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Approximate solution of linear optimal control problem with fixed ends

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Abstract

We consider solution of the linear general problem with fixed ends and apply two methods of its solution: Pontryagin maximum principle and Galerkin method. We developed the method of obtaining approximate solution for general problem with special form of objective function and explored its basic properties.

Keywords and Phrases. Optimal control problem, Pontryagin maximum principle, Galerkin method, residual function, trial function.

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1. INTRODUCTION

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is considered as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists, including Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems are different from variation problems due to the additional requirements needed to find a desired solution, and these requirements are sometimes difficult and even impossible to take into consideration when the methods for the calculus of variations are applied. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content. The central results of the theory – the Pontryagin Maximum Principle and the dynamic programming method of Bellman – became widely known in the scientific and engineering community, and these are now widely used in various academic fields.

Optimal control problems are classified on several types: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1,3,6] Our interest is related with the linear general problem with fixed ends and special form of objective function. We consider two methods of solution of this problem: Pontryagin maximum principle and Galerkin method and compare obtained results.

2. STATEMENT OF THE PROBLEM

We consider the problem of optimal control in the form

$$J = \int_{t_0}^{t_1} (c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + \alpha u^2)\,dt \to \min ,$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ldots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ \ldots \\ B_n \end{pmatrix} u , \qquad (1)$$

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \dots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \dots \\ x_n^0 \end{pmatrix}, \quad \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \\ \dots \\ x_n(t_1) \end{pmatrix} = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \dots \\ x_n^1 \end{pmatrix}, \ u \in U \subseteq R,$$

where $u(t)$ is control variable, $x(t) \in R^n$ is state variable, $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \dots \\ x_n^0 \end{pmatrix}$, $x^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \dots \\ x_n^1 \end{pmatrix}$ are fixed ends of trajectory, c_1, c_2 , … , c_n and α are constants, t_0, t_1 are fixed moments of time.

Or in a brief form

$$J = \int_{t_0}^{t_1} (c^T x + \alpha u^2) dt \to \min, \ \dot{x} = Ax + Bu, \ x(t_0) = x^0, \ x(t_1) = x^1, \ u \in U \subseteq R, \quad (2)$$

where A and B are $n \times n$ and $n \times 1$ matrices respectively.

Note that if $c = 0$ minimum of functional in problem (1) (or (2)) corresponds to minimum of energy spending on control.

Definition. Pair functions $x(t)$, $u(t)$ satisfying all conditions of the problem (2) and minimizing the objective function J is called its solution.

The purpose of research is to obtain a solution of the problem (2). For general case of a mathematical model the problem of obtaining the exact solutions is very complicated. The most numerical methods using for solution of the problem (2): Newton's method, Gradient method and so on [9,10] can give an approximate solution with some acceptable or not acceptable accuracy. And the problem of convergence of applied methods has very important role. Numerical solution of the problem (2) was developed by a number of researchers, for instance, L.T. Aschepkov [1,5], F.P. Vasiliev [4,10], R.P. Fedorenko [7]. In general case, when we solve optimal control problem, there isn't a formula for getting unknown initial values for conjugate variables. It's worth to mention that there is a bad convergence of initial approximation for the values of conjugate variables to the values that put zeros for residual functions because of permanent getting by them their local minimum [8,9,10]. The latter means that neither Newton's method nor Gradient method don't give a good result.

In this paper, we show that utilizing of the Galerkin method with the proper choice of trial functions gives an exact solution of the problem (2) that we obtain by means of Pontryagin maximum principle. This advantage of Galerkin method can be generalized on the other types of optimal control problems.

3. MAIN RESULTS

By classification, the problem (2) is the general problem of optimal control [1,5,6], that is, the problem that has mobile ends of an integral curve. It has the form

$$\begin{aligned} &J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min \\ &J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1) \begin{cases} \le 0, i = 1,2,...,m_0, \\ = 0, i = m_0 + 1,..., m, \end{cases} \\ &\dot{x} = f(x,u,t),\ u \in U,\ t_1 \ge t_0. \end{aligned} \tag{3}$$

Here $\Phi_0, \Phi_1,..., \Phi_m$ are the given functions of the class $C_1(R^n \times R^n \times R \times R \to R)$, m_0 is an integer nonnegative number, and m is a natural number. If $m_0 = 0$ or $m_0 = m$, then the general problem only has constraints-equalities $J_i = 0, i = 0,1,..., m$, or only constraints-inequalities $J_i \le 0, i = 0,1,..., m$, respectively. The *process* is said to be a quaternion $x(t), u(t), t_0, t_1$ that satisfies all conditions of the general problem except, possibly, the first condition. A process $x(t), u(t), t_0, t_1$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t), \tilde{t}_0, \tilde{t}_1$, the following inequality is true

$$\Phi_0(x(t_0), x(t_1), t_0, t_1) \le \Phi_0(\tilde{x}(t_0), \tilde{x}(t_1), \tilde{t}_0, \tilde{t}_1).$$

The general problem consists of determining the optimal process.

The necessary conditions of optimality are defined by the Pontryagin maximum principle [1,6].

Theorem. (maximum principle for the general problem). Let $x(t), u(t), t_0, t_1$ be an optimal process of the general problem. Then there exists a vector $\lambda = (\lambda_0, \lambda_1,..., \lambda_m)$ and a continuous solution $\psi(t)$ of a conjugate system of differential equations $\dot{\psi} = -H(\psi, x(t), u(t), t)$, satisfying conditions:

1) non-triviality, non-negativity, and complementary slackness

$$\lambda \ne 0,\ \lambda_i \ge 0, i = 0,1,..., m_0,\ \lambda_i \Phi_i(x(t_0), x(t_1), t_0, t_1) = 0, i = 1,2,..., m_0;$$

2) transversality

$$\begin{aligned} &\psi(t_0) = L_{x(t_0)}(\lambda, x(t_0), x(t_1), t_0, t_1),\ \psi(t_1) = -L_{x(t_1)}(\lambda, x(t_0), x(t_1), t_0, t_1), \\ &\frac{d}{dt_0} L(\lambda, x(t_0), x(t_1), t_0, t_1) = 0,\ \frac{d}{dt_1} L(\lambda, x(t_0), x(t_1), t_0, t_1) = 0; \end{aligned}$$

3) maximum of Hamiltonian

$$H(\psi(t), x(t), u(t), t) = \max_{v \in U} H(\psi(t), x(t), v, t), \quad t_0 \le t \le t_1$$

with Lagrange and Hamilton functions

$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \sum_{i=0}^{m} \lambda_i \Phi_i(x(t_0), x(t_1), t_0, t_1), \; H(\psi, x, u, t) = \sum_{j=1}^{n} \psi_j f_j(x, u, t).$$

Transforming the problem (2) to standard form (3), we get

$$\begin{aligned} & x_{n+1}(t_1) \to \min, , \\ & \dot{x} = Ax + Bu, \; \dot{x}_{n+1} = c^T x + \alpha u^2, \\ & x(t_0) = x^0, \; x_{n+1}(t_0) = 0 \\ & x(t_1) = x^1, \; u \in U \subseteq R, \end{aligned} \tag{4}$$

We form Lagrange and Hamilton functions

$H(\psi, x, u, t) = \psi(Ax + Bu) + \psi_{n+1}(c^T x + \alpha u^2)$,

$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_{n+1}(t_1) + \sum_{i=1}^{n} \lambda_i (x_i(t_0) - x_i^0) + \sum_{i=n+1}^{2n} \lambda_i (x_i(t_1) - x_i^1)$.

Since the left end of a trajectory and times moments t_0 and t_1 are fixed, corresponding transversality conditions are satisfied automatically (see [1]) and we can simplify second, in above, function omitting the terms related with equalities $x(0) = x^0$ and $x_{n+1}(0) = 0$. That is,
the Lagrange function can be written

$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_{n+1}(t_1) + \sum_{i=1}^{n} \lambda_i (x_i(t_1) - x_i^1).$$

We get the conjugate system

$$\begin{cases} \dot{\psi} = -A^T \psi - c^T \psi_{n+1} \\ \quad \dot{\psi}_{n+1} = 0 \end{cases}.$$

Its solution is $\psi(t, C_1, C_2, ..., C_n, k)$, $\psi_{n+1} = k$, where $C_1, C_2, ..., C_n, k$ are constants of integration.
Maximum of Hamiltonian condition is reduced to solution of the following extreme problem

$$H(\psi, x, u, t) = ... + \psi^T Bu + k\alpha u^2 \to \max_{u \in R}, \quad t_0 \le t \le t_1,$$

where we denote the terms that don't depend on control variables by three dot.
Analysis of the latter problem arrive us at the following optimal control function:

$$u^{opt}(t) = \frac{\psi^T B}{2k\alpha}, \; k\alpha < 0,$$

or in coordinate form

$$u^{opt}(t) = \frac{1}{2k\alpha}\sum_{j=1}^{n} B_j \psi_j(t)\,. \tag{5}$$

If $k\alpha \geq 0$ we obtain unusable control $u(t) = \pm\infty$.

To obtain solution of the problem (4), we solve the system 2n differential equations

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j + \frac{1}{2k\alpha} B_i \sum_{j=1}^{n} B_j \psi_j\,,$$

$$\dot{\psi}_i = \frac{d\psi_i}{dt} = -\sum_{j=1}^{n} A_{ji}\psi_j\,, \quad i = \overline{1,n} \tag{6}$$

with 2n boundary conditions

$$x(t_0) = x^0\,,\ x(t_1) = x^1\,.$$

Differential equation $\dot{x}_{n+1} = c^T x + \alpha u^2$ and initial condition $x_{n+1}(t_0) = 0$ here can be omitted since they define an objective function of the original problem.

According to Galerkin method [2], we try approximate solution

$$x_j \approx \hat{x}_j = x_j^0 + \sum_{k=1}^{M} a_{j,k}\varphi_{j,k}(t)\,,$$

$$\psi_j \approx \hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k}\varphi_{n+j,k}(t), \quad j = \overline{1,n}\,. \tag{7}$$

Here $\varphi_{j,k}(t), \quad j = \overline{1,2n}\ \ , k = \overline{1,M}$ are trial functions satisfying the following conditions

$$\varphi_{j,k}(t_0) = 0, \quad j = \overline{1,n}\ \ , k = \overline{1,M}\ , \tag{8}$$

and

$$\forall j = \overline{1,n},\ \ \exists k,l,m : \varphi_{j,k}(t_1) \neq 0,\ \ \varphi_{n+j,l}(t_0) \neq 0,\ \ \varphi_{n+j,m}(t_1) \neq 0\,.$$

First condition guarantees that $x(t_0) = x^0$ and second one ensures non-triviality of a conjugate function $\psi(t)$.
The choice of a set of trial functions is critical for realization of Galerkin method. The basic requirement is: the functions $\varphi_{j,k}(t), \quad j = \overline{1,2n}\ \ , k = \overline{1,M}$ must be linearly independent on given interval $[t_0, t_1]$.

Assuming that trial functions are continuous and differentiable, we have

$$\frac{d}{dt}x_j \approx \frac{d}{dt}\hat{x}_j = \sum_{k=1}^{M} a_{j,k}\frac{d}{dt}\varphi_{j,k}(t), \tag{9}$$
$$\frac{d}{dt}\psi_j \approx \frac{d}{dt}\hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k}\frac{d}{dt}\varphi_{n+j,k}(t), \quad j=\overline{1,n}.$$

Substituting (7) into (6) and equating corresponding derivatives in (6) and (9), we obtain

$$\sum_{k=1}^{M} a_{i,k}\frac{d}{dt}\varphi_{i,k}(t) \approx \sum_{j=1}^{n} A_{ij}\big(x_j^0 + \sum_{k=1}^{M} a_{j,k}\varphi_{j.k}(t)\big) + \frac{1}{2k\alpha}B_i\sum_{j=1}^{n} B_j\sum_{k=1}^{M} a_{n+j,k}\varphi_{n+j,k}(t),$$
$$\sum_{k=1}^{M} a_{n+i,k}\frac{d}{dt}\varphi_{n+i,k}(t) \approx -\sum_{j=1}^{n} A_{ji}\sum_{k=1}^{M} a_{n+j,k}\varphi_{n+j,k}(t), \quad i=\overline{1,n}.$$

Residual functions for x_i and ψ_i variables are accordingly

$$R_{[t_0,t_1]}^{x_i} = \sum_{k=1}^{M} a_{i,k}\frac{d}{dt}\varphi_{i,k}(t) - \sum_{j=1}^{n} A_{ij}\big(x_j^0 + \sum_{k=1}^{M} a_{j,k}\varphi_{j.k}(t)\big) - \frac{1}{2k\alpha}B_i\sum_{j=1}^{n} B_j\sum_{k=1}^{M} a_{n+j,k}\varphi_{n+j,k}(t),$$
$$R_{[t_0,t_1]}^{\psi_i} = \sum_{k=1}^{M} a_{n+i,k}\frac{d}{dt}\varphi_{n+i,k}(t) + \sum_{j=1}^{n} A_{ji}\sum_{k=1}^{M} a_{n+j,k}\varphi_{n+j,k}(t), \quad i=\overline{1,n}.$$

By Galerkin method [2], we have

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{x_s} W_{s,k}\,dt = 0, \quad s=\overline{1,n}; \quad k=\overline{1,M},$$
$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{\psi_{s-n}} W_{s,k}\,dt = 0, \quad s=\overline{n+1,2n}; \quad k=\overline{1,M},$$

where $W_{s,k}$ are weight functions. For simplicity, we choose weight functions $W_{s,k}$ as

$$W_{s,k} = \varphi_{s,k}, \quad s=\overline{1,2n}; \quad k=\overline{1,M}.$$

To satisfy the condition $x(t_1) = x^1$, we add the term

$$(\hat{x}_{s-n} - x_{s-n}^1)\widetilde{W}_{s,k}\Big|_{t=t_1}, \quad s=\overline{n+1,2n}; \quad k=\overline{1,M}$$

into the second equation. It is reasonable, for convenience, to take

$$\widetilde{W}_{s,k} = -\varphi_{s,k}, \quad s=\overline{n+1,2n}; \quad k=\overline{1,M}.$$

Finally, we get the following linear system of 2nM equations in 2nM variables $a_{j,k}$:

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{x_s} \varphi_{s,k} dt = 0\,,\ s = \overline{1,n};\quad k = \overline{1,M}\,, \tag{10}$$

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{\psi_{s-n}} \varphi_{s,k} dt + (\hat{x}_{s-n}^0 + \sum_{k=1}^{M} a_{s-n,k}\varphi_{s-n,k}(t_1) - x_{s-n}^1)\tilde{W}_{s,k}(t_1) = 0\,,\ s = \overline{n+1,2n};\quad k = \overline{1,M}\,.$$

Solution of the linear system (10) gives an approximate solution (7) of the problem (4). The accuracy of approximation depends on the choice of trial functions and the exact solution of optimal control problem. Convergence of the method is related with the solvability of the system (10). Proper choice of weight and trial functions allows us to get an exact solution.

4. APPLICATION

We solve general optimal control problem

$$\begin{gathered} J = \int_0^1 u^2 dt \to \min\,, \\ \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ x_1(0) = x_2(0) = 0\,, \\ x_1(1) = 2\,,\ x_2(1) = 1\,, \\ u \in R\,. \end{gathered} \tag{11}$$

by means of two methods – using maximum principle and Galerkin method. For simplicity, we take constants $c_1 = c_2 = ... = c_n = 0$ and $\alpha = 1$ in objective function in the problem (2).

Method 1. We use Pontryagin maximum principle.

Solution. We form Lagrange and Hamilton functions

$$H(\psi, x, u, t) = \psi_1 x_2 + \psi_2 u + \psi_3 u^2\,,$$
$$L(\lambda, x^0, x^1, t_0, t_1) = \lambda_0 x_3(1) + \lambda_1(x_1(1) - 2) + \lambda_2(x_2(1) - 1)\,,$$

conjugate system $\begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = -\psi_1 \\ \dot{\psi}_3 = 0 \end{cases}.$

Its solution is

$$\begin{cases} \psi_1(t) = C_1 \\ \psi_2(t) = -C_1 t + C_2 \\ \psi_3(t) = C_3 \end{cases}. \tag{12}$$

Conditions 1 and 2 of the theorem are:

$$1)\ \lambda_0 \geq 0,\ \lambda_0 + |\lambda_1| + |\lambda_2| \neq 0;$$

$$2)\ \begin{cases} \psi_1(1) = -\lambda_1 \\ \psi_2(1) = -\lambda_2 \\ \psi_3(t) = -\lambda_0 \end{cases}. \tag{13}$$

From (12) and (13), we get

$$\begin{cases} \psi_1(t) = -\lambda_1 \\ \psi_2(t) = \lambda_1(t-1) - \lambda_2 \\ \psi_3(t) = -\lambda_0 \end{cases}. \tag{14}$$

Case (a): $\lambda_0 = 0$, $\lambda_1 = 0$, $\lambda_2 \neq 0$. Then (14) has the form $\begin{cases} \psi_1(t) = 0 \\ \psi_2(t) = -\lambda_2 \\ \psi_3(t) = 0 \end{cases}$.

And condition 3 of the theorem becomes

$$H(\psi, x, u, t) = -\lambda_2 u \to \max_{u \in R},\quad 0 \leq t \leq 1.$$

Solution of the last extreme problem is $u(t) = \pm\infty$. Unusable solution.

Case (b): $\lambda_0 = 0$, $\lambda_1 \neq 0$, $\lambda_2 = 0$. And (14) becomes $\begin{cases} \psi_1(t) = -\lambda_1 \\ \psi_2(t) = \lambda_1(t-1) \\ \psi_3(t) = 0 \end{cases}$.

Solution of the extreme problem

$$H(\psi, x, u, t) = \lambda_1(t-1)u \to \max_{u \in R},\quad 0 \leq t \leq 1$$

is $u(t) = \pm\infty$. Again, unusable solution.

Case (c): $\lambda_0 > 0$ ($\lambda_0 = 1$), $\lambda_1 - \forall$, $\lambda_2 - \forall$. System (14) arrives at

$$\begin{cases} \psi_1(t) = -\lambda_1 \\ \psi_2(t) = \lambda_1(t-1) - \lambda_2 \\ \psi_3(t) = -1 \end{cases}.$$

Condition 3 of the theorem has the form

$$H(\psi, x, u, t) = (\lambda_1(t-1) - \lambda_2)u - u^2 \to \max_{u \in R}, \quad 0 \le t \le 1.$$

Its solution (optimal control) is $u^{opt}(t) = \dfrac{\lambda_1(t-1) - \lambda_2}{2}$.

Substitution of the latter into (11) and application of boundary conditions $x_1(0) = x_2(0) = 0$ and $x_1(1) = 2$, $x_2(1) = 1$ give the optimal trajectory and explicit form of optimal control ($\lambda_1 = -36$, $\lambda_2 = 16$):

$$\begin{cases} x_1(t) = -3t^3 + 5t^2 \\ x_2(t) = -9t^2 + 10t \end{cases}, \quad u(t) = -18t + 10, \quad 0 \le t \le 1.$$

Method 2. Using of Galerkin method.

We take the following trial functions:

$$\varphi_{1,1}(t) = \varphi_{2,1} = t, \;\; \varphi_{1,2}(t) = \varphi_{2,2} = t^2, \;\; \varphi_{1,3}(t) = \varphi_{2,3} = t^3;$$
$$\varphi_{3,1}(t) = \varphi_{4,1} = 1, \;\; \varphi_{3,2}(t) = \varphi_{4,2} = t, \;\; \varphi_{3,3}(t) = \varphi_{4,3} = t^2,$$

satisfying the conditions (8). We form approximate solution (7) of the system (6)

$$\begin{aligned} \hat{x}_1 &= 0 + a_{1,1}t + a_{1,2}t^2 + a_{1,3}t^3, \\ \hat{x}_1 &= 0 + a_{2,1}t + a_{2,2}t^2 + a_{2,3}t^3, \\ \hat{\psi}_1 &= a_{3,1} + a_{3,2}t + a_{3,3}t^2, \\ \hat{\psi}_2 &= a_{4,1} + a_{4,2}t + a_{4,3}t^2. \end{aligned} \tag{15}$$

According to (5), optimal control is $u^{opt}(t) = \dfrac{1}{2}\psi^T B = \dfrac{1}{2}(\psi_1, \psi_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dfrac{1}{2}\psi_2$.

Substituting (15) into (6), forming residual functions and integrating expressions (10), yield the following linear system:

$\frac{1}{3}a_{1,1}-\frac{1}{4}a_{2,1}+\frac{1}{2}a_{1,2}-\frac{2}{5}a_{2,2}+\frac{3}{5}a_{1,3}-\frac{1}{6}a_{2,3}=0$,

$\frac{1}{3}a_{2,1}-\frac{1}{6}a_{4,1}+\frac{1}{2}a_{2,2}+\frac{1}{8}a_{4,2}+\frac{3}{5}a_{2,3}-\frac{1}{10}a_{4,3}=0$,

$\frac{1}{2}a_{3,2}+\frac{2}{3}a_{3,3}-a_{1,1}-a_{1,2}-a_{1,3}=-2$,

$\frac{1}{2}a_{3,1}+\frac{1}{2}a_{4,2}+\frac{1}{3}a_{3,2}+\frac{2}{3}a_{4,3}+\frac{1}{4}a_{3,3}-a_{2,1}-a_{2,2}-a_{2,3}=-1$,

$\frac{1}{2}a_{1,1}+\frac{2}{3}a_{1,2}+\frac{3}{4}a_{1,3}-\frac{1}{3}a_{2,1}-\frac{1}{4}a_{2,2}-\frac{1}{5}a_{2,3}=0$,

$\frac{1}{3}a_{1,1}+\frac{1}{2}a_{1,2}+\frac{3}{5}a_{1,3}-\frac{1}{4}a_{2,1}-\frac{1}{5}a_{2,2}-\frac{1}{6}a_{2,3}=0$,

$\frac{1}{4}a_{1,1}+\frac{2}{5}a_{1,2}+\frac{1}{2}a_{1,3}-\frac{1}{5}a_{2,1}-\frac{1}{6}a_{2,2}-\frac{1}{7}a_{2,3}=0$,

$\frac{1}{2}a_{2,1}+\frac{2}{3}a_{2,2}+\frac{3}{4}a_{2,3}-\frac{1}{4}a_{4,1}-\frac{1}{6}a_{4,2}-\frac{1}{8}a_{4,3}=0$,

$\frac{1}{3}a_{2,1}+\frac{1}{2}a_{2,2}+\frac{3}{5}a_{2,3}-\frac{1}{6}a_{4,1}-\frac{1}{8}a_{4,2}-\frac{1}{10}a_{4,3}=0$,

$\frac{1}{4}a_{2,1}+\frac{2}{5}a_{2,2}+\frac{1}{2}a_{2,3}-\frac{1}{8}a_{4,1}-\frac{1}{10}a_{4,2}-\frac{1}{12}a_{4,3}=0$,

$a_{3,2}+2a_{3,3}+a_{1,1}+a_{1,2}+a_{1,3}-2=0$,

$\frac{1}{2}a_{3,2}+a_{3,3}+a_{1,1}+a_{1,2}+a_{1,3}-2=0$,

$\frac{1}{3}a_{3,2}+\frac{2}{3}a_{3,3}+a_{1,1}+a_{1,2}+a_{1,3}-2=0$,

$a_{3,1}+a_{4,2}+\frac{1}{2}a_{3,2}+a_{4,3}+\frac{1}{3}a_{3,3}+a_{2,1}+a_{2,2}+a_{2,3}-1=0$,

$\frac{1}{2}a_{3,1}+\frac{1}{2}a_{4,2}+\frac{1}{3}a_{3,2}+\frac{2}{3}a_{4,3}+\frac{1}{4}a_{3,3}+a_{2,1}+a_{2,2}+a_{2,3}-1=0$,

$\frac{1}{3}a_{3,1}+\frac{1}{3}a_{4,2}+\frac{1}{4}a_{3,2}+\frac{1}{2}a_{4,3}+\frac{1}{5}a_{3,3}+a_{2,1}+a_{2,2}+a_{2,3}-1=0$.

Solution of this system is

$$a_{1,1}=0,\quad a_{1,2}=5,\quad a_{1,3}=-3$$
$$a_{2,1}=10,\quad a_{2,2}=-9,\quad a_{2,3}=0$$
$$a_{3,1}=36,\quad a_{3,2}=0,\quad a_{3,3}=0$$
$$a_{4,1}=20,\quad a_{4,2}=-36,\quad a_{4,3}=0$$

And finally, approximate solution is

$$\begin{cases}\hat{x}_1(t)=-3t^3+5t^2\\ \hat{x}_2(t)=-9t^2+10t\end{cases},\quad u(t)=-18t+10,\ 0\le t\le 1.$$

Since the exact solution has polynomial form and we took sufficient number of trial functions of the same form as well, approximate and exact solutions coincide.

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