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Contra-harmonic mean derivative - based closed newton Cotes quadrature

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Abstract

A new set of Contra-harmonic mean derivative - based closed Newton Cotes quadrature (CHMDCNC) formula is demonstrated for the valuation of the definite integral in which the derivative value is admitted in addition to the existing function value on uniformly spaced intervals. These derivative value is assessed by using the Contra harmonic mean at the terminal points. Besides, the error terms are set up by applying the concept of precision. The order of accuracy of these numerical integration formula is more eminent than the existing closed Newton Cotes quadrature (CNC) formula. Ultimately, the strength of the proposed formula is analyzed by applying the numerical examples.

Keywords - Closed Newton-Cotes formula, Definite integral, Contra harmonic mean, Precision, Numerical examples.

I. INTRODUCTION

Many authors are focused to exploit on the existing newton cotes quadrature formula to increase the order of accuracy of the value of numerical integration. Dehghan et al., derived the improved form of the closed, open and semi-open [5,6,7] Newton Cotes formulas, which improve the order of accuracy by two by letting in the location of the boundaries of the interval as two extra parameters and returning the original integral to fit the optimal boundary locations. They have utilized the same method to Gauss Legendre, Gauss Lobatto and Gauss Chebyshev quadrature rules [2,8,9].Burg and his associates proposed derivative based - closed ,open and Midpoint Newto-Cotes quadrature rules[3,4,10]by admitting the first and higher order derivatives. Also, Weijing Zhao and Hongxing Li [16] have improved the closed newton cotes quadrature formula by letting in the derivative value at the terminal points. Lately, we proposed Geometric mean [11], Harmonic mean [12], Heronian mean [14] and centroidal mean [15] derivative - based closed Newton cotes quadrature rule and the comparability of the arithmetic mean, geometric mean and harmonic mean derivative - based closed Newton Cotes quadrature rules [13].

The general pattern of the numerical integration formula for the valuation of the definite integral over [a, b] is

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \qquad (1.1)$$

Where there are (n+1) uniformly separated integration points $a=x_0, x_1, \ldots, x_n=b$ and are (n+1) weights w_i, i=0,1,,2,...,n and so $x_i=x_0+ih$, i=0,1,2,..n, where $h = \frac{b-a}{n}$.
Then the closed Newton Cotes quadrature formula for the evaluation of the definite integral over [a,b] is

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \qquad (1.2)$$

These weight can be derived in several different ways. One of the method is based on the precision of a quadrature formula. Pick out the values for w_i, i=0,1,...n. So that the error of approximation in the quadrature formula is zero, that is

$$E_n[f] = \int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = 0,$$

$$\text{for } f(x) = x^j \quad j = 0,1, \ldots n \qquad (1.3)$$

A. Definition

An integrated method of the form (1.1) is said to be of order P, if it gets accurate results ($E_n[f] = 0$) for all polynomials of degree less than or equal to P[1].

In that respect are various subclasses of newton Cotes formulas are obtained by giving various values of n in a general newton cotes quadrature formula.

Trapezoidal rule(n=1)

$$\int_a^b f(x)dx = \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f''(\xi),$$
$$where\ \xi \in (a,b) \quad (1.4)$$

Simpson's 1/3rd rule (n=2)

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$-\frac{(b-a)^5}{2880} f^{(4)}(\xi),\ where\ \xi \in (a,b) \quad (1.5)$$

Simpson's 3/8th rule (n=3)

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi),$$
$$where\ \xi \in (a,b) \quad (1.6)$$

Boole's rule (n=4)

$$\int_a^b f(x)dx = \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]$$
$$-\frac{(b-a)^7}{1935360} f^{(6)}(\xi), \quad where\ \xi \in (a,b) \quad (1.7)$$

It is recognized that the degree of precision is (n+1) for even value of n and n for odd values of n.

In this paper, contra-harmonic mean derivative-based closed Newton cotes quadrature formula is shown in which the contra harmonic mean of terminal points at the derivative is admitted to add-on the existing closed Newton Cotes quadrature formula. The order of accuracy of the proposed scheme has improved by a single order. That is, the degree of precision is (n+2) for even n and is (n+1) for odd n. Besides, the error terms are obtained by applying the concept of precision. Lately, numerical solutions are shown to show the accuracy of the proposed scheme.

II. CONTRA-HARMONIC MEAN DERIVATIVE - BASED CLOSED NEWTON COTES QUADRATURE RULE

In this section, contra-harmonic mean derivative - based closed Newton cotes quadrature formula is derived by using the Contra harmonic mean value at the terminal points[a, b] for the valuation of a definite integral.

A.Theorem

Closed Trapezoidal rule (n=1) using Contra-harmonic mean derivative is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12} f''\left(\frac{a^2+b^2}{a+b}\right), \quad (2.1)$$

The precision of this method is 2.

Proof: For f (x) = x^2

$$\text{The Exact value of} \int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3);$$

$$(2.1) \Rightarrow \frac{b-a}{2}(a^2+b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3 - a^3).$$

It indicates that the solution is exact. Thus, the precision of the closed Trapezoidal rule with Contra-harmonic mean derivative is 2 where as the precision of the existing Trapezoidal rule (1.4) is 1.

B. Theorem

Closed Simpson's1/3rd rule with Contra-harmonic mean derivative (n=2) is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a^2+b^2}{a+b}\right), \quad (2.2)$$

The precision of this method is 4.

Proof: For f (x) = x^4.

$$\text{The Exact value of} \int_a^b x^4\,dx = \frac{1}{5}(b^5-a^5);$$

$$(2.2) \Rightarrow \left(\frac{b-a}{6}\right)\left[a^4+4\left(\frac{a+b}{2}\right)^4+b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5 - a^5).$$

It indicates that the solution is exact. Thus, the precision of the closed Simpson's1/3rd rule with Contra-harmonic mean derivative is 4 where as the precision of the existing Simpson's1/3rd rule (1.5) is 3.

C. Theorem

Closed Simpson's3/8rd rule with Contra-harmonic mean derivative (n=3) is

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a^2+b^2}{a+b}\right), \quad (2.3)$$

The precision of this method is 4.

Proof: For f (x) = x^4.

$$\text{The Exact value of } \int_a^b x^4\, dx = \frac{1}{5}(b^5 - a^5);$$

$$(2.3 \Rightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5 - a^5).$$

It indicates that the solution is exact. Thus, the precision of the closed Simpson's3/8th rule with Contra-harmonic mean derivative is 4 where as the precision of the existing Simpson's3/8th rule (1.6) is 3.

D. Theorem

Closed Boole's rule with Contra-harmonic mean derivative (n=4) is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a^2+b^2}{a+b}\right) \quad (2.4)$$

The precision of this method is 6.

Proof: For f (x) = x^6.

$$\text{The Exact value of } \int_a^b x^6\, dx = \frac{1}{7}(b^7 - a^7);$$

$$(2.4) \Rightarrow \left(\frac{b-a}{90}\right)\left[\begin{matrix} 7a^6 + 32\left(\frac{3a+b}{4}\right)^6 \\ +12\left(\frac{a+b}{2}\right)^6 + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6 \end{matrix}\right] + \frac{720(b-a)^7}{1935360} = \frac{1}{7}(b^7 - a^7).$$

It indicates that the solution is exact. Thus, the precision of the closed Boole's rule with Contra-harmonic mean derivative is 6 where as the precision of the existing Boole's rule (1.7) is 5.

III. THE ERROR TERMS of CONTRA-HARMONIC MEAN DERIVATIVE - BASED CLOSED NEWTON COTES QUADRATURE RULE

In this section, the error terms for the Contra-harmonic mean derivative -based closed Newton cotes quadrature formula is derived by using the reminder between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

A. Theorem

Contra-harmonic mean derivative-based closed Trapezoidal rule (n=1)with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f''\left(\frac{a^2+b^2}{a+b}\right) + \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi), \tag{3.1}$$

where ξ ∈ (a, b).The order of accuracy is 5 with the error term

$$E_1[f] = \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$$

Proof:

Let $f(x) = \frac{x^3}{3!}$,

The Exact value of $\frac{1}{3!}\int_a^b x^3\,dx = \frac{1}{24}(b^4 - a^4)$;

$$(2.1) \Rightarrow \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a^2+b^2}{a+b}\right)\right),$$

Hence,

$$\frac{1}{24}(b^4 - a^4) - \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a^2+b^2}{a+b}\right)\right) = \frac{(b-a)^5}{24(a+b)} .$$

Hence the error term is,

$$E_1[f] = \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$$

B. Theorem

Contra-harmonic mean derivative-based closed Simpson's1/3rd rule (n=2)with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$-\frac{(b-a)^5}{2880}f^{(4)}\left(\frac{a^2+b^2}{a+b}\right)$$
$$+\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi), \qquad (3.2)$$

where ξ ∈ (a, b). The order of accuracy is 7 with the error term

$$E_2[f] = \frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi).$$

Proof:

$$\text{Let } f(x) = \frac{x^5}{5!},$$

$$\text{The Exact value of } \frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6);$$

$$(2.2) \Rightarrow \frac{b-a}{5!.48}\begin{pmatrix} 8a^5 + (a+b)^5 + 8b^5 \\ -2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\end{pmatrix},$$

Hence,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.48}\begin{pmatrix} 8a^5 + (a+b)^5 + 8b^5 \\ -2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\end{pmatrix} = \frac{(b-a)^7}{5760(a+b)}.$$

Hence the error term is,

$$E_2[f] = \frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi).$$

C. Theorem

Contra-harmonic mean derivative-based closed Simpson's 3/8th rule (n=3) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$
$$-\frac{(b-a)^5}{6480}f^{(4)}\left(\frac{a^2+b^2}{a+b}\right)$$
$$+\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi), \qquad (3.3)$$

Where ξ ∈ (a, b). The order of accuracy is 7 with the error term

$$E_3[f] = \frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$$

Proof:

Let $f(x) = \frac{x^5}{5!}$,

The Exact value of $\frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6)$;

$$(2.3) \Rightarrow \frac{b-a}{5!.648}\begin{pmatrix} 81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5 \\ -12(b-a)^4\left(\frac{a^2+b^2}{a+b}\right) \end{pmatrix},$$

Hence,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.648}\begin{pmatrix} 81a^5 + (2a+b)^5 + (a+2b)^5 \\ +81b^5 - 12(b-a)^4\left(\frac{a^2+b^2}{a+b}\right) \end{pmatrix}$$

$$= \frac{(b-a)^7}{12960(a+b)}.$$

Hence the error term is,

$$E_3[f] = \frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi).$$

D.Theorem

Contra-harmonic mean derivative-based closed Boole's rule (n=4) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$$

$$-\frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a^2+b^2}{a+b}\right) + \frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi), \qquad (3.4)$$

where ξ ∈ (a, b). The order of accuracy is 8 with the error term

$$E_4[f] = \frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi).$$

Proof:

Let $f(x) = \frac{x^7}{7!}$,

The Exact value of $\frac{1}{7!}\int_a^b x^7\,dx = \frac{1}{40320}(b^6 - a^6)$;

$$(2.4) \Rightarrow \frac{b-a}{7!.768}(97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6 + 97b^7 - 2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)),$$

Hence,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{7!.768}(97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6 + 97b^7 - 2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)), = \frac{(b-a)^9}{3870720(a+b)}.$$

Hence the error term is,

$$E_4[f] = \frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi).$$

The summary of precision, the orders and the error terms for Contra-harmonic mean derivative based closed Newton- Cotes quadrature is shown in Table I.

TABLE I: COMPARISON OF ERROR TERMS

Rules	**Precision**	**Order**	**Error terms**
Trapezoidal rule (n=1)	2	5	$\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$
Simpson's 1/3rd rule (n=2)	4	7	$\frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi).$
Simpson's 3/8th rule (n=3)	4	7	$\frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi).$
Boole's rule (n=4)	6	9	$\frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi)$

IV. NUMERICAL EXAMPLES

To compare the efficiency of the closed Newton- Cotes quadrature formula and Contra- harmonic mean derivative - based closed Newton- Cotes quadrature formula, the following integrals are evaluated and the results are compared and are exhibited in Table II and II$\left(\int_0^1 3^x\,dx \text{ and } \int_1^2 e^x\,dx\right)$. The result indicates that the suggested scheme gives a better solution than the existing scheme.

A.Example

solve $\int_0^1 3^x \, dx$ and compare the answers with the CNC and CHMDCNC rules.

Solution:

Exact value of $\int_0^1 3^x \, dx$=1.820478453.

TABLE II : COMPARISON OF CNC AND CHMDCNC RULES

Value of n	CNC		CHMDCNC	
	App.	Error	App.	Error
n = 1	2.0000000	0.1795215	1.6982627	0.1222156
n = 2	1.8213672	0.0008887	1.8198497	0.0006286
n = 3	1.8208750	0.0003965	1.8202006	0.0002778
n = 4	1.8204800	0.0000015	1.8204773	0.0000011

B. Example

solve $\int_1^2 e^x \, dx$ and compare the answers with the CNC and CHMDCNC rules.

Solution:

Exact value of $\int_1^2 e^x \, dx$=4.67077427.

TABLE III : COMPARISON OF CNC AND CHMDCNC RULES

Value of n	CNC		CHMDCNC	
	App.	Error	App.valu	Error
n = 1	5.0536689	0.3828946	4.6124614	0.0583128
n = 2	4.6723490	0.0015747	4.6705106	0.0002636
n = 3	4.6714764	0.0007022	4.6706594	0.0001148
n = 4	4.6707766	0.0000023	4.6707738	0.0000003

V. CONCLUSION

In this paper, contra-harmonic mean derivative - based closed Newton - Cotes quadrature formulas were demonstrated, which let in the derivative value to increase the order of accuracy of the existing closed Newton - Cotes quadrature formula. These derivative value is evaluated by using the contra harmonic mean value of the terminal points. The error bounds for these quadrature formulas were derived by applying the concept of precision. Finally, numerical examples show that the suggested scheme gives a better solution than the existing closed Newton - Cotes quadrature formula.

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