.
2. Obtaining the fuzzy comparison matrices.
3. Calculation of fuzzy synthetic extents.
4. Comparison of fuzzy synthetic extents
5. Evaluation of the minimum degree of possibilities.

2.4. Fuzzy synthetic extent values

**Definition 6.** Let \( X = \{x_1, x_2, ..., x_n\} \) be an object set and \( U = \{u_1, u_2, ..., u_m\} \) be a goal set. Then using the method of extent analysis, each object is taken and performs extent analysis for each goal respectively. Therefore, we have \( m \) extent analysis values for each object with the following notations:

\[
M_{g_1}^{x_1}, M_{g_2}^{x_1}, ..., M_{g_m}^{x_1}, i = 1, 2, ..., n
\]

where all the \( M_{g_j}^{x_i}(j = 1, 2, ..., m) \) are triangular fuzzy numbers.
Definition 7. Let $M_{gi}^1, M_{gi}^2, ..., M_{gi}^m$ be values of extent analysis of the $i^{th}$ object for m goals. Then the value of fuzzy synthetic extent with respect to $i^{th}$ object can be determined by using the algebraic operations on triangular fuzzy numbers as follows:

$$S_i = \sum_{j=1}^{m} M_{gi}^j \otimes [\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^j]^{-1}$$

(2)

3. CHANG’S EXTENT ANALYSIS METHOD

The Chang’s extent analysis on Fuzzy AHP is based on degree of possibilities of each criterion. Firstly, triangular fuzzy numbers are taken into consideration for the pairwise comparison scale of Fuzzy AHP. Afterwards, the following steps of Chang’s analysis are used in order to complete the whole procedure

Step1. The fuzzy synthetic extent values for $i^{th}$ object can be computed using equation (2), which involves computation of $\sum_{j=1}^{m} M_{gi}^j$ and $[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^j]^{-1}$.

Step2. The degree of possibility of $M_2$ greater than equal to $M_1$ is defined as follows:

$$V(M_2 \geq M_1) = \max_{y \geq x} [\min(\mu_{M_1}(x), \mu_{M_2}(y))]$$

(3)

where x and y are the values on the axis of membership function of each criterion. This expression can be equivalently written as follows:

$$V(M_2 \geq M_1) = \begin{cases} 
1, & \text{if } m_2 \geq m_1 \\
0, & \text{if } l_1 \geq u_2 \\
\frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)}, & \text{otherwise}
\end{cases}$$

(4)

Step3. The degree of possibility for a convex fuzzy number $M$ to be greater than k convex fuzzy numbers $M_i$ ($i=1, 2, 3, ..., k$) can be defined as follows:

$$V(M \geq M_1, M_2, ..., M_k) = V[(M \geq M_1) \land (M \geq M_2) \land ... \land (M \geq M_k)]$$

$$= \min V(M \geq M_i), i = 1, 2, ..., k$$

(5)

Step4. Assume that $d'(A_i) = \min V(S_i \geq S_k)$ for $k = 1, 2, ..., n; k \neq i$. Then the weight vector is given by

$$W' = \left(d'(A_1), d'(A_2), ..., d'(A_n)\right)^T$$

(6)
where $A_i (i = 1, 2, ..., n)$ are $n$ elements.

**Step 5.** Then via normalization process, we have obtained the following normalized weight vectors

$$W = \left(d(A_1), d(A_2), ..., d(A_n)\right)^T$$

(7)

4. NUMERICAL EXAMPLE

Suppose that at a university in a competitive examination, the three students obtained the same marks. We will call them $ST_1$, $ST_2$ and $ST_3$. A committee has formed for finding the ordering relation or deciding the rank among students. The committee has three members and they have identified the following decision criteria:

$DC_1$ - Academic performance.

$DC_2$ - Self confidence.

$DC_3$ - Ability to deal with complex problems.

$DC_4$ - Human maturity.

**First level of decision criteria**

According to the step 2 of fuzzy AHP, the fuzzy pairwise comparison matrix $R$ is constructed (See Table 1)

<table>
<thead>
<tr>
<th></th>
<th>$DC_1$</th>
<th>$DC_2$</th>
<th>$DC_3$</th>
<th>$DC_4$</th>
<th>$W_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$DC_1$</td>
<td>(1, 1, 1)</td>
<td>(0.9, 1.2, 1.5)</td>
<td>(0.6, 1, 1.4)</td>
<td>(0.35, 0.45, 0.55)</td>
<td>0.17</td>
</tr>
<tr>
<td>$DC_2$</td>
<td>(0.5, 0.8, 1.1)</td>
<td>(1, 1, 1)</td>
<td>(2.49, 2.99, 3.49)</td>
<td>(0.8, 1.3, 1.8)</td>
<td>0.32</td>
</tr>
<tr>
<td>$DC_3$</td>
<td>(0.7, 1.1, 1.5)</td>
<td>(0.23, 0.3, 0.37)</td>
<td>(1, 1, 1)</td>
<td>(0.32, 0.49, 0.66)</td>
<td>0.46</td>
</tr>
<tr>
<td>$DC_4$</td>
<td>(2.4, 2.8, 3.2)</td>
<td>(0.4, 0.7, 1)</td>
<td>(1.7, 2.1, 2.5)</td>
<td>(1, 1, 1)</td>
<td>0.41</td>
</tr>
</tbody>
</table>
A Study on Fuzzy AHP method and its applications in a “tie-breaking procedure”

By using formula (7), we can obtained the following fuzzy synthetic extent values

\[ S_1 = (2.85, 3.65, 4.45) \left( \frac{1}{23.07}, \frac{1}{18.23}, \frac{1}{14.39} \right) = (0.123, 0.20, 0.31) \]

\[ S_2 = (4.79, 5.09, 7.39) \left( \frac{1}{23.07}, \frac{1}{18.23}, \frac{1}{14.39} \right) = (0.21, 0.28, 0.51) \]

\[ S_3 = (2.25, 2.89, 3.47) \left( \frac{1}{23.07}, \frac{1}{18.23}, \frac{1}{14.39} \right) = (0.1, 0.16, 0.24) \]

\[ S_4 = (4.5, 6.6, 7.7) \left( \frac{1}{23.07}, \frac{1}{18.23}, \frac{1}{14.39} \right) = (0.19, 0.36, 0.53) \]

The degree of possibility for comparison of any two fuzzy synthetic extent values is defined as follows:

\[ V(S_1 \geq S_2) = \frac{0.21 - 0.31}{0.20 - 0.31} = 0.55 \]

\[ V(S_1 \geq S_3) = 1 \]

\[ V(S_1 \geq S_4) = \frac{0.19 - 0.31}{0.20 - 0.31} = 0.43 \]

\[ V(S_2 \geq S_1) = 1, V(S_2 \geq S_3) = 1 \]

\[ V(S_2 \geq S_4) = \frac{0.19 - 0.51}{0.28 - 0.51} = 0.8 \]

\[ V(S_3 \geq S_1) = \frac{0.123 - 0.24}{0.16 - 0.24} = 0.74 \]

\[ V(S_3 \geq S_2) = \frac{0.21 - 0.24}{0.16 - 0.24} = 0.2 \]

\[ V(S_3 \geq S_4) = \frac{0.19 - 0.24}{0.16 - 0.24} = 0.2 \]

\[ V(S_4 \geq S_1) = 1, V(S_4 \geq S_2) = 1, V(S_4 \geq S_3) = 1 \]
Using these values the minimum degree of possibilities are calculated as follows:

\[ d'(DC_1) = V(S_1 \geq S_2, S_3, S_4) = \min(0.55, 1, 0.43) = 0.43 \]

\[ d'(DC_2) = V(S_2 \geq S_1, S_3, S_4) = \min(1, 1, 0.8) = 0.8 \]

\[ d'(DC_3) = V(S_3 \geq S_1, S_2, S_4) = \min(0.74, 0.2, 0.2) = 0.2 \]

\[ d'(DC_4) = V(S_4 \geq S_1, S_2, S_3) = \min(1, 1, 1) = 1 \]

Therefore, the weight vectors can be generated as:

\[ W' = \begin{pmatrix} d'(DC_1), d'(DC_2), d'(DC_3) \end{pmatrix}^T = (0.43, 0.8, 0.2, 1)^T \]

Via normalization, the normalized weight vectors for the decision criteria DC_1, DC_2, DC_3 and DC_4 are calculated as follows:

\[ W = \frac{W'}{\sum_{i=1}^{n} d'(DC_i)} = (0.17, 0.32, 0.46, 0.41)^T \]

Second level of decision criteria

At the second level, the committee compares students ST_1, ST_2 and ST_3 for each criteria separately and formed the fuzzy comparison matrices R_1, R_2, R_3 and R_4 as listed below (See Tables 2-5)

**Table 2:** The fuzzy pairwise comparison matrix R_4 of alternatives under decision criteria DC_1

<table>
<thead>
<tr>
<th>Criteria-DC_1</th>
<th>ST_1</th>
<th>ST_2</th>
<th>ST_3</th>
<th>W_{DC_1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST_1</td>
<td>(1, 1, 1)</td>
<td>(0.6, 1, 1.4)</td>
<td>(0.55, 0.75, 1.2)</td>
<td>0.28</td>
</tr>
<tr>
<td>ST_2</td>
<td>(0.6, 1, 1.4)</td>
<td>(1, 1, 1)</td>
<td>(0.45, 0.55, 0.65)</td>
<td>0.21</td>
</tr>
<tr>
<td>ST_3</td>
<td>(0.9, 1.32, 1.84)</td>
<td>(1.4, 2, 2.6)</td>
<td>(1, 1, 1)</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Table 3: The fuzzy pairwise comparison matrix $R_2$ for the alternatives under the decision criteria-DC$_2$

<table>
<thead>
<tr>
<th>Criteria-DC$_2$</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_1$</td>
<td>(1, 1, 1)</td>
<td>(2.8, 3, 3.2)</td>
<td>(1.8, 2.2, 2.6)</td>
</tr>
<tr>
<td>$ST_2$</td>
<td>(2.5, 3, 3.5)</td>
<td>(1, 1, 1)</td>
<td>(0.8, 1, 1.2)</td>
</tr>
<tr>
<td>$ST_3$</td>
<td>(0.4, 0.5, 0.6)</td>
<td>(0.8, 1, 1.2)</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

In Table 3, there are some elements such that $l_i - u_j > 0$, then the elements of the given matrix must be normalized in order to find the fuzzy synthetic extent values, minimum degree of possibilities and determining the normalized weight vectors.

Table 3’: The normalized fuzzy pairwise comparison matrix $R'_2$ for the alternatives under decision criteria-DC$_2$

<table>
<thead>
<tr>
<th>Criteria-DC$_2$</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
<th>$W_{DC_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_1$</td>
<td>(0.33, 0.33, 0.34)</td>
<td>(0.31, 0.33, 0.36)</td>
<td>(0.27, 0.33, 0.40)</td>
<td>0.33</td>
</tr>
<tr>
<td>$ST_2$</td>
<td>(0.27, 0.34, 0.39)</td>
<td>(0.33, 0.33, 0.34)</td>
<td>(0.26, 0.33, 0.41)</td>
<td>0.35</td>
</tr>
<tr>
<td>$ST_3$</td>
<td>(0.2, 0.3, 0.5)</td>
<td>(0.26, 0.33, 0.41)</td>
<td>(0.33, 0.33, 0.34)</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Table 4: The fuzzy pairwise comparison matrix $R_3$ for the alternatives under the decision criteria-DC$_3$

<table>
<thead>
<tr>
<th>Criteria-DC$_3$</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_1$</td>
<td>(1, 1, 1)</td>
<td>(2.1, 2.6, 3.1)</td>
<td>(2.7, 3.1, 3.5)</td>
</tr>
<tr>
<td>$ST_2$</td>
<td>(0.3, 1.1, 1.4)</td>
<td>(1, 1, 1)</td>
<td>(0.6, 1.2, 1.8)</td>
</tr>
<tr>
<td>$ST_3$</td>
<td>(0.65, 0.8, 0.95)</td>
<td>(0.63, 1, 1.4)</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

Similarly, there are some elements in Table 4 such that $l_i - u_j > 0$, then the elements of the given matrix must be normalized in order to find the fuzzy synthetic extent values, minimum degree of possibilities and determining the normalized weight vectors.
Table 4: The normalized fuzzy pairwise comparison matrix $R'_3$ for the alternatives under the decision criteria-DC3

<table>
<thead>
<tr>
<th>Criteria-DC3</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
<th>$W_{DC3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_1$</td>
<td>(0.33, 0.33, 0.34)</td>
<td>(0.27, 0.33, 0.4)</td>
<td>(0.29, 0.33, 0.38)</td>
<td>0.32</td>
</tr>
<tr>
<td>$ST_2$</td>
<td>(0.11, 0.39, 0.5)</td>
<td>(0.33, 0.33, 0.34)</td>
<td>(0.16, 0.34, 0.5)</td>
<td>0.35</td>
</tr>
<tr>
<td>$ST_3$</td>
<td>(0.27, 0.33, 0.4)</td>
<td>(0.2, 0.34, 0.46)</td>
<td>(0.33, 0.33, 0.34)</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 5: The fuzzy pairwise comparison matrix $R_4$ for the alternatives under the decision criteria-DC4

<table>
<thead>
<tr>
<th>Criteria-DC4</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
<th>$W_{DC4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ST_1$</td>
<td>(1, 1, 1)</td>
<td>(0.9, 1.1, 1.3)</td>
<td>(0.95, 1.25, 1.55)</td>
<td>0.34</td>
</tr>
<tr>
<td>$ST_2$</td>
<td>(0.55, 0.85, 1.5)</td>
<td>(1, 1, 1)</td>
<td>(1.7, 2, 2.3)</td>
<td>0.43</td>
</tr>
<tr>
<td>$ST_3$</td>
<td>(0.91, 1.25, 1.54)</td>
<td>(0.41, 0.52, 0.67)</td>
<td>(1, 1, 1)</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Third level of decision criteria

At the third level, the final scores of all the students are obtained by taking the sum of product of weights per candidate and weights of the corresponding criteria. The results are shown in the Tables 6 and 7.

Table 6

<table>
<thead>
<tr>
<th>Criterion\Alternatives</th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$DC_1$</td>
<td>0.28</td>
<td>0.21</td>
<td>0.50</td>
</tr>
<tr>
<td>$DC_2$</td>
<td>0.33</td>
<td>0.35</td>
<td>0.32</td>
</tr>
<tr>
<td>$DC_3$</td>
<td>0.32</td>
<td>0.35</td>
<td>0.33</td>
</tr>
<tr>
<td>$DC_4$</td>
<td>0.34</td>
<td>0.43</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th></th>
<th>$ST_1$</th>
<th>$ST_2$</th>
<th>$ST_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Scores</td>
<td>0.44</td>
<td>0.48</td>
<td>0.43</td>
</tr>
</tbody>
</table>
5. RESULTS AND CONCLUSIONS

In this work, the Fuzzy AHP method is used for breaking the tie situation and deciding the rank among the students, when they have obtained the same marks in a competitive examination. This method of ranking (or ordering relation) between the students is same as in [5]. According to the obtained final scores (see Table 5), it is concluded that student $ST_2$ have obtained rank 1, whereas students $ST_1$ and $ST_3$ have rank 2 and 3, respectively.

6. ACKNOWLEDGEMENT

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REFERENCES


