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Two Variables Generalization of Jacobi Polynomials

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Abstract

In the present investigation a two variable generalization of the Jacobi Polynomials is introduced and some generating functions are also derived. Further Bateman's generating function, Brafman's generating function, Rodrigues formula, Relation between Legendre and Jacobi polynomials are obtained and some applications of Jacobi polynomials are also studied.

Keywords: Jacobi Polynomials, generating functions, Rodrigues formula.
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1. INTRODUCTION

The classical Jacobi polynomials have been used extensively in mathematical analysis and practical applications (cf. [1, 15-17]). In particular, the Legendre and Chebyshev polynomials have played an important role in spectral methods for partial differential equations (cf. [5-9]). Recently, there have been renewed interests in using the Jacobi polynomials in spectral approximations, especially for problems with degenerated or singular coefficient. For instance, Bernardi and Maday [4] considered spectral approximations using the ultra-spherical polynomials in weighted Sobolev spaces. Guo [10-12] developed Jacobi approximations in certain Hilbert spaces with their applications to singular differential equations and some problems on infinite intervals.

The Jacobi approximations were also used to obtain optimal error estimates for p-version of finite element methods (cf. [2, 3]). Another application of Jacobi type polynomials is in irrationality measures for certain values of binomial functions and definite integrals of some rational functions [13].

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined as ([15]; p. 254 (1))

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n; & \frac{1-x}{2} \\ 1+\alpha; & \end{matrix}\right]. \tag{1.1}$$

When $\alpha=\beta=0$, the polynomials in eq.(1.1) reduce to the Legendre Polynomials.

An elementary generating function of Jacobi polynomial is given by

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x).t^n = \rho^{-1}\left(\frac{2}{1+t+\rho}\right)^{\beta}\left(\frac{2}{1-t+\rho}\right)^{\alpha}, \tag{1.2}$$

or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha}, \tag{1.3}$$

where $\rho = (1-2xt+t^2)^{\frac{1}{2}}$.

Khan and Abukhammash [14] define the two variable Legendre Polynomials $P_n(x,y)$ by series as

$$P_n(x,y) = \sum_{k=0}^{[n/2]} \frac{(-y)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!(n-2k)!}. \tag{1.4}$$

and the generating function for $P_n(x,y)$ is given by

$$\sum_{n=0}^{\infty} P_n(x,y)t^n = (1-2xt+yt^2)^{-\frac{1}{2}}. \tag{1.5}$$

Motivated by their work we introduce two variable generalization of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and many interesting results have been obtained.

2. JACOBI POLYNOMIALS OF TWO VARIABLES:

Jacobi Polynomials of two variables $P_n^{(\alpha,\beta)}(x,y)$ may be defined as:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{y}t+\rho\right)^{-\beta}\left(1-\sqrt{y}t+\rho\right)^{-\alpha}, \tag{2.1}$$

where

$$\rho = \left(1-2xt+yt^2\right)^{\frac{1}{2}}$$

Derivation: Jacobi Polynomials are defined as ([15]; P. 270(2))

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = F_4\left(1+\beta, 1+\alpha; 1+\alpha, 1+\beta; \frac{1}{2}t(x-1), \frac{1}{2}t(x+1)\right). \tag{2.2}$$

Further, we recall the result:

If neither a nor *b* is zero or negative integer,

$$F_4\left(a,b;b,a;\frac{-u}{(1-u)(1-v)},\frac{-v}{(1-u)(1-v)}\right) = (1-uv)^{-1}(1-u)^a(1-v)^b, \tag{2.3}$$

where a =1+β, b = 1+α

$$\frac{-u}{(1-u)(1-v)} = \frac{t(x-1)}{2}, \frac{-v}{(1-u)(1-v)} = \frac{t(x+1)}{2}. \tag{2.4}$$

For two variables extension of Jacobi polynomials, let us consider

$$\rho = \left(1-2xt+yt^2\right)^{\frac{1}{2}}, \quad \text{and}$$

$$u = 1-\frac{2}{1+\sqrt{y}t+\rho}, \quad v = 1-\frac{2}{1-\sqrt{y}t+\rho} \quad . \tag{2.5}$$

Now from eq. (2.5)

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{1-v}\left(1-\frac{1}{1-u}\right)$$

$$= \frac{1-\sqrt{y}t+\rho}{2}\left(1-\frac{1+\sqrt{y}t+\rho}{2}\right)$$

$$= \frac{\left(1-\sqrt{y}t+\rho\right)\left(1-\sqrt{y}t-\rho\right)}{4}$$

$$= \frac{\left(1-\sqrt{yt}\right)^2 - \rho^2}{4}$$

$$= \frac{t\left(x-\sqrt{y}\right)}{2}. \tag{2.6}$$

Similarly

$$\frac{-v}{(1-u)(1-v)} = \frac{t\left(x+\sqrt{y}\right)}{2}. \tag{2.7}$$

Further

$$\frac{1}{1-u} = \frac{1}{2}\left(1+\sqrt{yt}+\rho\right) \quad \text{and} \quad \frac{1}{1-v} = \frac{1}{2}\left(1-\sqrt{yt}+\rho\right),$$

from which we can have

$$\rho = \frac{1}{1-u} + \frac{1}{1-v} - 1$$

$$= \frac{1-uv}{(1-u)(1-v)}.$$

and hence

$$(1-uv)^{-1}(1-u)^a(1-v)^b = \rho^{-1}(1-u)^{a-1}(1-v)^{b-1}.$$

Thus from eq.(2.2) and eq.(2.3) we obtain

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = \rho^{-1}\left(\frac{2}{1+\sqrt{yt}+\rho}\right)^{\beta}\left(\frac{2}{1-\sqrt{yt}+\rho}\right)^{\alpha}$$

or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{yt}+\rho\right)^{-\beta}\left(1-\sqrt{yt}+\rho\right)^{-\alpha}.$$

where

$$\rho = \left(1-2xt+yt^2\right)^{\frac{1}{2}}$$

Which completes the proof.

Further from eq.(2.2), eq.(2.6) and eq.(2.7), we obtain

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = F_4\left(1+\beta,1+\alpha;\ 1+\alpha,1+\beta;\frac{t}{2}\left(x-\sqrt{y}\right),\frac{t}{2}\left(x+\sqrt{y}\right)\right). \tag{2.8}$$

Eq.(2.8) can be put in the form

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n,k=0}^{\infty} \frac{(1+\alpha)_{n+k}(1+\beta)_{n+k} \frac{1}{2}\left(x-\sqrt{y}\right)^k \frac{1}{2}\left(x+\sqrt{y}\right)^n t^n}{k!\ n!\ (1+\alpha)_k (1+\beta)_n}$$

which can be simplified to obtain the generating functions for $P_n^{(\alpha,\beta)}(x,y)$ as:

$$P_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\beta)_n}{k!\ (n-k)!(1+\alpha)_k (1+\beta)_{n-k}} \left(\frac{x-\sqrt{y}}{2}\right)^k \left(\frac{x+\sqrt{y}}{2}\right)^{n-k}, \tag{2.9}$$

$$P_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+\sqrt{y}}{2}\right)^n 2F_1\left[\begin{matrix} -n,-\beta-n; & \frac{x-\sqrt{y}}{x+\sqrt{y}} \\ 1+\alpha; & \end{matrix}\right], \tag{2.10}$$

and

$$P_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n}{n!} 2F_1\left[\begin{matrix} -n,1+\alpha+\beta+n; & \frac{\sqrt{y}-x}{2} \\ 1+\alpha; & \end{matrix}\right]. \tag{2.11}$$

For α = β = 0, (2.10) reduce to Legendre polynomials of two variable $P_n(x,y)$

$$P_n(x,y) = \left(\frac{x+\sqrt{y}}{2}\right)^n 2F_1\left[\begin{matrix} -n,-n; & \frac{x-\sqrt{y}}{x+\sqrt{y}} \\ 1; & \end{matrix}\right]. \tag{2.12}$$

Further eq. (2.11) yields a finite series form for $P_n^{\alpha,\beta}(x,y)$ as

$$P_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\alpha)_k (1+\alpha+\beta)_n} \left(\frac{x-\sqrt{y}}{2}\right)^k. \tag{2.13}$$

To derive a generating function for the Jacobi polynomials from eq. (2.13), we consider the series

$$\psi = \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1+\alpha+\beta)_{n+k}\left(\frac{x}{2}-\frac{\sqrt{y}}{2}\right)^k t^n}{k!(n-k)!(1+\alpha)_k}$$

$$= \sum_{n,k=0}^{\infty} \frac{(1+\alpha+\beta)_{n+2k}\left(x-\sqrt{y}\right)^k t^{n+k}}{k!\, n!(1+\alpha)_k\, 2^k}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+2k)_n t^n}{n!} \frac{(1+\alpha+\beta)_{2k}\left(x-\sqrt{y}\right)^k t^k}{k!\,(1+\alpha)_k\, 2^k}$$

$$= \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_{2k}\left(x-\sqrt{y}\right)^k t^k}{k!\, 2^k\,(1+\alpha)_k (1-t)^{1+\alpha+\beta+2k}}.$$

It is well known that:

$$(a)_{2k} = 2^{2k}\left(\frac{1}{2}a\right)_k \left(\frac{a}{2}+\frac{1}{2}\right)_k.$$

Thus we obtain,

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n} = (1-t)^{-1-\alpha-\beta}\, 2F_1\left[\begin{matrix}\frac{1}{2}(1+\alpha+\beta), \frac{1}{2}(2+\alpha+\beta); & \frac{2t\left(x-\sqrt{y}\right)}{(1-t)^2}\\ (1+\alpha); & \end{matrix}\right]. \quad (2.14)$$

which is another generating relation for the Jacobi polynomials.

3. Bateman's generating function for $P_n^{(\alpha,\beta)}(x,y)$

From eq. (2.9), we obtained

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left[\frac{1}{2}\left(x-\sqrt{y}\right)\right]^k \left[\frac{1}{2}\left(x+\sqrt{y}\right)\right]^{n-k} t^n}{k!(n-k)!(1+\alpha)_k(1+\beta)_{n-k}}$$

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} = \left[\sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(x-\sqrt{y}\right)^n t^n}{n!(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(x+\sqrt{y}\right)^n t^n}{n!(1+\beta)_n}\right]$$

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} = 0F_1\left[\begin{matrix}---; & \frac{t}{2}\left(x-\sqrt{y}\right)\\ (1+\alpha); & \end{matrix}\right]\; 0F_1\left[\begin{matrix}---; & \frac{t}{2}\left(x+\sqrt{y}\right)\\ (1+\beta); & \end{matrix}\right].$$

(3.1) which is Bateman's generating function.

4. THE RODRIGUES FORMULA

Eq. (2.9) can be written as

$$P_n^{(\alpha,\beta)}(x,y)=\sum_{k=0}^{n}\frac{(1+\alpha)_n(1+\beta)_n\left(x-\sqrt{y}\right)^k\left(x+\sqrt{y}\right)^{n-k}}{2^n k!(n-k)!(1+\alpha)_k(1+\beta)_{n-k}} \tag{4.1}$$

We recall that

$$D^s x^{m+\alpha}=\frac{(1+\alpha)_m x^{m-s+\alpha}}{(1+\alpha)_{m-s}}$$

From which we obtained

$$D^k\left(x+\sqrt{y}\right)^{n+\beta}=\frac{(1+\beta)_n\left(x+\sqrt{y}\right)^{n-k+\beta}}{(1+\beta)_{n-k}}$$

and

$$D^{n-k}\left(x-\sqrt{y}\right)^{n+\alpha}=\frac{(1+\alpha)_n\left(x-\sqrt{y}\right)^{k+\alpha}}{(1+\alpha)_k}$$

Thus from eq.(4.1) we can have

$$P_n^{(\alpha,\beta)}(x,y)=\frac{\left(x-\sqrt{y}\right)^{-\alpha}\left(x+\sqrt{y}\right)^{-\beta}}{2^n n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\bullet$$

$$\left[D^{n-k}\left(x-\sqrt{y}\right)^{n+\alpha}\right]\left[D^k\left(x+\sqrt{y}\right)^{n+\beta}\right]$$

Applying Leibnitz rule for derivative of a product we obtain the Rodrigues formula.

$$P_n^{(\alpha,\beta)}(x,y)=\frac{\left(x-\sqrt{y}\right)^{-\alpha}\left(x+\sqrt{y}\right)^{-\beta}}{2^n n!}D^n\left[\left(x-\sqrt{y}\right)^{n+\alpha}\left(x+\sqrt{y}\right)^{n+\beta}\right] \tag{4.2}$$

or

$$P_n^{(\alpha,\beta)}(x,y)=\frac{(-1)^n\left(\sqrt{y}-x\right)^{-\alpha}\left(\sqrt{y}+x\right)^{-\beta}}{2^n n!}D^n\left[\left(\sqrt{y}-x\right)^{n+\alpha}\left(\sqrt{y}+x\right)^{n+\beta}\right]. \tag{4.3}$$

Eq. (4.3) is desirable when we work in the interval $-1 < x < 1$.

5. BRAFMAN'S GENERATING FUNCTIONS.

To obtain the Brafman's generating function consider the sum

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n(\delta)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\gamma)_n(\delta)_n\left[\frac{1}{2}\left(x-\sqrt{y}\right)\right]^k\left[\frac{1}{2}\left(x+\sqrt{y}\right)\right]^{n-k}t^n}{k!(n-k)!(1+\alpha)_k(1+\beta)_{n-k}}$$

$$=\sum_{n,k=0}^{\infty}\frac{(\gamma)_{n+k}(\delta)_{n+k}\left[\frac{1}{2}t\left(x-\sqrt{y}\right)\right]^k\left[\frac{1}{2}t\left(x+\sqrt{y}\right)\right]^n}{k!\ n!(1+\alpha)_k(1+\beta)_n}$$

$$=F_4\left[\gamma,\delta,1+\alpha,1+\beta;\frac{1}{2}t\left(x-\sqrt{y}\right),\frac{1}{2}t\left(x+\sqrt{y}\right)\right]. \tag{5.1}$$

We recall the Theorem ([15], P. 269)

$$F_4\left(a,b,c,1-c+a+b;\frac{-u}{(1-u)(1-v)},\frac{-v}{(1-u)(1-v)}\right)$$

$$=2F_1\left[\begin{matrix}a,b; & \frac{-u}{1-u}\\ c; & \end{matrix}\right]2F_1\left[\begin{matrix}a,b; & \frac{-v}{1-v}\\ 1-c+a+b; & \end{matrix}\right].$$

The F_4 in eq.(5.1) will fit into this theorem if we choose

$$1+\beta=1-(1+\alpha)+\gamma+\delta,$$

or $\delta=1+\alpha+\beta-\gamma$, and

$$\frac{-u}{(1-u)(1-v)}=\frac{t\left(x-\sqrt{y}\right)}{2},\ \frac{-v}{(1-u)(1-v)}=\frac{t\left(x+\sqrt{y}\right)}{2}$$

where u and v are of the section 2 and

$$\frac{-u}{1-u}=1-\frac{1}{1-u}=1-\frac{1}{2}\left(1+\sqrt{y}t+\rho\right)=\frac{1}{2}\left(1-\sqrt{y}t-\rho\right)$$

$$\frac{-v}{1-v}=1-\frac{1}{2}\left(1-\sqrt{y}t+\rho\right)=\frac{1}{2}\left(1+\sqrt{y}t-\rho\right)$$

Hence eq. (5.1) becomes

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n(1+\alpha+\beta-\gamma)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n}$$

$$= {}_2F_1\left[\begin{matrix}\gamma, 1+\alpha+\beta-\gamma; & \dfrac{1-\sqrt{y}t-\rho}{2}\\ 1+\alpha; & \end{matrix}\right] \bullet$$

$$ {}_2F_1\left[\begin{matrix}\gamma, 1+\alpha+\beta-\gamma; & \dfrac{1+\sqrt{y}t-\rho}{2}\\ 1+\beta; & \end{matrix}\right],$$

where ρ = $\left(1-2xt+yt^2\right)^{\frac{1}{2}}$ and γ is arbitrary, which is Brafman's generating function.

6. RELATION BETWEEN LEGENDRE AND JACOBI POLYNOMIALS.

Consider the two variables Jacobi polynomial $P_n^{(\alpha,\beta)}(x,y)$

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{y}t+\rho\right)^{-\beta}\left(1-\sqrt{y}t+\rho\right)^{-\alpha}$$

where $\rho = \left(1-2xt+yt^2\right)^{\frac{1}{2}}$

or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = 2^{\alpha+\beta}\left(1-2xt+yt^2\right)^{-\frac{1}{2}}\left(1+\sqrt{y}t+\rho\right)^{-\beta}\left(1-\sqrt{y}t+\rho\right)^{-\alpha}$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty} P_n(x,y)t^n\left(\frac{1+\sqrt{y}t+\rho}{2}\right)^{-\beta}\left(\frac{1-\sqrt{y}t+\rho}{2}\right)^{-\alpha}$$

$$P_n^{(\alpha,\beta)}(x,y) = P_n(x,y)\left(\frac{1+\sqrt{y}t+\rho}{2}\right)^{-\beta}\left(\frac{1-\sqrt{y}t+\rho}{2}\right)^{-\alpha}. \tag{6.1}$$

Taking, $y = 1$

$$P_n^{(\alpha,\beta)}(x) = P_n(x)\left(\frac{1+t+\rho}{2}\right)^{-\beta}\left(\frac{1-t+\rho}{2}\right)^{-\alpha}. \tag{6.2}$$

Eqs. (6.1) and (6.2) gives a relation between Jacobi polynomials and Legendre polynomials.

7. APPLICATIONS

The formalism developed in the 2nd section can be used to obtain some interesting results involving $P_n^{(\alpha,\beta)}(x,y)$

1. u , v and ρ are chosen as

$$\rho = \left(\xi - 2xt + yt^2\right)^{\frac{1}{2}} \quad ,$$

$$u = 1 - \frac{2\sqrt{\xi}}{\sqrt{\xi} + \frac{\sqrt{yt}}{\sqrt{\xi}} + \rho} \qquad \textit{and} \qquad v = 1 - \frac{2\sqrt{\xi}}{\sqrt{\xi} - \frac{\sqrt{yt}}{\sqrt{\xi}} + \rho}$$

proceeding in the same way as in section 2 , from eq.(2.2), we obtain

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y;\xi)t^n = F_4\left[1+\beta, 1+\alpha; 1+\alpha, 1+\beta; \frac{t}{2}\left(x - \sqrt{\xi}\sqrt{y}\right), \frac{t}{2}\left(x + \sqrt{\xi}\sqrt{y}\right)\right]$$

or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y;\xi)t^n = \left(2\sqrt{\xi}\right)^{\alpha+\beta}\rho^{-1}\left(\sqrt{\xi} + \sqrt{yt} + \rho\right)^{-\beta}\left(\sqrt{\xi} - \sqrt{yt} + \rho\right)^{-\alpha}. \tag{7.1}$$

2. If $\rho = \left(1 - 2(x_1 + x_2)t + yt^2\right)^{\frac{1}{2}}$,

$$u = 1 - \frac{2}{1 + \sqrt{yt} + \rho}, \qquad v = 1 - \frac{2}{1 - \sqrt{yt} + \rho}$$

Then,

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{1-v}\left(1 - \frac{1}{1-u}\right) = \left(\frac{1 - \sqrt{yt} + \rho}{2}\right)\left(1 - \frac{1 + \sqrt{yt} + \rho}{2}\right)$$

$$= \left(\frac{1 - \sqrt{yt} + \rho}{2}\right)\left(\frac{1 - \sqrt{yt} - \rho}{2}\right)$$

$$\frac{t}{2}\left((x_1 + x_2) - \sqrt{y}\right) \tag{i}$$

similarly

$$-\frac{v}{(1-u)(1-v)} = \frac{t}{2}\left((x_1 + x_2) + \sqrt{y}\right) \tag{ii}$$

Using (i) & (ii) in eq.(2.2) we obtained,

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x_1+x_2,y)t^n = F_4\left(1+\beta,1+\alpha;1+\alpha,1+\beta;\frac{t}{2}(x_1+x_2-\sqrt{y}),\frac{t}{2}(x_1+x_2+\sqrt{y})\right)$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x_1+x_2,y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{y}t+\rho\right)^{-\beta}\left(1-\sqrt{y}t+\rho\right)^{-\alpha} \quad (7.2)$$

3. If $\rho = \left(1-2xt+(y_1+y_2)t^2\right)^{\frac{1}{2}}$,

$$u = 1-\frac{2}{1+\left(\sqrt{y_1+y_2}\right)t+\rho}, \qquad v = 1-\frac{2}{1-\left(\sqrt{y_1+y_2}\right)t+\rho}$$

then

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{1-v}\left(1-\frac{1}{1-u}\right)$$

$$= \left(\frac{1-\left(\sqrt{y_1+y_2}\right)t+\rho}{2}\right)\left(1-\frac{1+\left(\sqrt{y_1+y_2}\right)t+\rho}{2}\right)$$

$$= \left(\frac{1-\left(\sqrt{y_1+y_2}\right)t+\rho}{2}\right)\left(\frac{1-\left(\sqrt{y_1+y_2}\right)t-\rho}{2}\right)$$

$$= \frac{1}{4}\left[\left(1-\left(\sqrt{y_1+y_2}\right)t\right)^2-\rho^2\right]$$

$$= \frac{t}{2}\left(x-\sqrt{y_1+y_2}\right) \quad \text{(iii)}$$

similarly

$$\frac{-v}{(1-u)(1-v)} = \frac{t}{2}\left(x+\sqrt{y_1+y_2}\right) \quad \text{(iv)}$$

Now using (iii) & (iv) in eq.(2.2), we get

$$P_n^{(\alpha,\beta)}(x,y_1+y_2)t^n = F_4\left[1+\beta,1+\alpha;1+\alpha,1+\beta;\frac{t}{2}\left(x-\sqrt{y_1+y_2}\right),\frac{t}{2}\left(x+\sqrt{y_1+y_2}\right)\right]$$

$$P_n^{(\alpha,\beta)}(x,y_1+y_2)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\left(\sqrt{y_1+y_2}\right)t+\rho\right)^{-\beta}\left(\left(1-\sqrt{y_1+y_2}\right)t+\rho\right)^{-\alpha}. \quad (7.3)$$

4. If $\rho = \left(1 - 2\lambda x + \mu y t^2\right)^{\frac{1}{2}}$, then from eq.(2.2) we get,

$$P_n^{\alpha,\beta}(\lambda x, \mu y)t^n = F_4\left[1+\beta, 1+\alpha; 1+\alpha, 1+\beta; \frac{t}{2}\left(\lambda x - \mu\sqrt{y}\right), \frac{t}{2}\left(\lambda x + \mu\sqrt{y}\right)\right]$$

$$P_n^{(\alpha,\beta)}(\lambda x, \mu y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\mu\sqrt{y}t+\rho\right)^{-\beta}\left(1-\mu\sqrt{y}t+\rho\right)^{-\alpha}. \qquad (7.4)$$

In this article, generalized Jacobi polynomials and some new generating relations are introduced by using the technique which is flexible to be used for other polynomials also.

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