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On covered Γ-ideals in Γ-semigroups

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Abstract

The concept of covered ideal in semigroups has been introduced by I. Fabrici [1]. In this paper, we introduce covered Γ-ideal in Γ-semigroups. We study some results based on covered Γ-ideals in Γ-semigroup.

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1. PRELIMINARIES

To start with, we need the following definition:

Definition 1.1 [1]. An ideal M of a semigroup S is called covered ideal if $M \subset S(S \setminus M)\, S$.

We study some properties of Γ-semigroups containing covered Γ-ideals. In fact the class of covered Γ-ideals in Γ-semigroups are a generalization of the class of covered ideals in semigroups.

For subsets A, B of a Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \; and \; \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$. For properties of Γ-semigroups, we refer [2], [3]. For $s \in S$, the principal Γ-ideal generated by s is of the form $I(s) = s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S$. If there is no way for any ambiguity, we denote Γ-semigroup $(S, \Gamma, \cdot)$ by S.

Green's relation $\mathfrak{J}$ is defined on S by, for any $a, b \in S$, $a\mathfrak{J}b$ if and only if $I(a) = I(b)$. A $\mathfrak{J}$-class containing an element a of S will be denoted by J_a. The $\mathfrak{J}$-classes of S is a quasi-ordered set where the quasi-order $\mathfrak{J}$ is defined as follows: For any a, b ∈ S, $J_a \leq J_b$ if and only if $I(a) \subseteq I(b)$. The symbol $J_a \prec J_b$ means $J_a \leq J_b$, but $J_a \neq J_b$. Throughout the paper, for the sake of clarity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

2. MAIN RESULTS

Lemma 2.1. Let s be any element of a Γ-semigroup $(S, \Gamma, \cdot)$. If $I(s)$ is not a proper subset of any principal ideal of S, then J_s is maximal.

Proof. This is obvious.

Lemma 2.2. Let J be a subset of a Γ-semigroup $(S, \Gamma, \cdot)$. Then J is a maximal $\mathfrak{J}$-class of S if and only if $S \setminus J$ is a maximal Γ-ideal of S.

Proof. Let J be a maximal $\mathfrak{J}$-class of S. Then $J = J_s$ for some s in S. We obtain $S\Gamma(S \setminus J_s) \subseteq S \setminus J_s$ and $(S \setminus J_s)\Gamma S \subseteq S \setminus J_s$. Let $a \in S \setminus J_s$ and $b \in S$ be such that $b \leq a$. Then $J_b \leq J_a$. If $b \in J_s$, then J_s is a maximal $\mathfrak{J}$ -class of S, and therefore $J_s = J_a$. This is a contradiction. Hence $b \in S \setminus J_s$. This implies that $S \setminus J_s$ is an ideal of S. To prove that $S \setminus J_s$ is a maximal Γ-ideal of S, we prove that I is an ideal of S such that $(S \setminus J_s) \subset I$. Then there exists $z \in I \setminus (S \setminus J_s)$, and so $z \in J_s$. If $b \in J_s$, then $I(b) = I(s) = I(z) \subseteq I$, and therefore $J_s \subseteq I$. Hence $S = I$. Conversely, let $S \setminus J$ is a maximal ideal of S. Set s ∈ S \ (S \ J). If $a \in J_s$, then $I(a) = I(s)$; hence $a \in J$. So $J_s \subseteq J$. As $S \setminus J \subset (S \setminus J) \cup I(s)$, it follows by the hypothesis that $(S \setminus J) \cup I(s) = S$. It follows that $I(a) = I(b)$ for all a, b ∈ J. Therefore $a \in J$ implies $a \in J_s$. Then $J \subseteq J_s$. Hence $J = J_s$. If J_s is not maximal, then there exists $b \in S$ such that $J_s \prec J_b$. It implies that $I(s) \subset I(b)$. It further implies $I(b) \subseteq S \setminus J$. So, s ∈ S \ J. This is a contradiction. This completes the proof.

We now define covered Γ-ideals of Γ-semigroup.

Definition 2.1 A proper ideal M of Γ-semigroup $(S, \Gamma, \cdot)$ is called covered Γ-ideal of S if $M \subseteq S\Gamma(S \setminus M)\Gamma S$.

Proposition 2.1 If M_1 and M_2 are different proper Γ-ideals of Γ-semigroup S such that $M_1 \cup M_2 = S$, then both M_1 and M_2 are covered Γ-ideals of S.

Proof. As $M_1 \cup M_2 = S$, it implies that $S \setminus M_1 \subseteq M_2$ and $S \setminus M_2 \subseteq M_1$. If M_1 is a covered Γ-ideal of S, then $M_1 \subseteq S\Gamma(S \setminus M_1)\Gamma S \subseteq S\Gamma M_2 \Gamma S \subseteq M_2$. Therefore $S = M_2$. This is impossible. Hence the Proposition is established.

Next Corollary is a consequence of Proposition 2.1 .

Corollary 2.1. If a Γ-semigroup $(S, \Gamma, \cdot)$ contains more than one maximal Γ-ideal, then none of them is a covered Γ-ideal of S.

Proposition 2.2. Suppose S is a Γ-semigroup. If M_1 and M_2 are covered Γ-ideals of S, then $M_1 \cup M_2$ is a covered Γ-ideal of S.

Proof. Let M_1 and M_2 be covered Γ-ideals of S. Then $M_1 \subseteq S\Gamma(S \setminus M_1)\Gamma S$ and $M_2 \subseteq S\Gamma(S \setminus M_2)\Gamma S$. Let $x \in M_1 \cup M_2$. If $x \in M_1$, then $x \in S\Gamma a \Gamma S$ for some $a \in S \setminus M_1$. If $a \in S \setminus (M_1 \cup M_2)$, then $x \in S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S$. If $a \in M_1 \cup M_2$, then $a \in M_2$. Hence $a \in S\Gamma b \Gamma S$ for some $b \in S \setminus M_2$. We have $x \in S\Gamma s \Gamma S \subseteq S\Gamma S\Gamma b\Gamma S\Gamma S = S\Gamma S\Gamma b\Gamma S\Gamma S \subseteq S\Gamma b\Gamma S$. If $b \in M_1$, then $a \in M_1$. This is a contradiction. Therefore $b \in S \setminus (M_1 \cup M_2)$, and therefore $x \in S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S$. In a similar fashion, $x \in M_2$ implies $x \in S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S$. This shows that $M_1 \cup M_2$ is a covered Γ-ideal of S.

Proposition 2.3. Suppose M is an ideal of Γ-semigroup S. If M_1 is a covered Γ-ideal of S, then $M_1 \cap M$ is a covered Γ-ideal of S.

Proof. If M_1 is a covered Γ-ideal of S, then $M_1 \subseteq S\Gamma(S \setminus M_1)\Gamma S$. Hence $M_1 \cap M \subseteq M_1 \subseteq S\Gamma(S \setminus M_1)\Gamma S \subseteq S\Gamma(S \setminus (M_1 \cap M))\Gamma S$. Hence $M_1 \cap M$ is a covered Γ-ideal of S.

Corollary 2.2. Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup. If M_1 and M_2 are covered Γ- ideals of S, then $M_1 \cap M_2$ is a covered Γ-ideal of S.

Using Proposition 2.2 and Corollary 2.1, we obtain the following:

Theorem 2.1. The set of all covered Γ-ideals of a Γ-semigroup $(S, \Gamma, \cdot)$ is a sublattice of the lattice of all ideals of S.

Theorem 2.2. Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup. If S is not simple, then S contains a covered Γ-ideal.

Proof. As S is not simple, S contains a proper Γ-ideal T. Since $T \cap S\Gamma(S \setminus T)\Gamma S$ is a proper Γ-ideal of S and $T \cap S\Gamma(S \setminus T)\Gamma S \subseteq S\Gamma(S \setminus T)\Gamma S \subseteq (S\Gamma S \setminus (S \setminus (T \cap S\Gamma(S \setminus T)\Gamma S))\Gamma S$. It follows that $T \cap S\Gamma(S \setminus T)\Gamma S$ is a covered Γ-ideal of S.

Definition 2.2. A subset A of a Γ-semigroup $(S, \Gamma, \cdot)$ is called a two-sided base of S if it satisfies the following:

(i) $S = A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S$;

(ii) If B is a subset of A such that $S = B \cup B\Gamma S \cup S\Gamma B \cup S\Gamma B\Gamma S$, then $B = A$.

A covered Γ-ideal M of a Γ-semigroup $(S, \Gamma, \cdot)$ is called the greatest covered Γ-ideal of S if it contains every covered Γ-ideal of S. If a Γ-semigroup $(S, \Gamma, \cdot)$ contains the greatest covered Γ-ideal, we identify it by M^g.

To give a necessary condition so that a Γ-semigroup $(S, \Gamma, \cdot)$ contains a two-sided base, we need the following lemma.

Lemma 2.3. Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup containing the greatest covered Γ-ideal M^g. If $M^g \subset S^3$, then the following assertions hold:

(i) Every $\mathfrak{J}$ -class in $S^3 \setminus M^g$ is maximal;

(ii) $I(s) = S\Gamma s\Gamma S$ for all s in $S^3 \setminus M^g$.

Proof. (ii) Suppose that $M^g \subset S^3$. Then $S^3 \setminus M^g$ is nonempty. Suppose $s \in S^3 \setminus M^g$. As M^g is a Γ-ideal of S, it follows that $J_s \subseteq S^3 \setminus M^g$. Then $s \in S\Gamma b\Gamma S$ for some $b \in S$, and therefore $S\Gamma s\Gamma S \subseteq S\Gamma b\Gamma S$. As $S\Gamma b\Gamma S \subseteq I(b)$, we obtain $I(s) \subseteq I(b)$.

Let b be not contained in J_s; $so\, J_s \neq J_b$. If $b \in I(s)$, then $I(s) = I(b)$. So $J_s = J_b$. This is impossible. Then $b \in S \setminus I(s)$. It follows that $I(s) \subseteq S\Gamma(S \setminus I(s))\Gamma S$, and $I(s)$ is a covered Γ-ideal of S. By Proposition 2.2, $M^g \cup I(s)$ is a covered Γ-ideal of S. As s is not contained in M^g, therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Therefore $b \in J_s$. Hence $I(s) \subseteq S\Gamma b\Gamma S \subseteq I(b) = I(s)$. Then $I(s) = S\Gamma b\Gamma S = I(b)$. Obviously, $S\Gamma s\Gamma S \subseteq I(s)$. If $b \leq s$, then $I(s) = S\Gamma b\Gamma S \subseteq S\Gamma s\Gamma S$. Thus $I(s) \subseteq S\Gamma s\Gamma S$. If $b \leq s$ is false, then $b \in S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S$, so $S\Gamma b\Gamma S \subseteq S\Gamma S\Gamma s\Gamma S \subseteq S\Gamma S\Gamma s\Gamma S \subseteq S\Gamma S\Gamma s\Gamma S \subseteq S\Gamma s\Gamma S$. In a similar fashion, if $b \in s\Gamma S$ or $b \in S\Gamma s\Gamma S$, then $S\Gamma b\Gamma S \subseteq S\Gamma s\Gamma S$. So, $I(s) = I(b) = S\Gamma b\Gamma S \subseteq S\Gamma s\Gamma S$.

(i) Let J_s be a $\mathfrak{J}$ -class in $S^3 \setminus M^g$. Let J_s be not maximal. By Lemma 1.1, we have $I(s) \subset I(c)$ for some c in S. Then $s \in I(c)$. So $s \in c$ or $s \in S\Gamma c$ or $s \in c\Gamma S$ or $s \in S\Gamma c\Gamma S$. Each of the cases implies $S\Gamma s\Gamma S \subseteq S\Gamma c\Gamma S$, and therefore $I(s) \subseteq S\Gamma c\Gamma S$. As $c \in S \setminus I(s)$, it shows that $I(s)$ is a covered Γ-ideal of S. Therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Thus any $\mathfrak{J}$ -class in $S^3 \setminus M^g$ is maximal.

Theorem 2.3. Suppose $(S, \Gamma, .)$ is a Γ-semigroup containing the greatest covered Γ-ideal M^g Then S contains a two-sided base if it satisfies the following:

(i) $M^g \subset S^3$;

(ii) For any two elements $a, b \in S \setminus S^2$, neither $J_a \leq J_b$ nor $J_b \leq J_a$.

Proof. Let $M^g \subset S^3$and any two elements $a, b \in S \setminus S^2$ are incomparable. By $M^g \subseteq S\Gamma(S \setminus M^g)\Gamma S \subseteq S^3 \subseteq S^2 \subseteq S$, there are three families of $\mathfrak{J}$-classes to consider: $c_1 = \{J_a \mid a \in S \setminus S_2\}$, $c_2 = \{J_a \mid a \in S \setminus S_3\}$, and $C_3 = \{J_a \mid a \in S^3 \setminus M^g\}$. Consider one element from each $\mathfrak{J}$ -class from c_1 and c_3. Suppose A is the set of all elements we take, we claim that A is a two-sided base of S. Furthermore, suppose $I(A) = A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S$. To prove that $S = I(A)$, it is sufficient to prove that M^g , $S^3 \setminus M^g$, $S^2 \setminus S^3$, and $S \setminus S^2$ are subsets of $I(A)$.

(a) Let $x \in M^g$. Then $x \in S\Gamma(S \setminus M^g)\Gamma S$, or equivalently $x \in S\Gamma b\Gamma S$ for some $b \in S \setminus M^g$. We obtain $b \in J_a$ for some a $\in S \setminus S^2$ or $a \in S^2 \setminus S^3$ or $a\; S^3 \setminus M^g$. If $a \in S \setminus S^2$ or $a \in S^3 \setminus M^g$, then by constructing A, we obtain $b \in I(A)$. Hence

$x \in I(A)$. Let $a \in S^2 \setminus S^3$. Then $a \le c\gamma d$ for some $c, d \in S$ and $\gamma \in \Gamma$. As a is not contained in S^3 , it shows that $c, d \in S \setminus S^2$. It follows that $a \in I(A)$, and so $b \in I(A)$. Therefore $x \in I(A)$.

(b) If $x \in S^3 \setminus M^g$, then there exists $a_1 \in A$ such that $x \in I(a_1)$. Therefore $x \in I(A)$.

(c) If $x \in S^2 \setminus S^3$, then the proof is similar to (a).

(d) If $x \in S \setminus S^2$, then there exists $a_2 \in A$ such that $x \in I(a_2) \subseteq I(A)$. Finally, we prove that A is the minimal subset of S such that $S = I(A)$. By Lemma 2.1, it follows that $J_a \in C_3$ is maximal. Moreover, every $J_a \in C_1$ is also maximal as for any elements a, $b \in S \setminus S^2$, neither $J_a \le J_b$ nor $J_b \le J_a$. Now suppose B is a proper subset of A such that $S = B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$. Suppose $x \in A \setminus B$. Then $x \le y$ for some $y \in B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$. As $y \in I(b)$ for some $b \in B$, it implies that $I(x) \subset I(b)$. This contradicts to the construction of A. This completes the proof.

Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup. An ideal M of S is called the greatest Γ-ideal of S if it contains every proper Γ-ideal of S. If a Γ-semigroup (S, Γ, ·) contains the greatest Γ-ideal, we denote it by M^*.

Theorem 2.4. Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup containing only one maximal Γ-ideal M. If M is a covered Γ-ideal of S, then M is the greatest Γ-ideal of S.

Proof. This is clear to see since if T is a proper Γ-ideal of S, then $T \subseteq M$. Hence $M = M^*$ by Proposition 2.1.

Theorem 2.5. Suppose $(S, \Gamma, \cdot)$ is a Γ-semigroup with the property that every proper Γ-ideal of S is a covered Γ-ideal of S. Then either one of the following statements hold:

(1) S contains M^*;

(2) $S = S^2$ and for any proper Γ-ideal M and for every Γ-ideal $I(a) \subseteq M$, there is b in $S \setminus M$ such that $I(a) \subset I(b) \subset S$.

Proof. Let J_x and J_y be maximal $\mathfrak{J}$ -classes of S such that $J_x \neq J_y$. Then by Lemma 2.1, $M_x = S \setminus J_x$ and $M_y = S \setminus J_y$ are maximal proper Γ-ideals of S such that none of them is a covered Γ-ideal of S. This is a contradiction. Then S contains no different maximal $\mathfrak{J}$ -class. Therefore S contains one maximal $\mathfrak{J}$ -class or S does not contain maximal $\mathfrak{J}$ -class. If S contains one maximal $\mathfrak{J}$ -class J_x. Then $M_x = S \setminus J_x$ is a maximal proper Γ-ideal of S. By the hypothesis, M_x is a covered Γ-ideal of S. By Theorem 2.3, $M_x = M^*$. Suppose S does not contain maximal $\mathfrak{J}$ -class. To prove that $S = S^2$. Assume $S^2 \subset S$. Then there exists s in $S \setminus S^2$. If $I(s) = S$, then S contains a maximal $\mathfrak{J}$ -class. This is impossible. Then $I(s) \subset S$, and so $I(s) \subseteq S\Gamma(S \setminus I(s))\Gamma S$. Then $s \in S^3 \subseteq S^2$. This is a contradiction. Let M be a proper Γ-ideal of S, and suppose $I(a) \subseteq M$. As $M \subseteq S\Gamma(S \setminus M)\Gamma S$, there exists $b \in S \setminus M$ so that $a \in S\Gamma b\Gamma S$, and hence $I(a) \subseteq I(b) \subseteq S$. As $b \in S \setminus M$, so $I(a) \subset I(b)$. By the hypothesis, $I(b) \subset S$.

Theorem 2.6. Suppose that a Γ-semigroup $(S, \Gamma, \cdot)$ satisfies one of the following:

(1) S contains M^* which is a covered Γ-ideal of S;

(2) $S = S^2$, and for any proper Γ-ideal M and for every Γ-ideal $I(a) \subseteq M$, there is b in $S \setminus M$ such that $I(a) \subseteq I(b)$. Then every proper Γ-ideal of S is a covered Γ-ideal of S.

Proof. Suppose M is a proper Γ-ideal of S. Suppose S satisfies (1). Then $M \subseteq M^*$. As $S \setminus M^* \subseteq S \setminus M$, it implies that $M \subseteq M^* \subseteq S\Gamma(S \setminus M^*)\Gamma S \subseteq S\Gamma(S \setminus M)\Gamma S$. Then M is a covered Γ-ideal of S.

Now suppose that the condition (2) holds. Let $x \in M$; so $I(x) \subseteq M$. We have $I(x) \subset I(b)$. As $S = S^2$, therefore $S = S^3$. Hence $b \in S\Gamma d\Gamma S$ for some $d \in S$. As $b \in S \setminus M$, therefore $d \in S \setminus M$. Hence, $x \in S\Gamma d\Gamma S \subseteq S\Gamma(S \setminus M)\Gamma S$. It implies that $M \subseteq S\Gamma(S \setminus M)\Gamma S$.

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