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Investigation of Haar Wavelet Collocation Method to Solve Ninth Order Boundary Value Problems

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Abstract

In this paper, numerical scheme is described to approximate the solution of ninth order boundary problems by Haar wavelet collocation method (HWCM). The validation and comparison of the scheme is done through the simulation of three test problems, which are arise in the study of astrophysics, hydrodynamics and hydro magnetic stability. Estimation of error bound and order of convergence are discussed. Efficient solutions obtained by increasing the level of resolutions are seems to be in better agreement with the exact and other numerical methods such as Modified decomposition method(MDM), Homotopy perturbation method (HPM), Petrov-Galerkin Method with Quintic B-splines as Basis Functions (PGM).

Keywords: Haar wavelets, Ninth order boundary value problems, Collocation method, Convergence analysis.

INTRODUCTION

In the recent years the wavelet approach is becoming more popular in the domain of numerical estimation. The several types of wavelets and approximate functions have been used for this purpose. A short introduction to the Haar wavelets and applications can be found in [1]. The Haar wavelet method has some preferences as mathematical simplicity, fast convergence, possibility to implement standard algorithms and high accuracy for small number of grid points [1, 2]. Alfred Haar [3] demonstrated the concept of wavelets and they placed a imported role for the numerical solution of differential and integral equations. At present there are two approaches to applying the Haar wavelet for integrating ordinary differential equations (ODE). In case of the first method for integrating ODE concept of operational matrix is introduced by Chen and Hsiao [4, 5]. Another approach is called direct method due to Lepik [6] where the Haar functions are integrated directly.

The boundary value problems of ninth order have been developed due to their mathematical importance and the potential applications in the study of astrophysics, hydrodynamics and hydro-magnetic stability[7]. It's not so easy to determine the analytical solution for such type of BVPs but existence and uniqueness of solution for these type of problems have been discussed in the book written by Agarwal[8]. The solutions of differential equations have a major role in the field of science and engineering. Many mathematical formulations of physical phenomena contain ninth order BVPs.

Over the years many researchers have worked on ninth order BVPs by using different methods for numerical solutions. Chawla and Katti [9] have used finite difference method for solving two-point boundary value problems involving higher order differential equations. Wazwaz [10] had employed Modified decomposition method (MDM) for solving higher order BVPs. Abdel and Vedat [11] have found the solution of different types of linear and nonlinear higher order BVPs by differential transformation method (DTM). Syed and Ahmet[12, 13] used homotopy perturbation method(HPM) and variational iteration method (VIM) to find the solution of tenth and ninth order BVPs interms of convergent series. Luma and Samaher[14] have solved higher order boundary value problems using Semi-Analytic techniques(SAT). Bellal and Shafiqul [15] have established a novel numerical approach for odd higher order BVPs. Samir kumar[16] has used Tchebychev polynomial approximations for m^{th} order BVPs.

Many researchers have worked on Haar wavelet method to solve various orders of ODEs. Siraj ul-I et al.[17] have solved second order BVPs. Fazal et al.[18, 19] have found the solutions for fourth order and sixth order BVPs. Reddy et al.[20, 21] have approximated the solutions for fifth and seventh order ODEs. This motivated us to solve ninth order boundary value problems arise in the astrophysics, hydrodynamics and hydro magnetic stability.

The following form of ninth order BVP is considered

$$u^{(9)}(x) = f(x, u, u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}, u^{(8)}), \quad x \in (c, d), \tag{1}$$

subject to the following type of boundary conditions:

$$u(c) = \eta_1,\ u^{(1)}(c) = \eta_2,\ u^{(2)}(c) = \eta_3,\ u^{(3)}(c) = \eta_4,\ u^{(4)}(c) = \eta_5,\ u(d) = \eta_6,\ u^{(1)}(d) = \eta_7,$$
$$u^{(2)}(d) = \eta_8,\ u^{(3)}(d) = \eta_9. \tag{2}$$

Where n_i's, c and d are real constants for $i = 1, 2, \ldots, 9$.

The organization of this article is as follows. In section 2, Haar wavelets and their integrals are introduced. In section 3, a general formulation for the numerical algorithm based on Haar wavelets is presented. Convergence of Haar wavelet discretization method(HWDM) is briefly presented in section 4. Few problems are solved in section 5 to test the effectiveness of the method and finally conclusion has been discussed in the last part of this paper.

HAAR WAVELETS AND THEIR INTEGRALS

In this section, we obtain orthogonal basis for the subspaces of $L^2[c,d]$ called Haar wavelet family. For this notations introduced in Ref. [2] are used. The interval $[c,d]$ is divided into 2^{J+1} subintervals of equal length $\left(\Delta t = \frac{(d-c)}{2^{J+1}}\right)$, where J is called maximal level of resolution. We have coarser resolution values $j = 0,1,2,\ldots,J-1$ and translation parameter $k = 0,1,2,\ldots,2^j - 1$. With these two parameters i^{th} Haar wavelet in Haar family is defined as

$$h_i(t) = \begin{cases} 1, & \text{for } t \in [\xi_1(\mathrm{i}), \xi_2(\mathrm{i})), \\ -1, & \text{for } t \in [\xi_2(\mathrm{i}), \xi_3(\mathrm{i})), \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

here $i = m + k + 1, \xi_1(i) = c + 2k\mu\Delta t, \xi_2(i) = c + (2k+1)\mu\Delta t$ and $\xi_3(i) = c + 2(k+1)\mu\Delta t$, where $\mu = 2^{J-j}$.

Above equations are valid for $i > 2$. $h_1(t)$ and $h_2(t)$ are called father and mother wavelets in Haar wavelet family and are given by

$$h_1(t) = \begin{cases} 1, & for\ t \in [c,d), \\ 0, & otherwise, \end{cases} \tag{4}$$

$$h_2(t) = \begin{cases} 1, & for\ t \in \left[c,\ p\ \right), \\ -1, & for\ t \in \left[p,d\right), \\ 0, & otherwise, \end{cases} \tag{5}$$

where, $p = \frac{c+d}{2}$.

Any function which is having finite energy on $[c,d]$, i.e. $u \in L^2[c,d]$ can be decomposed as infinite sum of Haar wavelets:

$$u(x) = \sum_{i=1}^{\infty} b_i h_i(x), \tag{6}$$

where $b_i\text{'}s$ are called Haar coefficients. If f is either piecewise constant or wish to approximate by piecewise constant on each subinterval then the above infinite series will be terminated at a finite number of terms. Since, we have explicit expression for each member of Haar family (3-5). We can integrate as many times depend upon the context. The following notations are used for τ times of integration of members in the family defined on $[c,d)$:

$$P_{\tau,i}(t)=\int_c^t\int_c^t.....\int_c^t h_i(x)\,dx^{\tau}, \tag{7}$$

$$G_{\tau,i}=\int_c^d P_{\tau,i}(t)dt. \tag{8}$$

For $i=1$, (7) becomes

$$P_{\tau,1}(t)=\frac{1}{\tau!}(t-c)^{\tau}, \tag{9}$$

for $i\geq 2$, we have

$$P_{\tau,i}(t)=\begin{cases}0, & \text{if } t\in[c,\xi_1(i)),\\ \frac{1}{\tau!}(t-\xi_1(i))^{\tau}, & \text{if } t\in[\xi_1(i),\xi_2(i)),\\ \frac{1}{\tau!}\left\{(t-\xi_1(i))^{\tau}-2(t-\xi_2(i))^{\tau}\right\}, & \text{if } t\in[\xi_2(i),\xi_3(i)),\\ \frac{1}{\tau!}\left\{(t-\xi_1(i))^{\tau}-2(t-\xi_2(i))^{\tau}+(t-\xi_3(i))^{\tau}\right\}, & \text{if } t\in[\xi_3(i),d).\end{cases} \tag{10}$$

METHOD OF SOLUTION

3.1 Haar Wavelet collocation method:

The proposed method is as follows [2, 20, 21]

- Approximate highest order derivative by piecewise constant on each subinterval

$$u^{(9)}(x)=\sum_{i=1}^{2^{J+1}} b_i h_i(x). \tag{11}$$

- Decompose $u^{(8)}(x), u^{(7)}(x), ..., u(x)$ in terms of integrated Haar functions and replace these in to the given linear differential equation.
- Discritize equations obtained in above at collocation points $x_l=\frac{(\tilde{x}_{l-1}-\tilde{x}_l)}{2}, l=1,2,...,2^{J+1}$, where $\tilde{x}_n$ is the grid point given by $\tilde{x}_n=c+n\frac{(d-c)}{2^{J+1}}$, $n=0,1,2,...,2^{J+1}$. Resulting into $2^{J+1}\times 2^{J+1}$ linear algebraic system.

- Calculate the wavelet coefficients b_i's and obtain the Haar solution for unknown function u.

The proposed method is further simplified with the help of particular boundary conditions for BVPs: $c = 0, d = 1.$

3.2 The following type of boundary conditions are considered:

$$u(0) = \eta_1,\ u^{(1)}(0) = \eta_2,\ u^{(2)}(0) = \eta_3,\ u^{(3)}(0) = \eta_4,\ u^{(4)}(0) = \eta_5,\ u(1) = \eta_6,\ u^{(1)}(1) = \eta_7,$$
$$u^{(2)}(1) = \eta_8,\ u^{(3)}(1) = \eta_9. \tag{12}$$

The solution *u(x)* can be derived as

$$u(x) = \eta_1 + \eta_2 x + \eta_3 \frac{x^2}{2} + \eta_4 \frac{x^3}{6} + \eta_5 \frac{x^4}{24} + u^{(5)}(0)\frac{x^5}{120} + u^{(6)}(0)\frac{x^6}{720} + u^{(7)}(0)\frac{x^7}{5040}$$
$$+ u^{(8)}(0)\frac{x^8}{40320} + \sum_{i=1}^{2^{J+1}} b_i P_{9,i}(x). \tag{13}$$

Where the unknowns $u^{(5)}(0), u^{(6)}(0), u^{(7)}(0), u^{(8)}(0)$ can be found using boundary conditions (12) and given by

$$u^{(5)}(0) = -6720\eta_1 - 4200\eta_2 - 1200\eta_3 - 200\eta_4 - 20\eta_5 + 6720\eta_6 - 2520\eta_7 + 360\eta_8 - 20\eta_9 +$$
$$\sum_{i=1}^{2^{J+1}} b_i[-6720G_{9,i} + 2520G_{8,i} - 360G_{7,i} + 20G_{6,i}]. \tag{14}$$

$$u^{(6)}(0) = 100800\eta_1 + 60480\eta_2 + 16200\eta_3 + 2400\eta_4 + 180\eta_5 - 100800\eta_6 + 40320\eta_7 - 6120\eta_8 + 360\eta_9 +$$
$$\sum_{i=1}^{2^{J+1}} b_i[100800G_{9,i} - 40320G_{8,i} + 6120G_{7,i} - 36020G_{6,i}]. \tag{15}$$

$$u^{(7)}(0) = -604800\eta_1 - 352800\eta_2 - 90720\eta_3 - 12600\eta_4 - 840\eta_5 + 604800\eta_6 - 252000\eta_7 + 40320\eta_8 -$$
$$2520\eta_9 + \sum_{i=1}^{2^{J+1}} b_i[-604800G_{9,i} + 252000G_{8,i} - 40320G_{7,i} + 2520G_{6,i}]. \tag{16}$$

$$u^{(8)}(0) = 1411200\eta_1 + 806400\eta_2 + 201600\eta_3 + 26880\eta_4 + 1680\eta_5 - 1411200\eta_6 + 604800\eta_7 - 100800\eta_8$$
$$+ 6720\eta_9 + \sum_{i=1}^{2^{J+1}} b_i(1411200G_{9,i} - 604800G_{8,i} + 100800G_{7,i} - 6720G_{6,i}). \tag{17}$$

Where, $G_{6,i} = \int_0^1 P_{6,i}(x)dx,\quad G_{7,i} = \int_0^1 P_{7,i}(x)dx,\ G_{8,i} = \int_0^1 P_{8,i}(x)dx,\ G_{9,i} = \int_0^1 P_{9,i}(x)dx.$ (18)

CONVERGENCE ANALYSIS OF HAAR WAVELET DISCRETIZATION METHOD

The accuracy issues of the HWDM open from year 1997. This issue is clarified by J. Majak et al. [22] in 2015. The following results are due to notations introduced by J. Majak et al. [23]. General form of ninth order ODE is

$$f\left(x,u,u^{(1)},u^{(2)},u^{(3)},u^{(4)},u^{(5)},u^{(6)},u^{(7)},u^{(8)},u^{(9)}\right)=0 \tag{19}$$

Expand ninth order derivative into Haar wavelets as

$$\frac{d^9u(x)}{dx^9}=\sum_{i=1}^{\infty}b_ih_i(x) \tag{20}$$

$$=b_1h_1+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}b_{2^j+k+1}h_{2^j+k+1}. \tag{21}$$

In (20) and (21) $2^j+k+1=i,\ k=0,1,....,2^j-1$. Integrating (21) 9 times we obtain the solution of DE (19) as

$$u(x)=\frac{b_1}{9!}+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}b_{2^j+k+1}P_{9,\,2^j+k+1}(x)+B(x). \tag{22}$$

Here $P_{9,\,2^j+k+1}(x)$ represents the ninth order integrals of the Haar functions (9,10) and $B(x)$ is a boundary term.Let us assume that $\frac{d^9u(x)}{dx^9}\in L^2(R)$ is a continuous function and its next derivative is bounded on $[0,1]$,

$$\exists\ \zeta:\left|\frac{d^{10}u(x)}{dx^{10}}\right|\le\zeta$$

Let $u_{2^{J+1}}(x)=\frac{b_1}{9!}+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}b_{2^j+k+1}P_{9,2^j+k+1}(x)+B(x)$ be the approximation to the unknown u by integrated Haar wavelets. The absolute error at the J^{th} resolution is denoted $|E_{2^{J+1}}|$ and given by

$$\left|E_{2^{J+1}}\right|=\left|u(x)-u_{2^{J+1}}(x)\right|=\left|\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}b_{2^j+k+1}P_{9,2^j+k+1}(x)\right|.$$

Norm of the error in Hilbert space $L^2(R)$ [23] is defined as

$$\|E_{2^{J+1}}\|_2^2=\int_0^1\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\left(b_{2^j+k+1}P_{9,2^j+k+1}(x)\right)^2dx$$

$$= \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{r=J+1}^{\infty}\sum_{s=0}^{2^r-1} b_{2^j+k+1} b_{2^r+s+1} \int_0^1 P_{9,2^j+k+1}(x) P_{9,2^r+s+1}(x)\,dx, \tag{23}$$

where, $P_{9,i}(x)$ are the integrals of Haar functions J. Majak et al.[22] have shown that $b_i \le \frac{\zeta}{2^{j+1}}$, for $i = 2^j + k + 1$ and $P_{9,i}(x)$ are monotonically increasing on $[0,1)$. Therefore,

$$\left\|E_{2^{J+1}}\right\|_2^2 \le \frac{\zeta^2}{4} \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{r=J+1}^{\infty}\sum_{s=0}^{2^r-1} \frac{1}{2^j}\frac{1}{2^r} \times \left[\frac{1}{5040}\left(\frac{1}{2^{j+1}}\right)^2 + \frac{1}{1440}\left(\frac{1}{2^{j+1}}\right)^4 + \frac{1}{2160}\left(\frac{1}{2^{j+1}}\right)^6 + \frac{1}{20160}\left(\frac{1}{2^{j+1}}\right)^8\right]$$
$$\times \left[\frac{1}{5040}\left(\frac{1}{2^{r+1}}\right)^2 + \frac{1}{1440}\left(\frac{1}{2^{r+1}}\right)^4 + \frac{1}{2160}\left(\frac{1}{2^{r+1}}\right)^6 + \frac{1}{20160}\left(\frac{1}{2^{r+1}}\right)^8\right], \tag{24}$$

Above equation can be simplified as

$$\left(\begin{array}{l}\text{factrization and } \sum_{r=J+1}^{\infty}\left(\frac{1}{2^{r+1}}\right) = \left(\frac{1}{2^{2m}-1}\right)\times\left(\frac{1}{2^{J+1}}\right)^2, \\ m = 1, 2, 3, 4.\end{array}\right)$$

$$\left\|E_{2^{J+1}}\right\|_2 \le \frac{\zeta}{4320}\left[\frac{1}{7}\left(\frac{1}{2^{J+1}}\right)^2 + \frac{1}{10}\left(\frac{1}{2^{J+1}}\right)^4 + \frac{1}{63}\left(\frac{1}{2^{J+1}}\right)^6 + \frac{1}{2380}\left(\frac{1}{2^{J+1}}\right)^8\right], \tag{25}$$

$$\left\|E_{2^{J+1}}\right\|_2 = O\left[\left(\frac{1}{2^{J+1}}\right)^2\right]. \tag{26}$$

From above equation (26), we can conclude that the convergence is of order two.

NUMERICAL STUDIES

In this section, three numerical experiments are given. The approximate solution for each problem is devised by the HWCM. To show the accuracy of the present method, approximate solutions of the problems are compared with the exact and other numerical methods are available in the literature. All computations are carried out by MATLAB software.

Example 1: Consider the linear boundary value problem [10, 24],

$$u^{(9)}(x) + 9\mathrm{e}^x - u(x) = 0, \qquad x \in (0,1), \tag{27}$$

with boundary conditions:

$$u(0)=1,\ u^{(1)}(0)=0,\ u^{(2)}(0)=-1,\ u^{(3)}(0)=-2, u^{(4)}(0)=-3,\ u(1)=0,\ u^{(1)}(1)=-e,\ u^{(2)}(1)=-2e,$$
$$u^{(3)}(1)=-3e. \tag{28}$$

Its exact solution is $(1-x)e^{x}$

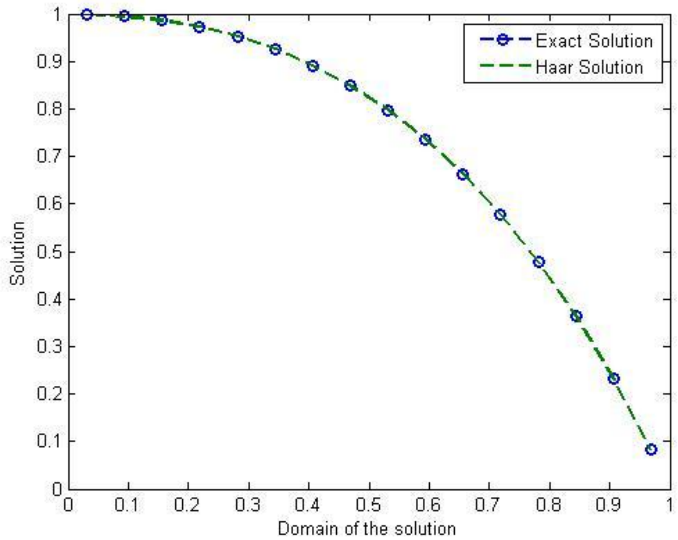

Figure 1: Comparison of exact and approximate solution for J=3 of example 1

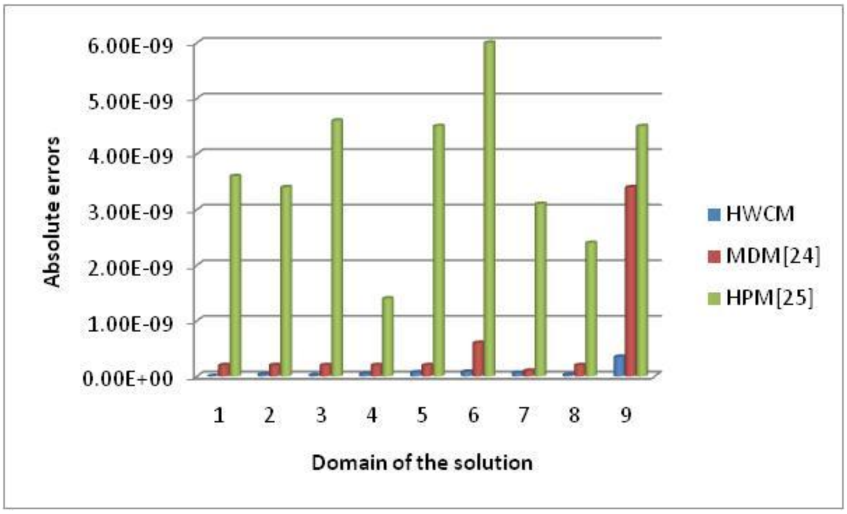

Figure 2: Comparison of absolute errors

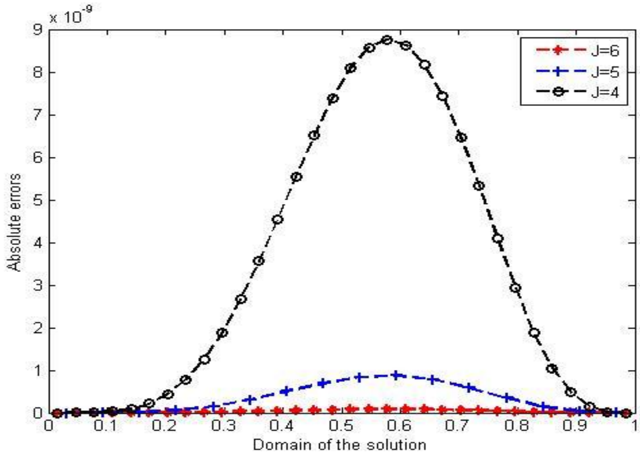

Figure 3: Absolute errors by HWCM with J=4, 5 and 6 for example 1

Example 2: Consider the boundary value problem [25],

$$u^{(9)}(x)+u^{(7)}(x)+xu^{(4)}(x)+u^{(3)}(x)+\sin(x)u^{(1)}(x)+u(x)=5x\sin(x)-\cos(x)+x^2\cos(x)-x\sin^2(x)$$
$$+\sin(x)\cos(x)+x\cos(x), \qquad x\in(0,1), \tag{29}$$

with boundary conditions:

$$u(0)=0,\ u^{(1)}(0)=1,\ u^{(2)}(0)=0,\ u^{(3)}(0)=-3, u^{(4)}(0)=0, u(1)=\cos(1), u^{(1)}(1)=\cos(1)-\sin(1),$$
$$u^{(2)}(1)=-2\sin(1)-\cos(1), u^{(3)}(1)=-3\cos(1)+\sin(1). \tag{30}$$

Its exact solution is x cos(x).

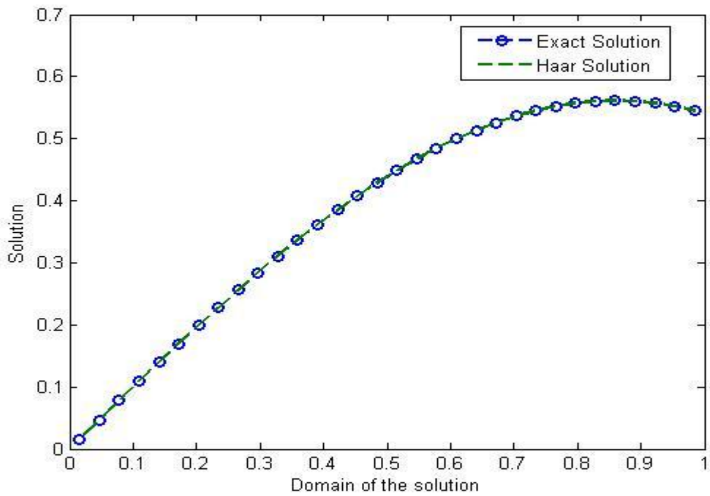

Figure 4: Comparison of exact and approximate solution for J=4 of example 2

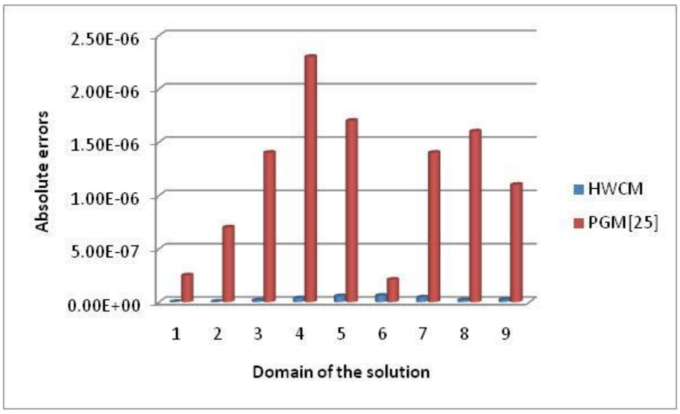

Figure 5: Comparison of absolute errors

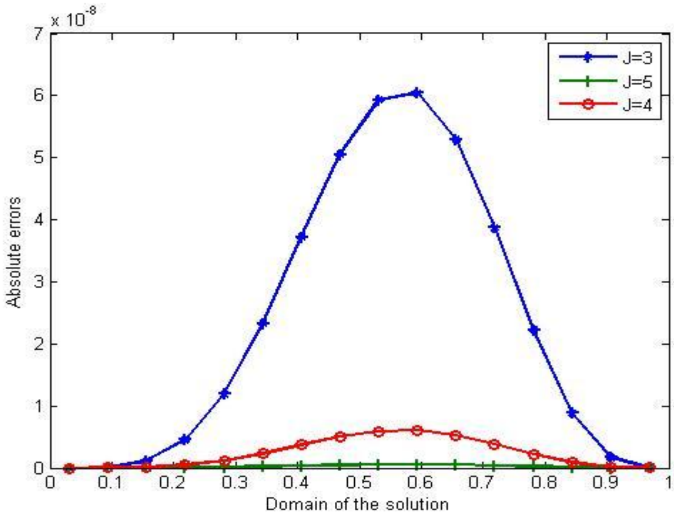

Figure 6: Absolute errors by HWCM with J=3,4 and 5 for example 2

Example 3: Consider the non linear boundary value problem [25],

$$u^{(9)}(x) - u^{(1)}(x)u^2(x) - \cos^3(x) = 0, \qquad x \in (0,1), \tag{31}$$

with boundary conditions:

$$u(0)=0,\ u^{(1)}(0)=1,\ u^{(2)}(0)=0, u^{(3)}(0)=-1, u^{(4)}(0)=0,\ u(1)=\sin(1),\ u^{(1)}(1)=\cos(1),$$
$$u^{(2)}(1)=-\sin(1), u^{(3)}(1)=-\cos(1). \tag{32}$$

Its exact solution is $u(x)=\sin(x)$. With the aid of quasilinearzation technique we converted nonlinear linear BVP(31) into a sequence of linear BVP[26] as

$$u^{(9)}_{(n+1)}(x) - u^2_{(n)}(x)\, u^{(1)}_{(n+1)}(x) - 2u_{(n)}(x)\, u^{(1)}_{(n)}(x)\, u_{(n+1)}(x) = \cos^3(x) - 2\, u^2_{(n)}(x)\, u^{(1)}_{(n)}(x), \quad n=0,1,\ldots \tag{33}$$

with the boundary conditions:

$$u_{(n+1)}(0)=0,\ u^{(1)}_{(n+1)}(0)=1,\ u^{(2)}_{(n+1)}(0)=0,\ u^{(3)}_{(n+1)}(0)=-1,\ u^{(1)}_{(n+1)}(0)=0,\ u_{(n+1)}(1)=\sin(1),\ u^{(1)}_{(n+1)}(1)=\cos(1),$$
$$u^{(2)}_{(n+1)}(1)=-\sin(1),\ u^{(3)}_{(n+1)}(1)=-\cos(1). \quad (34)$$

In this problem we assume that $u_0(x)$ has Maclarian series expansion and we calculated only one iteration i.e. $u_1(x)\approx u(x)$.

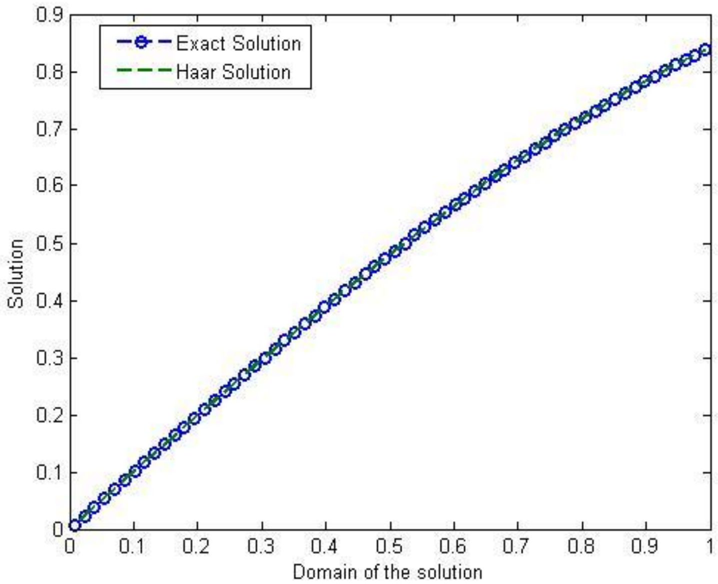

Figure 7: Comparison of exact and approximate solution for J=5 of example 3

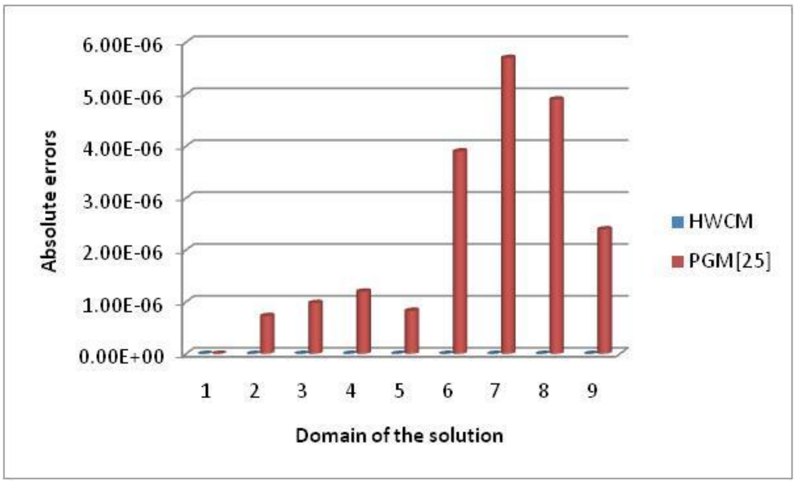

Figure 8: Comparison of absolute errors

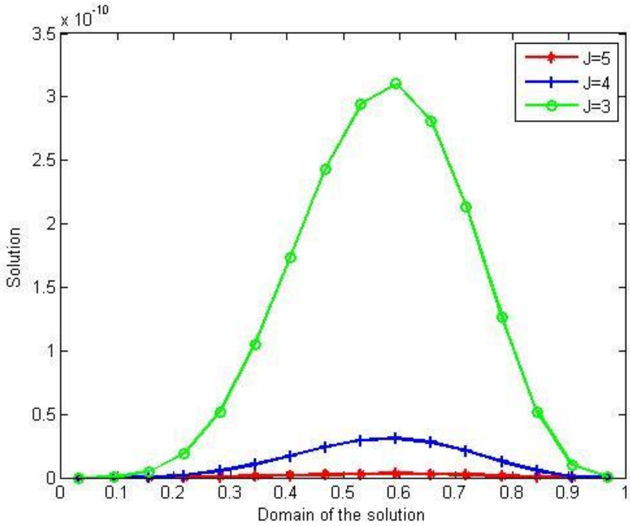

Figure 9: Absolute errors by HWCM with J=3,4 and 5 for example 3

RESULT AND DISCUSSION:

The comparison of approximate and exact solution at collocation points with **J=3, 4** and 5 for **Examples 1, 2 and 3** have been demonstrated in **Figures 1, 4 and 7** respectively. Here in each figure approximate solution coincided with the exact solution, this assures the exactness of HWCM results. **Figures 2, 5 and 8** indicate the comparison of absolute errors obtained by HWCM with MDM, HPM and PGM. These graphs exhibit the results that HWCM has given least absolute errors at each grid point. Absolute errors obtained for **Examples 1, 2 and 3** for various resolutions are drawn in **Figures 3, 6 and 9** conclude that as the resolution value increses absolute error curve approaches to x-axies (where the absolute errors are zero).

CONCLUSION

In this paper, we have employed a Haar wavelet collocation method to solve ninth order boundary value problems. The proposed method has been tested on two linear and one non-linear BVP. Convergence analysis shown that the HWCM is of order two. The numerical results obtained by this method are compared with MDM, HPM and PGM. The strength of the HWCM lies in its easy applicability, accuracy and efficiency to solve ninth order BVPs.

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REFERENCES

[1] Patrick, J. V., 2007, "Introduction: Why wavelets" in Discrete wavelet transformations*:* an elementary approach with applications, John wiely and sons, New Jersey, USA, pp. 1-13.

[2] Lepik, U. and Hein, H., 2014, "Haar wavelets with applications", 4, Springer, pp. 7-20.

[3] Haar A., 1910, "Zur theoric der orthogonalen Funktionsysteme, Math. Annal, 69, pp. 331-371.

[4] Chen, F. and Hsiao, C. H., 1999, " Wavelet approach to optimizing dynamic systems," IEEE Proc. Control Theory. Appl., pp. 146, 213−219.

[5] Chen, C. F., and Hsiao, C. F., 1997, "Haar wavelet method for solving lumped and distributed parameter systems," IEEE Proc. Control TheoryAppl., 144, pp. 87−94.

[6] Lepik, U., 2008,"Haar wavelet method for solving higher order differential equations," Int. J. Math. Comput., 1, pp. 84−94.

[7] Chandrasekhar, S., 1961, "Hydrodynamic and Hydromagnetic Stability", Clarendon Press.

[8] Agarwal, R. P., 1986, "Boundary Value Problems for Higher Order Differential Equations", World Scientific, Singapore.

[9] Chawla, M. M. and Katti C. C., 1979, "Finite difference methods for two-point boundary value problems involving high order differential equations", BIT19, 27-33.

[10] Wazwaz, A. M., 2000, "Approximate solutions to boundary value problems of higher order by the modified decomposition method", Comp. Mat.Appl.40 , 679-691.

[11] Abdel-H,. I. H. H. and Vedat, S. E., 2009, "Solutions of different types of the linear and nonlinear higher order boundary value problems by Differential transformation method", Eur. J. Pur. Appl.Math., 3, pp. 426-447.

[12] Syed, T. M. D. and Ahmet, Y., 2010, "Solution of tenth and ninth order boundary value problems by homotopy perturbation method", J.KSIAM 14, pp. 17-27.

[13] Syed, T. M. D and Ahmet, Y., 2010, "Solutions of tenth and ninth order boundary value problems by modified variational iterationmethod method", Appl.Appl. Math., 5, pp. 11-25.

[14] Luma, N. M. T. and Samaher, M. Y., 2013, "Solution of higher order boundary value problemsusing Semi-Analytic technique", Ibn Al-Haitham J.Pur.Appl.Sci., 1, pp. 281-291.

[15] Bellal, H. Md. and Shafiqul, I. Md., 2014, “A novel numerical approach for odd higher order boundary value problems”, Mat. Th.and Mod., 4, pp. 1-11.

[16] Samir, K. B., 2015 “Tchebychev polynomial approximations for m^{th} order boundary value problems”, Int. J. Pur.Appl. Math., 1, pp. 45-63.

[17] Siraj-ul-I., Imran, A. and Bozidar S., 2010, “The numerical solution of second order boundary value problems by collocation method with the Haar Wavelets”, Math. Comp. Model., 52, pp. 1577-1590.

[18] Fazal-i-Haq, 2011, “Numerical solution of fourth order boundary value problems using Haar wavelets,” Appl. Math. Sci., 63, pp. 3131−3146.

[19] Fazal-i-Haq, A. Arshed, and Iltaf, H., 2012, “ Numerical solution of sixth-order boundaryvalue problems by collocation method using Haar wavelets,”Appl. Math. Sci., 43, pp. 5729-5735.

[20] Padmanabha Reddy, A., Manjula, S. H., Sateesha, C. and Bujurke, N. M., 2016, “Haar wavelet approach for the solution of seventh order ordinary differential equations”, Math. Mod. Eng. Prob., 3, 108-114.

[21] Padmanabha Reddy, A., Sateesha, C., Manjula, S. H., 2016, “Application of Haar wavelet collocation method to solve the fifth order ordinary differential equations”, Int. J. Math. Archi., 7, pp. 53-62.

[22] Majak, J., Shvartsman, B. S., Kirs, M., Pohlak, M. and Herranen, M., 2015,“Convergence theorem for the Haar wavelet based discretization method”, Compos. Struct., 126, pp. 227-232.

[23] Majak, J., Shvartsman, B., Karjust, K., Mikola, M., Haavajoe, A. and Pohlak, M., 2015,“On the accuarcy of the Haar wavelet discretization method, Compos. Part B, 80, pp. 321-327.

[24] Jafar, S. N. and Shirin, Z., 2010, “Homotopy perturbation method for solving higher order boundary value problems”, Appl. Math.Comp. Sci. 2, pp. 199-224.

[25] KasiViswanadham, K. N. S., Reddy, S. M., 2015, “Numerical Solution of Ninth Order Boundary Value Problems by Petrov-GalerkinMethod with Quintic B-splines as Basis Functions and Septic B-splines as Weight Functions”, Elsevier, 127 , pp. 1227 – 1234.

[26] Bellman, R. E. and Kabala, R. E., 1965, “Quasilinearization and nonlinear boundary value problems”, Amer. Elsevier, New York, pp.13-55.