Sequence Dependent Flow Shop Scheduling With Job Block Criteria

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Abstract

The majority of research on scheduling assumes setup times negligible or part of the processing time. In this paper, a bicriteria scheduling with a sequence dependent setup time (SDST) and job block criteria is considered. The objective function of the problem is minimization of the total completion time and the rental cost of machines taken on rent under a specified rental policy. The processing time of attributes on these machines are associated with probabilities. The scheduling problems considering either of these objectives are NP-hard, so exact optimization techniques are impractical. A heuristic algorithm to find optimal or near optimal sequence of jobs processing is
discussed. The performance of the proposed algorithm is justified by bi-objective in-out flow table of jobs.

**Keywords:** Bicriteria Scheduling, Job Block, Sequence dependent setup time, Processing time, Rental Cost, Utilization time, etc.

I. INTRODUCTION

Scheduling problems exist almost everywhere in real industry world situations. Scheduling involves determination of the order of processing a set of tasks on resources or machines. Bi-criteria flow shop scheduling problem with sequence dependent setup time have been an escalating attention of researchers and managers in recent years. The bicriteria scheduling problems are motivated by the fact that they are more meaningful from practical point of view. In some applications setup time has a major impact on the performance measure considered for scheduling problem. Setup time includes work to prepare the machine, process or bench for product parts or the cycle. Scheduling problems involving setup times can be divided into two classes: the first class is sequence-independent and second is sequence-dependent setup times. The term “sequence-dependent” implies that the setup time depends on the sequence in which the jobs are processed on the machines. Each job $J_i$ is characterized by some attributes. The processing time of attribute of job $J_i$ on machine $k$ is denoted by $a_{i,k}$. If job $J_i$ is processed immediately after job $J_j$, a setup time $s_{ij,k}$ is required on machine $k$. Scheduling with sequence dependent setup time has received significant attention in recent years. Corwin and Esogbue [3] minimized makespan by considering sequence dependent setup time. Gupta [4] proposed a branch and bound algorithm to minimize setup cost in $n$ jobs and $m$ machines flowshop with sequence dependent set up time. The scheduling literature reveals that the research on bi-criteria is mainly focused on the single-machine or two machine problems without sequence dependent setup time. Rahimi-Vaheda et al. [12] considered a bicriteria no-wait flowshop scheduling problem in which weighted mean completion time and weighted mean tardiness are minimized. Some of the noteworthy heuristic approaches are due to Bagga and Bambi [1], Blazewicz et al.[2], Gajpal et al. [5], Gupta and Sharma [6,7], Lee and Jung [9], Sen and Gupta [12], Smith [13], Pugazhendi et al. [14], Van Wassenhove and Gelders [15].

The present paper is an attempt to extend the study made by Gupta, Sharma and Nailwal [8] by introducing the concept of job block in bicriteria scheduling with sequence dependent setup time. The idea of job block has a practical significance to create a balance between a cost of providing priority in service and cost of providing service with non priority customers, i.e. how much is to be charged extra from the priority customer(s) as compared to non priority customer(s). The two criteria of minimizing the maximum utilization of machines or rental cost and minimizing the maximum makespan are one of the combinations of our objective function reflecting the performance measure.
II. PRACTICAL SITUATIONS

Sequence dependent setup times are usually found in the situation where the facility is a multipurpose machine. Some examples of sequence dependent setup time flowshop scheduling problem include:

(a) **Textile industry**: where setup times are significant as fabric types are assigned to loom equipped with wrap chains, when the fabric type is changed on a machine, the wrap chain must be replaced and the time it takes depends on the previous and current fabric type;

(b) **Stamping plants**: used by most auto-makers, in such plants, sequence dependent setup time exists between manufacturing parts involves the changing of heavy dies;

(c) **Chemical compounds manufacturing**: where the extent of the cleansing depends on both the chemical most recently processed and the chemical about to be processed;

(d) **Printing industry**: where the cleaning and setting of the press for processing the next job depend on its difference from the colour of ink, size of paper and types used in the previous job;

The case of sequence dependent setups can be found in numerous other industrial systems also, like pharmaceutical, die changing, automobile industry and roll slitting in the paper industry.

Also, various practical situations occur in real life when one has got the assignments but does not have one’s own machine or does not have enough money or does not want to take risk of investing huge amount of money to purchase machine. Under these circumstances, the machines have to be taken on rent in order to complete the assignments. Renting of machines is an affordable and quick solution for an industrial setup, which are presently constrained by the availability of limited funds due to the recent global economic recession. Renting enables saving working capital, gives option for having the equipment, and allows up gradation to new technology.

III. ASSUMPTIONS AND NOTATIONS

The proposed algorithm is based on the following assumptions:
1. All the jobs and machines are available at the beginning of the processing.
2. Pre-emption of jobs is not allowed.
3. Machines never breakdown and are available throughout the scheduling process.
4. All processing times of the machines are deterministic, finite and independent of sequence of the jobs to be processed.
5. Each job is processed through each of the machine once and only once. A job is not available to the next machine until and unless processing on the current machine is completed.

The following notations will be used all the way through this present paper:

- $S$: Sequence of jobs 1, 2, 3... n
- $S_j$: Sequence obtained by applying Johnson’s procedure
- $M_k$: Machine $k$, $k=1,2$
\[ a_{i,k} \]: Processing time of \( i^{th} \) attribute on machine \( M_k \)
\[ p_{i,k} \]: Probability associated to the processing time \( a_{i,k} \)
\[ A_{i,k} \]: Expected processing time of \( i^{th} \) attribute on machine \( M_k \)
\[ J_i \]: \( i^{th} \) job, \( i = 1, 2, 3 \ldots n \)
\[ S_{ij,k} \]: Setup time if job \( i \) is processed immediately after job \( j \) on \( k^{th} \) machine
\[ L_d(S) \]: The latest time when machine \( M_k \) is taken on rent for sequence \( S \)
\[ t_{ij,k}(S) \]: Completion time of \( i^{th} \) job processed immediately after \( j^{th} \) job for sequence \( S \) on machine \( M_k \)
\[ t'_{ij,k}(S) \]: Completion time of \( i^{th} \) job processed immediately after \( j^{th} \) job for sequence \( S \) on machine \( M_k \) when machine \( M_k \) start processing jobs at time \( L_d(S) \)
\[ A_{i,k} = a_{i,k} \times p_{i,k} \times p \]: Expected processing time of \( i^{th} \) attribute on \( k^{th} \) machine for a particular job say \( J_n \).
\[ I_{i,k}(S) \]: Idle time of machine \( M_k \) for job \( i \) in the sequence \( S \)
\[ U_k(S) \]: Utilization time for which machine \( M_k \) is required, when \( M_k \) starts processing jobs at time \( L_d(S) \)
\[ R(S) \]: Total rental cost for the sequence \( S \) of the machines
\[ C_i \]: Rental cost of \( i^{th} \) machine
\[ \beta \]: Equivalent jobs for job block

**IV. PROBLEM FORMULATION**

The machines will be taken on rent as and when they are required and are returned back as and when they are no longer required, i.e. the first machine will be taken on rent in the starting of the processing the jobs, \( 2^{nd} \) machine will be taken on rent at time when \( 1^{st} \) job is completed on \( 1^{st} \) machine and is in ready mode for processing on \( 2^{nd} \) machine.

Completion time of \( i^{th} \) job processed immediately after \( j^{th} \) job for sequence \( S \) on machine \( M_k \) is defined as:

\[
t_{ij,k} = \max (t_{(i-1)j,k}, t_{ij,k-1}) + a_{i,k} \times p_{i,k} + S_{ij,k} \quad \text{for } k \geq 2.
\]

\[
t_{ij,k} = \max (t_{(i-1)j,k}, t_{ij,k-1}) + A_{i,k} + S_{ij,k},
\]

where; \( A_{i,k} = a_{i,k} \times p_{i,k} \) = Expected processing time of \( i^{th} \) attribute on \( k^{th} \) machine for a particular job say \( J_n \).

Also, Completion time if \( i^{th} \) job processed immediately after \( j^{th} \) job on machine \( M_k \) at latest time \( L_k \) is defined as

\[
t_{ij,k} = L_k + \sum_{q=1}^{i} A_{q,k} + \sum_{r=1}^{i-1} S_{q,j,k} = \sum_{q=1}^{i} I_{q,k} + \sum_{q=1}^{i} A_{q,k} + \sum_{r=1}^{i-1} S_{q,j,k}.
\]

Also, \( t'_{ij,k} = \max(t_{(i-1)j,k}, t'_{ij,k-1}) + A_{i,k} + S_{ij,k} \).

Let some job \( J_i \) (\( i = 1, 2 \ldots n \)) are to be processed on two machines \( M_k (k = 1, 2) \) under the specified rental policy. Let there are \( n \) attributes of jobs on machine \( M_1 \) and \( m \) attributes of jobs are on machine \( M_2 \). Let \( a_{j,k} \) be the processing time of \( j^{th} \) attribute on \( k^{th} \) machine with probabilities \( p_{j,k} \). Let \( A_{j,k} \) be the expected processing time and \( S_{j,k} \) be the setup time if job \( i \) is processed immediately after job \( j \) on machine \( k \). Our aim is to find the sequence \( \{S\} \) of the jobs which minimize the rental cost of the machines while minimizing total elapsed time.
The mathematical model of the problem in matrix form can be stated as:

**TABLE 1: ATTRIBUTES OF JOBS**

<table>
<thead>
<tr>
<th>Machine $M_2$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-</th>
<th>j</th>
<th>-</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$J_1$</td>
<td>-</td>
<td>$J_2$</td>
<td>$J_3$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>$J_4$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>$J_5$</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>$J_6$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>i</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>n</td>
<td>$J_{n-1}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>$J_n$</td>
</tr>
</tbody>
</table>

Each job is characterized by its first attribute (row) on the first machine and second attribute (column) on the second machine.

The processing times for various attributes on machine $M_1$ and $M_2$ are as shown in table 2.

**TABLE 2 PROCESSING TIME WITH PROBABILITIES**

<table>
<thead>
<tr>
<th>Machine $M_1$</th>
<th>Attributes</th>
<th>Machine $M_2$</th>
<th>Attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_{1,1}$</td>
<td>$p_{1,1}$</td>
<td>$a_{1,2}$</td>
</tr>
<tr>
<td>2</td>
<td>$a_{2,1}$</td>
<td>$p_{2,1}$</td>
<td>$a_{2,2}$</td>
</tr>
<tr>
<td>3</td>
<td>$a_{3,1}$</td>
<td>$p_{3,1}$</td>
<td>$a_{3,2}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>m</td>
<td>$a_{m,1}$</td>
<td>$p_{m,1}$</td>
<td>$a_{m,2}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>n</td>
<td>$a_{n,1}$</td>
<td>$p_{n,1}$</td>
<td>-</td>
</tr>
</tbody>
</table>

The setup times for various attributes on machine $M_k$ ($k=1,2$) is as shown in table 3.

**TABLE 3: SETUP TIME ON MACHINE $M_k$**

<table>
<thead>
<tr>
<th>Attributes</th>
<th>Machine $M_k$</th>
<th>Attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>$S_{12,k}$</td>
<td>$S_{13,k}$</td>
</tr>
<tr>
<td>3</td>
<td>$S_{31,k}$</td>
<td>$S_{32,k}$</td>
</tr>
<tr>
<td>i</td>
<td>$S_{i1,k}$</td>
<td>$S_{i2,k}$</td>
</tr>
<tr>
<td>n</td>
<td>$S_{n1,k}$</td>
<td>$S_{n2,k}$</td>
</tr>
</tbody>
</table>

(If the attribute in row $i$ is processed immediately after the attribute in column $j$)

Mathematically, the problem can be stated as
Minimize $U_k(S)$ and
Minimize $r(S) = t_{n,1} \times C_1 + U_k(S) \times C_2$
Subject to constraint: Rental Policy.

V. THEOREMS
The following theorems have been established to find the optimal sequence of jobs processing.

**Theorem 1:** In processing a schedule $S = \{J_1, J_2, J_3, \ldots, J_k, J_{k+1}, \ldots, J_n\}$ of $n$ jobs on two machines $M_1$ and $M_2$ in the order $M_1M_2$ with no passing allowed. The job block $(J_k, J_{k+1})$ having processing times $\{A_{k,1}, A_{k,2}, A_{(k+1),1}, A_{(k+1),2}\}$ is equal to the single job $\beta$. The processing time of job block $\beta$ on machine $M_1$ and $M_2$ denoted respectively by $A_{\beta,1}$ and $A_{\beta,2}$ are given by

$$A_{\beta,1} = A_{k,1} + A_{k+1,1} - \min\{A_{k,2}, A_{k+1,1}\}$$
$$A_{\beta,2} = A_{k,2} + A_{k+1,2} - \min\{A_{k,2}, A_{k+1,1}\}$$

Proof: Let $C_{i,j}$ denote the completion time of $i^{th}$ job $(k = 1, 2, 3, \ldots, n)$ on $l^{th}$ machine ($l = 1, 2$) for the sequence $S$ of jobs.

Therefore, by definition

$$C_{k,2} = \max\{C_{k,1}, C_{k-1,2}\} + A_{k,2} = \max\{C_{k,1} + A_{k,2}, C_{k-1,2} + A_{k,2}\}$$
$$C_{k+1,2} = \max\{C_{k+1,1}, C_{k,2}\} + A_{k+1,2}$$
$$= \max\{C_{k+1,1} + A_{k+1,2}, C_{k,1} + A_{k,2}, C_{k-1,2} + A_{k,2}\} + A_{k+1,2}$$
$$= \max\{C_{k+1,1} + A_{k+1,2}, C_{k,1} + A_{k,2}, C_{k-1,2} + A_{k,2} + A_{k+1,2}\}$$

Since, $C_{k+1,1} = C_{k,1} + A_{k+1,1}$

$$C_{k+1,2} = \max\left\{C_{k+1,1} + A_{k+1,2}, C_{k,1} + A_{k,2}, C_{k-1,2} + A_{k,2} + A_{k+1,2}\right\}$$

Also, $C_{k+2,2} = \max\{C_{k+2,1}, C_{k+1,2}\} + A_{k+2,2}$

$$= \max\left\{C_{k+2,1} + A_{k+2,1} + A_{k+1,2}, C_{k+1,1} + A_{k+1,2}, C_{k,1} + A_{k,2} + A_{k+1,2}\right\} + A_{k+2,2}$$

Since, $C_{k+2,1} = C_{k+1} + A_{k+1,1} + A_{k+2,1}$

Therefore, we have

$$C_{k+2,2} = \max\left\{C_{k+1,1} + A_{k+1,1} + A_{k+2,1}, C_{k,1} + A_{k,1} + A_{k+1,2}\right\} + A_{k+2,2}$$

$$= \max\left\{C_{k+1,1} + A_{k+1,1} + A_{k+2,1}, C_{k,1} + A_{k,1} + A_{k+1,2}\right\} + A_{k+2,2}$$

Since,

$$C_{k+1,1} = C_{k,1} + A_{k+1,1}$$

$$= \max\left\{C_{k+1,1} + A_{k+1,1}, C_{k,1} + A_{k,1}\right\} + A_{k+1,2}$$
Therefore, we have

\[ C_{k+2,2} = \max \left\{ \frac{C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}}{3}, \frac{C_{k+2,1} + A_{k+1,2} + A_{k+2,2}}{2} \right\} + A_{k+2,2} \]  --- (1)

Also, \[ C_{k+2,1} = C_{k-1,1} + A_{k,1} + A_{k+1,1} + A_{k+2,1} = C_{k,1} + A_{k+1,1} + A_{k+2,1} \]  --- (2)

Now, let us define a sequence \( \mathcal{S} \) of jobs as

\[ \mathcal{S} = \{ J_1, J_2, J_3, \ldots, J_{k-1}, \beta, J_{k+2}, \ldots, J_n \} \]

Where;

\[ A_{\beta,1} = A_{k,1} + A_{k+1,1} - c \]  --- (3)
\[ A_{\beta,2} = A_{k,2} + A_{k+1,2} - c ; \quad c \text{ is a constant.} \]  --- (4)

Let \( C_{k,1} \) denote the completion time of the \( k^{th} \) job \((k = 1, 2, 3, \ldots, n)\) on the \( l^{th} \) machine \((l = 1, 2)\) for the sequence \( \mathcal{S} \) of jobs.

Therefore, by definition

\[ C_{k+2,2} = \max \left\{ C_{k+1,1} + C_{\beta,2} + A_{k+2,2}, C_{k+1,2} + A_{k+2,2}, C_{k+1,2} + A_{k+2,2}, C_{k+1,2} + A_{k+2,2} \right\} \]

Since, \[ C_{k+2,1} = C_{k-1,1} + A_{\beta,1} + A_{k+1,1} \]

Also, \[ C_{\beta,2} = C_{k-1,1} + A_{\beta,1} + A_{k+1,1} - c = C_{k,1} + A_{k+1,1} - c \]  --- (7)

On combining the results (3), (4), (5), (6) and (7), we have

\[ C_{k+2,2} = \max \left\{ C_{k+1,1} + A_{k+1,1} - c + A_{k+2,1} + A_{k+2,2} + A_{k+1,2} - c + A_{k+2,2} + A_{k+1,2} - c + A_{k+2,2} + A_{k+1,2} - c \right\} + A_{k+2,2} \]  --- (8)

Let \( c = \min \{ A_{k+1,1}, A_{k,2} \} \), then

\[ A_{k+1,1} - c + A_{k,2} = A_{k+1,1} - \min \{ A_{k+1,1}, A_{k,2} \} + A_{k,2} = \max \{ A_{k+1,1}, A_{k,2} \} \]  --- (9)

Also, \[ C_{k+2,1} = C_{k-1,2} \]  --- (11)

On combining results (8), (9), (10) and (11), we have

\[ C_{k+2,2} = \max \left\{ C_{k,1} + A_{k+1,1} + A_{k+2,1} - c + A_{k+2,2} - c, C_{k,1} + A_{k+1,1} - c, C_{k,1} + A_{k+1,1} - c + A_{k+2,1} + A_{k+2,2} - c, A_{k+1,2} + A_{k+2,2} - c \right\} + A_{k+2,2} \]

\[ = \max \left\{ C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, A_{k+1,2} + A_{k+2,2} - c \right\} \]

\[ = \max \left\{ C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, C_{k,1} + A_{k+1,1} + A_{k+2,1} + A_{k+1,2} + A_{k+2,2}, A_{k+1,2} + A_{k+2,2} - c \right\} \]  --- (12)

From (1) and (12), we have

\[ C_{k+2,2} = C_{k+2,2} - c \]  --- (13)

From (2) and (6), we conclude that
\[ C'_{k+2,1} = C_{k+2,1} - c \quad \text{--- (14)} \]

From results (13) and (14), we observe that the replacement of job-block \((J_k, J_{k+1})\) in \(S\) by job \(\beta\) decreases the completion times of the later job \(J_{k+1}\) on both the machines by a constant \(c\) in \(S'\). i.e. if \(T\) and \(T'\) be the completion times of sequence \(S\) and \(S'\), then we have \(T = T' - c\), i.e. the completion times on both the machines are changed by a value which is independent of the particular sequence \(S\). Hence, the substitution does not change the relative merit of different sequences. Hence, job block \(\beta\) is equivalent to job block \((J_k, J_{k+1})\).

**Theorem 2:** The processing of jobs on \(M_2\) at time \(L_2 = \sum_{i=1}^{n} t_i\) keeps \(t_{nj,2}\) unaltered.

**Proof.** Let \(t'_{nj,2}\) be the completion time of \(n^{th}\) job processed immediately after \(j^{th}\) job when \(M_2\) starts processing of jobs at \(L_2\). We shall prove the theorem with the help of mathematical induction. Let \(P(n) : t'_{nj,2} = t_{nj,2}\)

**Basic step:** For \(n = 1\)

\[ t'_{1j,2} = L_2 + \sum_{i=1}^{1} A_{1,2} + S_{1j,2} = L_2 + \sum_{i=1}^{1} A_{1,2} + S_{1j,2} = \sum_{i=1}^{1} I_{1,2} + A_{1,2} = A_{1,1} + A_{1,2} = t_{1j,2} \]

\(\therefore\) \(P(1)\) is true.

**Induction step:** Let \(P(m)\) be true, i.e., \(t'_{mj,2} = t_{mj,2}\)

Now, we shall show that \(P(m+1)\) is also true, i.e., \(t'_{(m+1)j,2} = t_{(m+1)j,2}\)

Since \(t'_{(m+1)j,2} = \max\left(t_{(m+1)j,1}, t'_{mj,2}\right) + A_{m+1,2} + S_{mj,2}\)

\[ = \max\left(t_{(m+1)j,1}, L_2 + \sum_{i=1}^{m} A_{i,2} + \sum_{i=1}^{m-1} S_{ij,2}\right) + A_{m+1,2} + S_{mj,2} \]

\[ = \max\left(t_{(m+1)j,1}, \sum_{i=1}^{m} I_{i,2} + \sum_{i=1}^{m-1} A_{i,2} + \sum_{i=1}^{m-1} S_{ij,2} + I_{m+1}\right) + A_{m+1,2} + S_{mj,2} \]

\[ = \max\left(t_{(m+1)j,1}, I_{m+1}\right) + A_{m+1,2} + S_{mj,2} \]

\[ = \max\left(t_{(m+1)j,1}, t'_{mj,2}\right) + \max\left(t_{(m+1)j,1} - t'_{mj,2}, 0\right) + A_{m+1,2} + S_{mj,2} \]

\[ = \max\left(t_{(m+1)j,1}, t'_{mj,2}\right) + A_{m+1,2} + S_{mj,2} = t_{(m+1)j,2} \]

Therefore, \(P(m+1)\) is true whenever \(P(m)\) is true.

Hence, by the principle of mathematical induction \(P(n)\) is true for all \(n\) i.e. \(t'_{nj,2} = t_{nj,2}\) for all \(n\).
VI. ALGORITHM
The following algorithm is proposed to optimize the bicriteria in two stage flowshop scheduling in which the processing times are associated with probabilities under sequence dependent setup time. The bicriteria problem addressed in this research can be referred to as \( F_2 / S_{sd} / R(S), C_{max} \).

**Step 1:** Calculate the expected processing times of the given attributes on the two machines \( M_1 \) and \( M_2 \) as follows
\[
A_{i,j} = a_{i,j} \times p_{i,j} \quad \forall i, j \quad (i = 1, 2, \ldots, m; j = 1, 2)
\]

**Step 2:** Take equivalent job \( \beta(k_1, m_1) \) and calculate the processing time \( A_{\beta,1} \) and \( A_{\beta,2} \) on the guide lines of Maggu and Das [11] as follows
\[
A_{\beta,1} = A_{k_1,1} + A_{m_1,1} - \min(A_{m_1,1}, A_{k_1,2})
\]
\[
A_{\beta,2} = A_{k_1,2} + A_{m_1,2} - \min(A_{m_1,1}, A_{k_1,2})
\]

**Step 3:** Define a new reduced problem with the processing times \( A_{i,j} \) as defined in step 1 and jobs \( (k_1, m_1) \) are replaced by single equivalent job \( \beta \) with processing time \( A_{\beta,j} (j = 1, 2) \) as defined in step 2.

**Step 4:** Using Johnson’s technique (1954), obtain the sequences \( S_j \) having minimum total elapsed time. Let these be sequences be \( S_1, S_2, \ldots \).

**Step 5:** Compute total elapsed time \( t_{n,2}(S_j), j = 1, 2, 3, \ldots \) of second machine by preparing in-out tables for sequence \( S_j \).

**Step 6:** Compute \( L_2(S_j) \) for each sequence \( S_j \) as \( U_2(S_j) = t_{n,2}(S_j) - L_2(S_j) \).

**Step 7:** Find utilization time of 2nd machine for each sequence \( S_k \) as \( U_2(S_j) = t_{n,2}(S_j) - L_2(S_j) \).

**Step 8:** Find minimum of \( \{[U_2(S_j)] \}; j = 1, 2, 3, \ldots \).

Let it for sequence \( S_p \). Then \( S_p \) is the optimal sequence and minimum rental cost for the sequence \( S_p \) is \( R(S_p) = t_{n,1}(S_p)\times C_1 + U_2(S_p)\times C_2 \).

VII. NUMERICAL ILLUSTRATION
Consider a two stage furniture production system where each stage represents a machine. At stage one, sheets of raw materials (MDF, DDF, Plywood, Plyboard etc.) are cut and subsequently painted in the second stage according to the market demand. The painted pieces are then assembled on an assembly line and delivered to the customers. A setup change over is needed in cutting department when the thickness of two successive jobs differs substantially. In the painting department, a setup is required when the colour of two successive jobs changes. The setup times are sequence dependent. Further the machines \( M_1 \) and \( M_2 \) are taken on rent under rental policy \( P \).
Consider an instance consisting of seven jobs which are processed on two machines. On the first machine, there are four different attributes while the second machine is capable of handling six attributes. The attributes, processing times as well as setup times on the first and second machine are shown in tables 5, 6, 7 and 8 respectively.

**TABLE 5 : ATTRIBUTES OF JOBS**

<table>
<thead>
<tr>
<th>Machine M_2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine M_1</td>
<td>1</td>
<td>-</td>
<td>J_1</td>
<td>-</td>
<td>J_2</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>J_3</td>
<td>-</td>
<td>J_5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>J_6</td>
<td>-</td>
<td>-</td>
<td>J_4</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>J_7</td>
</tr>
</tbody>
</table>

**TABLE 6: PROCESSING TIMES OF ATTRIBUTES WITH PROBABILITIES**

<table>
<thead>
<tr>
<th>Attributes</th>
<th>Machine M_1</th>
<th>Machine M_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>0.2</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**TABLE 7: SETUP TIME ON M_1**

<table>
<thead>
<tr>
<th>Attributes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attributes</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

(If the attribute in row \(i\) is processed immediately after the attribute in column \(j\))
(If the attribute in row \(i\) is processed immediately after the attribute in column \(j\))

Let the jobs 3 and 5 are processed as a job block \(\beta = (3, 5)\). The rental cost per unit for the Machines \(M_1\) and \(M_2\) be 8 units and 10 units respectively. Our objective is to find the sequence of jobs processing with minimum possible rental cost, when the machines are taken on rent under the specified rental policy.

Solution: As per steps 1, 2 and 3: The expected processing times of the jobs on two machines for the possible attributes is

<table>
<thead>
<tr>
<th>Attributes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>-</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>2</td>
<td>8</td>
<td>6</td>
<td>5</td>
<td>-</td>
</tr>
</tbody>
</table>

As per step 4: Using Johnson’s technique [1], the sequence \(S_p\) having minimum total elapsed time is


The In-Out flow table of jobs for the sequence \(S_p\)

<table>
<thead>
<tr>
<th>Jobs</th>
<th>(J_1)</th>
<th>(J_2)</th>
<th>(J_\beta)</th>
<th>(J_4)</th>
<th>(J_6)</th>
<th>(J_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_1)</td>
<td>2.4</td>
<td>2.4</td>
<td>7.0</td>
<td>3.3</td>
<td>3.3</td>
<td>2.4</td>
</tr>
<tr>
<td>(M_2)</td>
<td>1.6</td>
<td>1.2</td>
<td>2.0</td>
<td>2.6</td>
<td>1.6</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Therefore, Total elapsed time \(t_{n,2}(S_p) = 34.0\) units
The latest time at which Machine $M_2$ should be taken on rent

$$L_2(S_p) = t_{n,2}(S_p) - \sum_{q=1}^{n} A_{q,2}(S_p) - \sum_{j=1}^{n-2} S_{j,2}(S_p)$$

$$= 34.0 - 15 - 14 = 5 \text{ units.}$$

The bi-objective In-Out flow table for the sequence $S$ is

<table>
<thead>
<tr>
<th>Jobs</th>
<th>Machine $M_1$</th>
<th>Machine $M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>In – Out</td>
<td>In – Out</td>
</tr>
<tr>
<td>$J_7$</td>
<td>0.0 – 2.4</td>
<td>5.0 – 8.0</td>
</tr>
<tr>
<td>$J_4$</td>
<td>4.4 – 7.7</td>
<td>13.0 – 15.6</td>
</tr>
<tr>
<td>$J_3$</td>
<td>8.7 – 12.7</td>
<td>18.6 – 21.6</td>
</tr>
<tr>
<td>$J_5$</td>
<td>12.7 – 16.7</td>
<td>22.6 – 24.6</td>
</tr>
<tr>
<td>$J_1$</td>
<td>18.7 – 21.1</td>
<td>26.6 – 28.2</td>
</tr>
<tr>
<td>$J_6$</td>
<td>23.1 – 26.4</td>
<td>28.2 – 29.8</td>
</tr>
<tr>
<td>$J_2$</td>
<td>30.4 – 32.8</td>
<td>32.8 – 34.0</td>
</tr>
</tbody>
</table>

Therefore, the utilization time of Machine $M_2$ is

$$U_2(S_p) = t_{n,2}(S_p) - L_2(S_p) = 34 - 5 = 29 \text{ units}$$

Total Minimum Rental Cost = $R(S_p) = t_{n,1}(S_p) \times C_1 + U_2(S_p) \times C_2 = 552.4 \text{ units.}$

**Remarks:** The utilization time of machine $M_2$ from table 10, by Johnson rule [9] is 31.6 units and the total rental cost of machines under the specified rental policy is 578.4 units.

**VII. CONCLUSION**

If the machine $M_2$ is taken on rent when it is required and is returned as soon as it completes the last job, the starting of processing of jobs at time

$$L_2(S) = t_{n,2}(S) - \sum_{i=1}^{n} A_{i,2}(S) - \sum_{i=1}^{n-1} S_{i,2}(S)$$

on $M_2$ will, reduce its utilization time. Therefore total rental cost of $M_2$ will be minimum. Also rental cost of $M_1$ will always be minimum as idle time of $M_1$ is minimum always due to our rental policy. The study may further be extending by introducing the concept of transportation time, Weightage of jobs, Breakdown Interval etc.
REFERENCES


