Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 4 (2017), pp. 1303–1317

http://www.ripublication.com/gjpam.htm

Correspondence and Isomorphism Theorems for L-Intuitionistic or L-Vague Sub Rings

G. Vasanti
Basic Science and Humanities Department,
Aditya Institute of Technology And Management
(An Autonomous Institution),
Tekkali, Andhra Pradesh, 532201.

Abstract

The aim of this paper is basically to generalize such results as First Isomorphism Theorem, Second Isomorphism Theorem, Third Isomorphism theorem, and Correspondence Theorem for the L-intuitionistic fuzzy or L-vague fuzzy sub rings of a ring under a crisp map.

AMS subject classification: 03F55, 06C05, 16D25.
Keywords: Intutionistic L-fuzzy or L-vague Sub ring, Correspondence Theorem, First (Second, Third) Isomorphism Theorem.

1. Introduction

Zadeh [1], in his pioneering paper, introduced the notion of Fuzzy Subset of a set X as a function μ from X to the closed interval [0,1] of real numbers. The function μ, he called, the membership function which assigns to each member x of X its membership value, μx in [0, 1].

In 1983, Atanassov [2] generalized the notion of Zadeh fuzzy subset of a set further by introducing an additional function ν which he called a non membership function with some natural conditions on μ and ν, calling these new generalized fuzzy subsets of a set, intutionistic fuzzy subsets. Thus according to him an intutionistic fuzzy subset of a set X, is a pair $A = (\mu_A, \nu_A)$, where μ_A, ν_A are functions from the set X to the closed interval [0, 1] of real numbers such that for each $x \in X$, $\mu x + \nu x \leq 1$, where μ_A is called the membership function of A and ν_A is called the non membership function of A. Later on in 1984, Atanassov and Stoeva [3], further generalized the notion of intuitionistic fuzzy subset to L-intuitionistic fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Thus an L-intutionistic fuzzy subset A of a set X, is a pair (μ_A, ν_A) where $\mu_A, \nu_A\colon X \to L$ are such that $\mu_A \le N\nu_A$. Let us recall that a complete order reversing involution is a map $N\colon L \to L$ such that (1) $N0_L = 1_L$ and $N1_L = 0_L$ (2) $\alpha \le \beta$ implies $N\beta \le N\alpha$ (3) $NN\alpha = \alpha$ (4) $N(\vee_{i\in I}\alpha_i) = \wedge_{i\in I}N\alpha_i$ and $N(\wedge_{i\in I}\alpha_i) = \vee_{i\in I}N\alpha_i$.

Liu [4] in 1982 introduced the concept of fuzzy ring and fuzzy ideal. In 1985, Ren [5] examined the concepts of fuzzy ideal and fuzzy quotient ring, which were actually an extension of Rosenfeld's [6] fuzzy group by starting with an ordinary ring and then define a fuzzy sub ring based on the ordinary operations of the given ring.

Coming to generalizations of substructures of rings on to the if/v-subsets, we briefly mention the following papers:

Banergee-Basnet [7] in 2003, introduced the concepts of if/v-sub rings and if/v-ideals of a ring. Hur-Kang-Song [8] introduced the concepts of if/v-sub ring of a ring, if/v-ideals and study of if/v-prime/maximal ideals, if/v-nil radicals etc. were made in Hur et al. [9], Jun-Ozturk-Park [10] and Wang-Lin [11] etc..

Palaniappan, Arjunan and Palanivelrajan [12] in 2008 studied the concepts of homomorphism and anti-homomorphism in intuitionistic L-fuzzy sub rings. In the same year, Meena and Thomas [13] presented properties of L-if/v-sub rings, L-if/v-ideals and introduced the quotient ring.

Murthy-Vasanti [14] presented a detailed study of the (lattice) algebraic properties of L-if/v-images and L-if/v-inverse images under a crisp map for L-if/v-subgroups that take their truth values in a complete lattice and Vasanti [15] gave an exclusive study of the (lattice) algebraic properties of L-if/v-images and L-if/v-inverse images under a crisp map for L-if/v-sub rings that take their truth values in a complete lattice.

In this paper, we apply parts of the above Theory to generalize such results as First Isomorphism Theorem, Second Isomorphism Theorem, Third Isomorphism theorem, Correspondence Theorem etc..

Hence, the set of all L-if/v-subsets of a set X be denoted by $A_L(X)$. For any pair of L-if/v-subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of X, $A \le B$ iff $\mu_A \le \mu_B$ and $\nu_B \le \nu_A$, $A_L(X)$ becomes a complete infinitely distributive lattice, provided L is also a complete infinitely distributive lattice. For any family $(A_i)_{i\in I}$ of L-if-subsets of X, $\mu_{\vee_{i\in I}A_i} = \vee_{i\in I}\mu_{A_i}$ and $\nu_{\vee_{i\in I}A_i} = \wedge_{i\in I}\nu_{A_i}$ where $\vee_{i\in I}A_i = (\mu_{\vee_{i\in I}A_i}, \nu_{\vee_{i\in I}A_i})$ and $\mu_{\wedge_{i\in I}A_i} = \wedge_{i\in I}\mu_{A_i}$ and $\nu_{\wedge_{i\in I}A_i} = \vee_{i\in I}\nu_{A_i}$ where $\wedge_{i\in I}A_i = (\mu_{\wedge_{i\in I}A_i}, \nu_{\wedge_{i\in I}A_i})$.

Let us recall that (a) a complete lattice L is a complete infinite distributive lattice iff for all subsets $(\alpha_i)_{i\in I}, (\beta_j)_{j\in J}$ and elements α, β of L (1) $\alpha \wedge (\vee_{j\in J}\beta_j) = \vee_{j\in J}(\alpha \wedge \beta_j)$, (2) $\beta \vee (\wedge_{i\in I}\alpha_i) = \wedge_{i\in I}(\beta \vee \alpha_i)$, hold.

Consequently, in a complete infinite distributive lattice, for all subsets $(\alpha_i)_{i\in I}$, $(\beta_j)_{j\in J}$, (3) $\vee_{i\in I} \vee_{j\in J} (\alpha_i \wedge \beta_j) = (\vee_{i\in I}\alpha_i) \wedge (\vee_{j\in J}\beta_j)$ and (4) $\wedge_{i\in I} \wedge_{j\in J} (\alpha_i \vee \beta_j) = (\wedge_{i\in I}\alpha_i) \vee (\wedge_{j\in J}\beta_j)$ hold and (b) a complete lattice with a unary complement operation is a complete de Morgan lattice iff for all subsets $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$ of L, (1) $(\vee_{i\in I}\alpha_i)^c = \wedge_{i\in I}\alpha_i^c$ and (2) $(\wedge_{j\in J}\beta_j)^c = \vee_{j\in J}\beta_j^c$ hold.

For any set X, the L-if-subset $(\mu_X, \nu_X) = (\overline{1}_X, \overline{0}_X)$, where $\overline{1}_X$ or simply 1 is the constant map assuming the value 1 of L for each $x \in X$ and $\overline{0}_X$ or simply 0 is the constant map assuming the value 0 of L for each $x \in X$, which turns out to be the largest element in $A_L(X)$. The L-if-empty subset ϕ of X is given by $(\mu_\phi, \nu_\phi) = (\overline{0}, \overline{1})$, which is the least element in $A_L(X)$. Also for any μ: $X \to L$, both $(\mu, N\mu)$, where $N\mu$: $X \to L$ is defined by $(N\mu)(x) = N(\mu x)$ $\forall x \in X$ and $(N\mu, \mu)$, define L-if-subsets of X because N is an involution on L. For $A = (\mu_A, \nu_A)$ an L-if-subset of X, since N is an order reversing involution, (ν_A, μ_A) is also an L-if-subset of X called as the L-if-complement of A, denoted by A^c. Observe that $A^c = X - A = X \wedge A^c$. Further for any pair A, B of L-if/v-subsets of X, we define B/A to be $B \wedge A^c$.

2. Basic Results

In this section, first we state some definitions and statements from [15], which are useful in the main results.

Definition 2.1.

(a) An L-if/v-subset A of R is called an *L-if/v-sub ring* of R iff:

(1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in R$.

(2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in R$.

(3) $\mu_A(-x) \geq \mu_A(x)$ and $\nu_A(-x) \leq \nu_A(x)$ for each $x \in R$.

(b) For any L-if/v-sub ring A of a ring R, $A_* = \{x \in R/\mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)\}$ and $A^* = \{x \in R/\mu_A(x) > 0$ and $\nu_A(x) < 1\}$.

(c) A complete lattice L is *strongly regular* iff 0 is $\wedge$ - prime or $\alpha > 0, \beta > 0$ implies $\alpha \wedge \beta > 0$ and 1 is $\vee$- prime or $\alpha < 1, \beta < 1$ implies $\alpha \vee \beta < 1$.

(d) Let X, Y be a pair of sets and let $f : X \to Y$ be a map. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be L-if/v-subsets of X and Y respectively.

(1) Let $\mu_D, \nu_D : Y \to L$ be defined by $\mu_D y = \vee \mu_A f^{-1} y$ and $\nu_D y = \wedge \nu_A f^{-1} y$ $\forall y \in Y$. Then D is a well defined *L-if/v-subset* of Y, called as the *L-intutionistic fuzzy/vague image* of A under f or simply the *L-if/v-image* of A under f.

(2) Let $\mu_C, \nu_C : Y \to L$ be defined by $\mu_C x = \mu_B f x$ and $\nu_C x = \nu_B f x, \forall x \in X$. Then C is a well defined *L-if/v-subset* of X, called as the *L-intutionistic fuzzy/vague inverse image* of B under f or simply the *L-if/v-inverse image* of B under f.

(e) Clearly, $\forall y \in Y$, $\mu_{fX} y = \vee \overline{1}_X f^{-1} y = \chi_{fX} y$ or $\mu_{fX} = \chi_{fX}$ and $\nu_{fX} y = \wedge \overline{0}_X f^{-1} y = N\chi_{fX} y$ or $\nu_{fX} = N\chi_{fX}$ implying $fX = (\chi_{fX}, N\chi_{fX})$.

(f) For any L-if/v-sub ring A of a ring R,

(1) A is an *L-if/v-left-ideal* of R iff $\mu_A(xy) \geq \mu_A y$ and $\nu_A(xy) \leq \nu_A(y)$ for each $x, y \in R$.

(2) A is an *L-if/v-right-ideal* of R iff $\mu_A(xy) \geq \mu_A x$ and $\nu_A(xy) \leq \nu_A(x)$ for each $x, y \in R$.

(3) A is an *L-if/v-ideal* of R iff the A is an *L-if/v-right-ideal* and *L-if/v-left-ideal* of R or ($\mu_A(xy) \geq \mu_A x$ and $\mu_A(xy) \geq \mu_A y$) or ($\mu_A(xy) \geq \mu_A x \vee \mu_A y$).

(g) A is an L-if/v-sub ring of R implies

(1) $\mu_A(0) \geq \mu_A(x)$ and (2) $\nu_A(0) \leq \nu_A(x)$ for each $x \in R$.

(h) A is an L-if/v-sub ring (left ideal, right ideal, ideal) of R implies

(1) $A_* = \{x \in R/\mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$ is a sub ring (left ideal, right ideal, ideal) of R;

(2) $A^* = \{x \in R/\mu_A(x) > 0, \nu_A(x) < 1\}$ is a sub ring (left ideal, right ideal, ideal) of R whenever L is strongly regular.

Lemma 2.2. For any pair of rings R and S and for any crisp homomorphism $f : R \to S$ the following are true:

1. A is an L-if/v-sub ring (left ideal, right ideal, ideal) of R implies $f(A)$ is an L-if/v-sub ring (left ideal, right ideal, ideal) of S, whenever L is a complete infinite distributive lattice (and f is onto)

2. B is an L-if/v-sub ring (left ideal, right ideal, ideal) of S implies $f^{-1}(B)$ is an L-if/v-sub ring (left ideal, right ideal, ideal) of R.

Lemma 2.3. For any family of L-if/v-sub rings (left ideals, right ideals, ideals) $(A_i)_{i\in I}$ of a ring R,

(1) $\wedge_{i\in I} A_i$ is an L-if/v-sub ring (left ideals, right ideals, ideals) of R.

(2) $\vee_{i\in I} A_i$ is an L-if/v-sub ring (left ideal, right ideal, ideal) of R whenever $(A_i)_{i\in I}$ is a sup/inf assuming chain.

Theorem 2.4. For any L-if/v-sub ring B of a ring R and for any crisp ideal N of R, the L-if/v-subset C: $R/N \to L$ where for each $x \in R$, $\mu_C(x + N) = \vee\mu_B(x + N)$ and $\nu_C(x + N) = \wedge\nu_B(x + N)$, is an L-if/v-sub ring of R/N when L is a complete infinite distributive lattice.

Definition 2.5. For any L-if/v-sub ring B of a ring R and for any ideal N of R, the L-if/v-sub ring $C : R/N \to L$, where L is a complete infinite distributive lattice, defined by $\mu_C(x + N) = \vee\mu_B(x + N)$ and $\nu_C(x + N) = \wedge\nu_B(x + N)$ for each $x \in R$, is called the *L-if/v-quotient sub ring* of R/N relative to B and is denoted by B/N or $\dfrac{B}{N}$.

In other words when N is an ideal of R and B is any L-if/v-sub ring of R, and L is a complete infinite distributive lattice, $\frac{B}{N}$: $\frac{R}{N} \to L$ is defined by $\mu_{\frac{B}{N}}(r+N) = \vee\mu_B(r+N)$ and $\nu_{\frac{B}{N}}(r+N) = \wedge\nu_B(r+N)$ for each $r \in R$.

For any pair of L-if/v-sub rings A and B of a ring R such that $A \leq B$,

(i) A is called an *L-if/v-left ideal* of B iff for each $x, r \in R$, $\mu_A(rx) \geq \mu_B(r) \wedge \mu_A(x)$ and $\nu_A(rx) \leq \nu_B(r) \vee \nu_A(x)$.

(ii) A is called an *L-if/v-right ideal* of B iff for each $x, r \in R$, $\mu_A(xr) \geq \mu_B(r) \wedge \mu_A(x)$ and $\nu_A(xr) \leq \nu_B(r) \vee \nu_A(x)$.

(iii) A is called an *L-if/v-ideal* of B iff A is both *L-if/v-left ideal* and *L-if/v-right ideal* of B.

Theorem 2.6. For any pair of L-if/v-sub rings A and B of a ring R such that A is an L-if/v-(left, right)ideal of B:

1. A_* is an (left, right) ideal of B_*.
2. A^* is an (left, right) ideal of B^* whenever L is strongly regular.

Lemma 2.7. For any pair of L-if/v-sub rings A and B of a ring R such that A is an L-if/v ideal of B, the L-if/v-subset C: $\frac{B^*}{A^*} \to L$ defined by, for each $b \in B^*$ $\mu_C(b+A^*) = \vee\mu_B(b+A^*)$ and $\nu_C(b+A^*) = \wedge\nu_B(b+A^*)$, is an L-if/v-sub ring of $\frac{B^*}{A^*}$, whenever L is a strongly regular complete infinite distributive lattice.

Definition 2.8. For any pair of L-if/v-sub rings A and B of a ring R such that A is an L-if/v ideal of B and L is a strongly regular complete infinite distributive lattice, the L-if/v-quotient sub ring of $B|B^*$ relative to A^*, denoted by B/A or $\frac{B}{A}$, is defined by B/A: $B^*/A^* \to L$ with $\mu_{B/A}(b+A^*) = \vee\mu_B(b+A^*)$ and $\nu_{B/A}(b+A^*) = \wedge\nu_B(b+A^*)$ for each $b \in B^*$ and is called *L-if/v-quotient sub ring of B relative to A*.

Lemma 2.9. For any pair of L-if/v-sub rings A and B of a ring R such that A is an L-if/v-ideal of B, $(B/A)^* = B^*/A^*$.

Theorem 2.10. For any L-if/v-(left, right) ideal A of R and an L-if/v-sub ring B of R, $A \wedge B$ is an L-if/v-(left, right) ideal of B.

3. Main Results

In this section, we begin with a result on L-if/v-(inverse) image of an L-if/v-ideal. Then we go on to introduce various notions of L-if/v-homo (iso) morphisms. Finally, in the end we show that $\eta A = B$ implies $\eta A^* = B^*$, under certain conditions. Then we generalize the First Isomorphism Theorem, Correspondence Theorem, the Second Isomorphism Theorem and the Third Isomorphism Theorem in that order.

Theorem 3.1. For any pair of rings R_1 and R_2 and for any pairs of L-if/v-sub rings A, B of R_1 and C, D of R_2, for any ring homomorphism η: $R_1 \to R_2$ such that A is an L-if/v-ideal of B and C is a L-if/v-ideal of D,

1. ηA is an L-if/v-ideal of ηB whenever L is a complete infinite distributive lattice
2. $\eta^{-1}C$ is an L-if/v-ideal of $\eta^{-1}D$.

Proof.

(1) Since A, B are L-if/v-subrings of R_1, by 2.2(1), $\eta A, \eta B$ are L-if/v-subrings of R_2. Further, $A \leq B$ implies $\eta A \leq \eta B$. So ηA is an L-if/v-subring of ηB.

Since (a) L is a complete infinite distributive lattice, (b) $\eta u = x$, $\eta v = y$ impliy $uv \in \eta^{-1}(xy)$ and (c) A is an L-if/v-ideal of B, we get that $\mu_{\eta A}(xy) = \vee_{z\in\eta^{-1}(xy)}\mu_A z \geq \vee_{u\in\eta^{-1}x, v\in\eta^{-1}y}\mu_A(uv) \geq \vee_{u\in\eta^{-1}x, v\in\eta^{-1}y}(\mu_B(u) \wedge \mu_A(v)) = (\vee_{v\in\eta^{-1}x}\mu_B(v)) \wedge (\vee_{u\in\eta^{-1}y}\mu_A(v)) = \mu_{\eta B}(x) \wedge \mu_{\eta A}(y)$ and $\nu_{\eta A}(xy) = \wedge_{z\in\eta^{-1}(xy)}\nu_A z \leq \wedge_{u\in\eta^{-1}x, v\in\eta^{-1}y}\nu_A(uv) \leq \wedge_{u\in\eta^{-1}x, v\in\eta^{-1}y}(\nu_B(u) \vee \nu_A(v)) = (\wedge_{u\in\eta^{-1}x}\nu_B(u)) \vee (\wedge_{v\in\eta^{-1}y}\nu_A(v)) = \nu_{\eta B}(x) \vee \nu_{\eta A}(y)$. Hence $\mu_{\eta A}(xy) \geq \mu_{\eta B}(x) \wedge \mu_{\eta A}(y)$ and $\nu_{\eta A}(xy) \leq \nu_{\eta B}(x) \vee \nu_{\eta A}(y)$. And similarly $\mu_{\eta A}(yx) \geq \mu_{\eta B}(x) \wedge \mu_{\eta A}(y)$ and $\nu_{\eta A}(yx) \leq \nu_{\eta B}(x) \vee \nu_{\eta A}(y)$.Hence ηA is an L-if/v-ideal of ηB whenever L is a complete infinite-distributive lattice.

(2) Since C, D are L-if/v-subrings of R_2, by 2.2(2), $\eta^{-1}C, \eta^{-1}D$ are L-if/v-subrings of R_1. Further, $C \leq D$ implies $\eta^{-1}C \leq \eta^{-1}D$. So, $\eta^{-1}C$ is an L-if/v-subring of $\eta^{-1}D$. Since C is an L-if/v-ideal of D, for each $x, y \in R_1$, $\mu_{\eta^{-1}C}(xy) = \mu_C\eta(xy) = \mu_C(\eta(x)\eta(y)) \geq \mu_D(\eta x) \wedge \mu_C(\eta y) = \mu_{\eta^{-1}D}(x) \wedge \mu_{\eta^{-1}C}(y)$ and $\nu_{\eta^{-1}C}(xy) = \nu_C\eta(xy) = \nu_C(\eta(x)\eta(y)) \leq \nu_D(\eta x) \vee \nu_C(\eta y) = \nu_{\eta^{-1}D}(x) \vee \nu_{\eta^{-1}C}(y)$. Hence $\mu_{\eta^{-1}A}(xy) \geq \mu_{\eta^{-1}D}(x) \wedge \mu_{\eta^{-1}C}(y)$ and $\nu_{\eta^{-1}C}(xy) \leq \nu_{\eta^{-1}D}(x) \vee \nu_{\eta^{-1}C}(y)$. And similarly $\mu_{\eta^{-1}C}(yx) \geq \mu_{\eta^{-1}D}(x) \wedge \mu_{\eta^{-1}C}(y)$ and $\nu_{\eta^{-1}C}(yx) \leq \nu_{\eta^{-1}D}(x) \vee \nu_{\eta^{-1}C}(y)$. Hence $\eta^{-1}C$ is an L-if/v-ideal of $\eta^{-1}D$. ■

Definition 3.2. Let R_1 and R_2 be rings and let A be an L-if/v-sub ring of R_1 and B be an L-if/v-sub ring of R_2.

(a) A homomorphism η: $R_1 \to R_2$ is called a *L-if/v-weak homomorphism* of A into B iff $\eta A \leq B$. If η is a L-if/v-weak homomorphism of A into B, then we say that A is *L-if/v-weak homomorphic* to B.

(b) An isomorphism η: $R_1 \to R_2$ is called an *L-if/v-weak isomorphism* of A into B iff $\eta A \leq B$. If η is an L-if/v-weak isomorphism of A into B, then we say that A is *L-if/v-weak isomorphic* to B.

(c) An epimorphism η: $R_1 \to R_2$ is called an *L-if/v-epimorphism* of A onto B iff $\eta A = B$. If η is an L-if/v-epimorphism of A onto B, then we say that A is *L-if/v-epimorphic* to B.

(d) An isomorphism η: $R_1 \to R_2$ is called an *L-if/v-isomorphism* of A onto B iff $\eta A = B$. If η is an L-if/v-isomorphism of A onto B, then we say that A is *L-if/v-isomorphic* to B.

In Isomorphism Theorems we need some kind of one-one ness of the CF-operator N at the lower extreme 0_L of L in the sense that $\alpha > 0$ implies $N\alpha < 1$. Note that, since 1_L is the largest element of L and 0_L is the smallest element of L, the former statement is equivalent to $N\alpha = 1 (= N0)$ implies $\alpha = 0$, as shown in the following:

Since $\alpha \geq 0_L, \alpha \neq 0$ implies $\alpha > 0$ which implies $N\alpha < 1$, which is a contradiction. On the other hand, since $N\alpha \leq 1_L$, $N\alpha \not< 1$ implies $N\alpha = 1$ but then $\alpha = 0$, which is a contradiction.

(e) $\alpha > 0$ implies $N\alpha < 1$ if and only if $N\alpha = 1$ implies $\alpha = 0$.

(f) For any complete lattice L with the CF operator N, N is *one-one at* 0_L for L if and only if $N\alpha (= N(0)) = 1$ implies $\alpha = 0$.

Theorem 3.3. For any pair of L-if/v-sub rings A and B of a ring R such that A is an L-if/v-ideal of B, $B|B^*$ is homomorphic to B/A, whenever L is a strongly regular complete infinite distributive lattice.

Proof. By 2.6, A is an L-if/v-ideal of B implies A^* is an ideal of B^*. Let $\eta: B^* \to B^*/A^*$ be the natural homomorphism which is onto. we will show that $\eta(B|B^*) = B/A$: $B^*/A^* \to L$.

(1) $\mu_{\eta(B|B^*)}(b + A^*) = \vee \mu_{B|B^*} \eta^{-1}(b + A^*) = \vee \mu_B(b + A^*) = \mu_{B/A}(b + A^*)$ for each $b \in B^*$ where the first equality is due to the definition of L-if/v-image, the second equality is due to $\eta^{-1}(b + A^*) = \{b' \in B^*/\eta b' = b + A^*\} = \{b' \in B^*/b' + A^* = b + A^*\} = b + A^*$ and the last equality is due to the definition 2.7 of B/A. Hence $\mu_{\eta(B|B^*)} = \mu_{B/A}$.

(2) $\nu_{\eta(B|B^*)}(b + A^*) = \wedge \nu_{B|B^*} \eta^{-1}(b + A^*) = \wedge \nu_B(b + A^*) = \nu_{B/A}(b + A^*)$ for each $b \in B^*$ or $\nu_{\eta(B|B^*)} = \nu_{B/A}$.
So, $\eta(B|B^*) = B/A$. Thus η: $B|B^* \to B/A$ is an onto homomorphism. ■

Lemma 3.4. For any epimorphism η: $R_1 \to R_2$ and for any pair of L-if/v-subsets A of R_1 and B of R_2 such that $\eta(A) = B$, $\eta(A^*) = B^*$ if the CF-operator N is one-one at 0_L for L.

Proof. Let $\eta(A) = B$. Then $\mu_{\eta(A)}(y) = \vee \mu_A \eta^{-1} y$ and $\nu_{\eta(A)}(y) = \wedge \nu_A \eta^{-1} y$. Now we show that $\eta(A^*) = B^*$.

$\alpha \in \eta(A^*)$ implies $\alpha = \eta a$, $a \in A^*$. Now $\mu_{\eta A}\alpha = \mu_{\eta A}\eta a = \vee \mu_A \eta^{-1}\eta a \geq \mu_A a > 0$ and $\nu_{\eta A}\alpha = \nu_{\eta A}\eta a = \wedge \nu_A \eta^{-1}\eta a \leq \nu_A a < 1$. Therefore $\alpha \in B^*$. Hence $\eta(A^*) \subseteq B^*$.

Suppose $\alpha \in B^*$. Then $\mu_B \alpha > 0$ which implies $\mu_{\eta(A)}\alpha > 0$ implying $\vee \mu_A \eta^{-1}\alpha > 0$. Since η is onto $\eta^{-1}\alpha \neq \phi$. If for each $x \in \eta^{-1}\alpha$, $\mu_A x = 0$, then $\vee \mu_A \eta^{-1}\alpha = 0$ which is a contradiction. Therefore, there exists $x_0 \in \eta^{-1}\alpha$ such that $\mu_A x_0 > 0$. Now, $0 < \mu_A x_0$ implies $N\mu_A x_0 < 1$ because N is one-one at 0_L for L. But $\nu_A x_0 \leq N\mu_A x_0$. So, $\nu_A x_0 < 1$. Therefore $x_0 \in A^*$ and $\alpha = \eta x_0 \in \eta A^*$ implying $B^* \subseteq \eta(A^*)$.Therefore $\eta(A^*) = B^*$. ■

4. First Isomorphism Theorem

Theorem 4.1. For any L-if/v-sub ring B of a ring R_1 and for any L-if/v-sub ring C of a ring R_2 such that B is L-if/v-epimorphic to C, there exists an L-if/v-ideal A of B such that B/A is L-if/v-isomorphic to $C|C^*$, whenever L is a strongly regular infinite distributive lattice with CF operator N which is one-one at 0_L for L.

Proof. Since (1) B is L-if/v-epimorphic to C (2) L is a strongly regular infinite distributive lattice with CF-operator N which is one-one at 0_L for L and (3) 3.2(c) and 3.4, there exists an epimorphism η: $R_1 \to R_2$ such that $\eta(B) = C$ and $\eta(B^*) = C^*$.

Define $A : R_1 \to L$ by $\mu_A x = \mu_B x$, $\nu_A x = \nu_B x$ when $x \in \text{Ker}\eta$ and $\mu_A x = 0_L$, $\nu_A x = 1_L$ when $x \notin \text{Ker}\eta$.

First we show that A is an L-if/v-subring of R_1 or to show that

(i) $\mu_A(xy) \geq \mu_A x \wedge \mu_A y$, $\nu_A(xy) \leq \nu_A x \vee \nu_A y$, (ii) $\mu_A(x - y) \geq \mu_A x \wedge \mu_A y$, $\nu_A(x - y) \leq \nu_A x \vee \nu_A y$.

Without loss of generality both x, y be in Kerη. Then $\mu_A x = \mu_B x$, $\nu_A x = \nu_B x$ and $\mu_A y = \mu_B y$, $\nu_A y = \nu_B y$ and xy, $x - y \in \text{Ker}\eta$. Thus

(i) $\mu_A(xy) = \mu_B(xy) \geq \mu_B x \wedge \mu_B y = \mu_A x \wedge \mu_A y$ and $\nu_A(xy) = \nu_B(xy) \leq \nu_B x \vee \nu_B y = \nu_A x \vee \nu_A y$

(ii) $\mu_A(x - y) = \mu_B(x - y) \geq \mu_B x \wedge \mu_B y = \mu_A x \wedge \mu_A y$ and $\nu_A(x - y) = \nu_B(x - y) \leq \nu_B x \vee \nu_B y = \nu_A x \vee \nu_A y$.

Now we show that A is an L-if/v-ideal of B or to show that

(i) $\mu_A(xy) \geq \mu_B x \wedge \mu_A y$ and $\nu_A(xy) \leq \nu_B x \vee \nu_A y$

(ii) $\mu_A(yx) \geq \mu_B x \wedge \mu_A y$ and $\nu_A(yx) \leq \nu_B x \vee \nu_A y$.

Once again without loss of generality both x, y be in Kerη. Then $\mu_A x = \mu_B x$, $\nu_A x = \nu_B x$ and $\mu_A y = \mu_B y$, $\nu_A y = \nu_B y$. Then $xy \in \text{Ker}\eta$. Thus

(i) $\mu_A(xy) = \mu_B(xy) \geq \mu_B x \wedge \mu_B y = \mu_B x \wedge \mu_A y$ and $\nu_A(xy) = \nu_B(xy) \leq \nu_B x \vee \nu_B y = \nu_B x \vee \nu_A y$

(ii) $\mu_A(yx) = \mu_B(yx) \geq \mu_B y \wedge \mu_B x = \mu_B x \wedge \mu_A y$ and $\nu_A(yx) = \nu_B(yx) \leq \nu_B y \vee \nu_B x = \nu_B x \vee \nu_A y$. Hence A is an L-if/v-ideal of B.

Since B is L-if/v-epimorphic to C, $\eta B = C$ and $\eta B^* = C^*$. Let $\phi = \eta | B^*$. Then $\phi : B^* \to C^*$ is an epimorphism and Kerϕ = A^* as follows:
$g \in$ Kerϕ implies $g \in B^*$ and $\eta g = \phi g = 0$ implies $g \in$ Kerη or $\mu_A g = \mu_B g$ and $\nu_A g = \nu_B g$. Since $g \in B^*$, $\mu_A g = \mu_B g > 0_L$ and $\nu_A g = \nu_B g < 1_L$. So, $g \in A^*$.

Conversely, $A^* \subseteq B^*$, $\phi = \eta | B^*$ and the definition of A imply if $g \in A^*$ then $\mu_A g > 0_L$ which implies $g \in$ Kerη. But $\eta g = 0$ implies $\phi g = 0$ or $g \in$ Kerϕ. Thus Kerϕ = A^*.

Thus by (crisp) First Isomorphism Theorem, there is an isomorphism $h : \dfrac{B^*}{A^*} = \dfrac{B^*}{Ker\phi} \to C^*$ such that $h(b + A^*) = \phi b$.

Now we show that $h(\dfrac{B}{A}) = C$. But first we show that for all $z \in C^*$, if $h^{-1}z = b_0 + A^*$ or $z = h(b_0 + A^*)$ then $\eta^{-1}z = b_0 + A^*$, as follows:
$b \in \eta^{-1}z$ implies $\eta b = z$ which implies $h(b + A^*) = \phi b = \eta b = z = h(b_0 + A^*)$ or $b + A^* = b_0 + A^*$ or $b \in b_0 + A^*$ because $0 \in A^*$. Thus $\eta^{-1}z \subseteq b_0 + A^*$.

Conversely, $b_0 + a \in b_0 + A^*$, $a \in A^*$ implies $\eta(b_0 + a) = \phi(b_0 + a) = h(b_0 + a + A^*) = h(b_0 + A^*) = z$. Thus for $a \in A^*$, $b_0 + a \in \eta^{-1}z$ or $b_0 + A^* \subseteq \eta^{-1}z$.

Now, since L is a strongly regular complete infinite distributive lattice, the definitions of the L-if/v-image and the L-if/v-quotient B/A imply:

$$\mu_{h(\frac{B}{A})}(z) = \vee \mu_{\frac{B}{A}}(h^{-1}z) = \mu_{\frac{B}{A}}(b_0 + A^*) = \vee \mu_B(b_0 + A^*) \text{ and}$$
$$\nu_{h(\frac{B}{A})}(z) = \wedge \nu_{\frac{B}{A}}(h^{-1}z) = \nu_{\frac{B}{A}}(b_0 + A^*) = \wedge \nu_B(b_0 + A^*).$$

On the other hand, the definition of the L-if/v-image, ηB implies:
$\mu_C z = \mu_{\eta B} z = \vee \mu_B(\eta^{-1}z)$ and $\nu_C z = \nu_{\eta B} z = \wedge \nu_B(\eta^{-1}z)$.
Now, since $h^{-1}z = b_0 + A^*$ implies $\eta^{-1}z = b_0 + A^*$, $\mu_{h(\frac{B}{A})}(z) = \mu_C z$ and $\nu_{h(\frac{B}{A})}(z) = \nu_C z$ or $h\left(\dfrac{B}{A}\right) = C$ or $\dfrac{B}{A}$ is L-if/v-isomorphic to $C|C^*$. ■

Lemma 4.2. For any L-if/v-sub ring A of a ring R,

(1) $\mu_A(x_1 - x_2) = \mu_A 0 \Rightarrow \mu_A(x_1) = \mu_A(x_2)$. (2) $\nu_A(x_1 - x_2) = \nu_A 0 \Rightarrow \nu_A(x_1) = \nu_A(x_2)$.

Proof. By 2.1(a) and 2.1(g),

(1) $\mu_A(x_1) = \mu_A(x_1 - x_2 + x_2) \geq \mu_A(x_1 - x_2) \wedge \mu_A(x_2) = \mu_A 0 \wedge \mu_A x_2 = \mu_A x_2$ because $\mu_A 0$ is the largest of $\mu_A R$. On the other hand, $\mu_A(x_2) = \mu_A(x_2 - x_1 + x_1) \geq \mu_A(x_2 - x_1) \wedge \mu_A(x_1) = \mu_A(-(x_1 - x_2)) \wedge \mu_A(x_1) = \mu_A(x_1 - x_2) \wedge \mu_A(x_1) = \mu_A 0 \wedge \mu_A x_1 = \mu_A x_1$. Therefore $\mu_A(x_1) = \mu_A(x_2)$.

(2) $\nu_A(x_1) = \nu_A(x_1 - x_2 + x_2) \leq \nu_A(x_1 - x_2) \vee \nu_A(x_2) = \nu_A 0 \vee \nu_A x_2 = \nu_A x_2$ because $\nu_A 0$ is the least of $\nu_A R$. On the other hand, $\nu_A(x_2) = \nu_A(x_2 - x_1 + x_1) \leq \nu_A(x_2 - x_1) \vee \nu_A(x_1) = \nu_A(-(x_1 - x_2)) \vee \nu_A(x_1) = \nu_A(x_1 - x_2) \vee \nu_A(x_1) = \nu_A 0 \vee \nu_A x_1 = \nu_A x_1$. Therefore $\nu_A(x_1) = \nu_A(x_2)$. ■

5. Correspondence Theorem

Theorem 5.1. For any pair of rings R_1 and R_2 and for any epimorphism $\eta{:}R_1 \to R_2$, $A \to \eta A$ defines a one-one correspondence between, the set $\zeta_K(R_1)$ of all L-if/v-sub rings of R_1 that are constant on the Ker(η) and the set $\zeta(R_2)$ of all L-if/v-subrings of R_2 in such a way that:

(a) $A_1 \le A_2$ iff $\eta A_1 \le \eta A_2$ for all $A_1, A_2 \in \zeta_K(R_1)$.

(b) A is an L-if/v-ideal of R_1 iff ηA is an L-if/v-ideal of R_2.

Proof.

(1) Since η is surjective, for each $B \in R_2$, $\eta\eta^{-1}B = B$.

(2) If A is an L-if/v-subring of R_1 and A is constant on Ker(η) then for each $x \in Ker(\eta)$, $\mu_A x = \mu_A 0$ and $\nu_A x = \nu_A 0$ because $0 \in Ker(\eta)$.

(3) Next for any $y \in R_2$, μ_A is constant on $\eta^{-1}y$ because, $x_1, x_2 \in \eta^{-1}y$ implies $\eta x_1 = \eta x_2$ which implies $x_1 - x_2 \in$ Ker(η) or $\mu_A(x_1 - x_2) = \mu_A 0$ or $\mu_A(x_1) = \mu_A(x_2)$, by the previous Lemma.

Thus for any $y \in R_2$, $\vee \mu_A \eta^{-1} y = \mu_A x$. Similarly $\wedge \nu_A \eta^{-1} y = \nu_A x$ where $x \in \eta^{-1}y$ is any element.

(4) Now we show that $\eta^{-1}\eta A = A$.

Let $B = \eta A$ and $C = \eta^{-1}B$. Then $\mu_B y = \vee \mu_A \eta^{-1} y$, $\nu_B y = \wedge \nu_A \eta^{-1} y$, for each $y \in R_2$.

And $\mu_C x = \mu_B \eta x = \vee \mu_A \eta^{-1}\eta x = \mu_A x$, $\nu_C x = \nu_B \eta x = \wedge \nu_A \eta^{-1}\eta x = \nu_A x$, since A is constant on $\eta^{-1}y$, for each $y \in R_2$.

Thus $\eta^{-1}\eta A = A$ when A is constant on $ker(\eta)$.

From (1) and (4) we shall get that $A \to \eta A$ is a one-one correspondence between the set $\zeta_K(R_1)$ of all L-if/v-subrings of R_1 that are constant on the Ker(η) and the set $\zeta(R_2)$ of all L-if/v-subrings of R_2.

(a) Let $A_1, A_2 \in \zeta_K(R_1)$. By monotonicity for L-if/v-images of L-if/v-subsets, $A_1 \le A_2$ implies $\eta A_1 \le \eta A_2$.

On the other hand, by monotonicity for L-if/v-inverse images of L-if/v-subsets, $\eta A_1 \le \eta A_2$ implies $\eta^{-1}\eta A_1 \le \eta^{-1}\eta A_2$, but since A_1, A_2 are constant on Ker(η), we shall get $A_1 = \eta^{-1}\eta A_1 \le \eta^{-1}\eta A_2 = A_2$.

Hence from the above we shall get that $A_1 \le A_2$ iff $\eta A_1 \le \eta A_2$.

(b) Let $A \in \zeta_K(R_1)$. By 2.2(1) if A is an L-if/v-ideal of R then $\eta(A)$ is an L-if/v-ideal of R_2 since η is onto homomorphism from $R_1 \to R_2$. Again by 2.2(b), if ηA is an L-if/v-ideal of R_2 then $\eta^{-1}\eta A = A$ is an L-if/v-ideal of R_1, since A is constant on Ker(η).

Thus A is an L-if/v-ideal of R_1 iff ηA is an L-if/v-ideal of R_2. ■

Corollary 5.2. The complete lattice of all L-if/v-(ideal) sub rings of R_1 that are constant on the Ker(η) is isomorphic to the complete lattice of all L-if/v-(ideal) subrings of R_2.

Proof. Since the set of all L-if/v-(ideal) subrings of R_1 is a complete lattice, the proof follows from the previous theorem. ■

6. Second Isomorphism Theorem

Theorem 6.1. For any ring R and for any pair of L-if/v-sub rings A and B of R such that A is an L-if/v-ideal of R and $A(0) = B(0)$, $B/(A \wedge B)$ is L-if/v-weak isomorphic to $(A + B)/A$, whenever L is a strongly regular complete infinite distributive lattice with CF operator N which is one-one at 0_L for L.

Proof. From 2.1(h), if A is an L-if/v-ideal of R then A^* is a ideal of R, where $A^* = \{x \in R/\mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$. Hence $A^* \cap B^*$ is an ideal of B^*, $A^* + B^*$ is a subring of R and A^* is an ideal of $A^* + B^*$.

By the second isomorphism theorem for crisp rings,
$\eta : B^*/(A^* \cap B^*) \to (A^* + B^*)/A^*$, defined by $\eta(b + (A^* \cap B^*)) = b + A^*$ is an isomorphism.

First we show that (a) $(A \wedge B)^* = (A^* \cap B^*)$ (b) $(A + B)^* = A^* + B^*$.

(a) Always $x \in (A \wedge B)^*$ implies $\mu_{A\wedge B}(x) > 0$ and $\nu_{A\wedge B}(x) < 1$ implying $\mu_A x$, $\mu_B x \geq \mu_A x \wedge \mu_B x = (\mu_A \wedge \mu_B)(x) = \mu_{A\wedge B}(x) > 0$ and $\nu_A x, \nu_B x \leq \nu_A x \vee \nu_B x = (\nu_A \vee \nu_B)(x) = \nu_{A\wedge B}(x) < 1$ or $x \in (A^* \cap B^*)$ which in turn implies that $(A \wedge B)^* \subseteq (A^* \cap B^*)$.

On the other hand, since L is strongly regular $x \in A^* \cap B^*$ implies $\mu_A x > 0$ and $\nu_A x < 1$, $\mu_B x > 0$ and $\nu_B x < 1$ which implies $\mu_A x \wedge \mu_B x > 0$ and $\nu_A x \vee \nu_B x < 1$ or $\mu_{A\wedge B}(x) > 0$ and $\nu_{A\wedge B}(x) < 1$ implying $x \in (A \wedge B)^*$ or $(A^* \cap B^*) \subseteq (A \wedge B)^*$.Hence $(A^* \cap B^*) = (A \wedge B)^*$.

(b) Let us recall that $(A + B)^* = \{x \in R/\mu_{A+B}(x) > 0 \text{ and } \nu_{A+B}(x) < 1\}$.

Let $x \in (A + B)^*$. Then $\mu_{A+B}(x) = \vee_{x=y+z}(\mu_A(y) \wedge \mu_B(z)) > 0$ and $\nu_{A+B}(x) = \wedge_{x=y+z}(\nu_A(y) \vee \nu_B(z)) < 1$.

Let $\alpha = \vee_{x=y+z}(\mu_A(y) \wedge \mu_B(z))$. There exists $y, z \in R$ such that $x = y + z$, otherwise $\alpha = \vee\phi = 0$ which is a contradiction. Now if for each $y, z \in R$ such that $x = y + z$, $\mu_A y \wedge \mu_B z = 0$ then again $\alpha = 0$ which is not true. So, there exist $y_1, z_1 \in R$ such that $x = y_1 + z_1$ and $(\mu_A(y_1) \wedge \mu_B(z_1)) > 0$. So, $\alpha, \beta \geq \alpha \wedge \beta > 0$ implies $\mu_A(y_1) > 0$, $\mu_B(z_1) > 0$, but N is a CF operator with $\alpha > 0$ implies $N\alpha < 1$. Therefore $\mu_A(y_1) > 0$ implies $\nu_A(y_1) \leq N\mu_A(y_1) < 1$ and $\mu_B(z_1) > 0$ implies $\nu_B(z_1) \leq N\nu_B(z_1) < 1$ where $x = y_1 + z_1$. Thus $x = y_1 + z_1$ with $y_1 \in A^*$, $z_1 \in B^*$ or $x \in A^* + B^*$. Thus $(A + B)^* \subseteq A^* + B^*$.

Let $x \in A^* + B^*$. Then $x = y + z$, $y \in A^*$, $z \in B^*$. Hence $\mu_A(y) > 0$, $\nu_A(y) < 1$, $\mu_B(z) > 0$ and $\nu_B(z) < 1$.

Since L is strongly regular, $(\mu_A(y) \wedge \mu_B(z)) > 0$, $(\nu_A(y) \vee \nu_B(z)) < 1$ and $x = y + z$. Hence $\mu_{A+B}(x) \geq \mu_A(y) \wedge \mu_B(z) > 0$ and $\nu_{A+B}(x) \leq \nu_A(y) \vee \nu_B(z) < 1$ which implies $x \in (A + B)^*$. Thus $A^* + B^* \subseteq (A + B)^*$.Hence $(A + B)^* = A^* + B^*$.

Consequent of (a) and (b), $\eta : B^*/(A \wedge B)^* \to (A + B)^*/A^*$, defined by $\eta(b + (A \wedge B)^*) = b + A^*$ for each $b \in B^*$ is the isomorphism.

Now we show that the above η, in fact, defines an L-if/v-weak isomorphism between the L-if/v-subrings $B/(A \wedge B)$ and $(A + B)/A$.

Observe that, since L is a strongly regular complete infinite-distributive lattice,

(1) $B/(A \wedge B) : B^*/(A \wedge B)^* \to L$ is defined by $\mu_{B/(A\wedge B)}(b + (A \wedge B)^*)$ $= \vee\mu_B(b + (A \wedge B)^*)$ and $\nu_{B/(A\wedge B)}(b + (A \wedge B)^*) = \wedge\nu_B(b + (A \wedge B)^*)$ for each $b \in B^*$.

(2) $(A + B)/A : (A + B)^*/A^* \to L$ defined by $\mu_{(A+B)/A}(x + A^*)$ $= \vee\mu_{(A+B)}(x + A^*)$ and $\nu_{(A+B)/A}(x + A^*) = \wedge\nu_{(A+B)}(x + A^*)$ for each $x \in (A + B)^*$.

(3) $A \wedge B \leq A$ implies $(A \wedge B)^* \subseteq A^*$ because $C \leq D$, $x \in C^*$ implies $\mu_D x \geq \mu_C x > 0$, $\nu_D x \leq \nu_C x < 1$ or $x \in D^*$.

Now we claim that, $\eta : B^*/(A \wedge B)^* \to (A + B)^*/A^*$ defined by $\eta(b + (A \wedge B)^*)$ $= b + A^*$, $b \in B^*$ is the L-if/v-weak isomorphism of $B/(A \wedge B)$ to $(A + B)/A$.

Let $b \in B^*$. Let $c \in (A \wedge B)^*$ be arbitrary. Then $z = b + c \in b + (A \wedge B)^* \subseteq b + B^* = B^*$ since $b \in B^*$. Since $A(0) = B(0)$, for each $z \in b + (A \wedge B)^*$, $\mu_{(A+B)}z = \mu_{(A+B)}(0+z)$ $\geq \mu_A 0 \wedge \mu_B z = \mu_B 0 \wedge \mu_B z = \mu_B z$ and $\nu_{(A+B)}z = \nu_{(A+B)}(0+z) \leq \nu_A 0 \vee \nu_B z = \nu_B 0 \vee \nu_B z$ $= \nu_B z$, implying thus $\vee\mu_{(A+B)}(b + (A \wedge B)^*) = \vee_{z \in b+(A\wedge B)^*}\mu_{(A+B)}z \geq \vee_{z\in b+(A\wedge B)^*}\mu_B z$ $= \vee\mu_B(b + (A \wedge B)^*)$ and $\wedge\nu_{(A+B)}(b + (A \wedge B)^*) \leq \wedge\nu_B(b + (A \wedge B)^*)$.

On the other hand since η is defined by $\eta(b + (A \wedge B)^*) = b + A^*$ for each $b \in B^*$, is one-one, we get that

(1) $\mu_{\eta(B/A\wedge B)}(b + A^*) = \vee\mu_{(B/A\wedge B)}\eta^{-1}(b + A^*) = \mu_{(B/A\wedge B)}(b + (A \wedge B)^*)$ $= \vee\mu_B(b + (A \wedge B)^*) \leq \vee\mu_{(A+B)}(b + (A \wedge B)^*) \leq \vee\mu_{(A+B)}(b + A^*)$ $= \mu_{((A+B)/A)}(b + A^*)$, because $(A \wedge B)^* \subseteq A^*$.

(2) $\nu_{\eta(B/A\wedge B)}(b + A^*) = \wedge\nu_{(B/A\wedge B)}\eta^{-1}(b + A^*) = \nu_{(B/A\wedge B)}(b + (A \wedge B)^*)$ $= \wedge\nu_B(b + (A \wedge B)^*) \geq \wedge\nu_{(A+B)}(b + (A \wedge B)^*) \geq \wedge\nu_{(A+B)}(b + A^*)$ $= \nu_{((A+B)/A)}(b + A^*)$, because $(A \wedge B)^* \subseteq A^*$.Hence $B/(A \wedge B)$ is L-if/v-weak isomorphic to $(A + B)/A$. ■

7. Third Isomorphism Theorem

Theorem 7.1. For any L-if/v-sub rings A, B and C of a ring R such that A is an L-if/v-ideal of B and A, B are L-if/v-ideals of C, the L-if/v-sub ring $\dfrac{C/A}{B/A}$ is L-if/v-isomorphic to C/B, whenever L is a strongly regular complete infinite distributive lattice.

Proof.

(1) Since A is an L-if/v-ideal of B and L is a strongly regular complete infinite distributive lattice, by 2.6(2), A^* is an ideal of B^*, by 2.8, the L-if/v-quotient subring $\dfrac{B}{A}$ exists and by 2.9, $(B/A)^* = B^*/A^*$.

Since A, B are L-if/v-ideals of C and L is a strongly regular complete infinite distributive lattice, by 2.6(2), A^*, B^* are ideals of C^*, by 2.8, the L-if/v-quotient subrings $\dfrac{C}{A}$ and $\dfrac{C}{B}$ exist and by 2.9, $(C/A)^* = C^*/A^*$ and $(C/B)^* = C^*/B^*$.

By Third Isomorphism Theorem for crisp rings, $\eta : \dfrac{C^*/A^*}{B^*/A^*} \to C^*/B^*$ defined by $\eta(x + A^*.\dfrac{B^*}{A^*}) = x + B^*$ for each $x \in C^*$ is the ring isomorphism.

(2) B/A is an L-if/v-ideal of C/A as follows:

Let us recall that $\dfrac{B}{A} : \dfrac{B^*}{A^*} \to L$ is defined by, for each $b \in B^*$, $\mu_{B/A}(b + A^*) = \vee\mu_B(b + A^*)$ and $\nu_{B/A}(b + A^*) = \wedge\nu_B(b + A^*)$ and $\dfrac{C}{A} : \dfrac{C^*}{A^*} \to L$ is defined by, for each $c \in C^*$, $\mu_{C/A}(c + A^*) = \vee\mu_C(c + A^*)$ and $\nu_{C/A}(c + A^*) = \wedge\nu_C(c + A^*)$.

Now (a) Since $B \leq C$ on R, for each $c \in C^*$, $\mu_{B/A}(c + A^*) = \vee\mu_B(c + A^*) \leq \vee\mu_C(c + A^*) = \mu_{C/A}(c + A^*)$ and $\nu_{C/A}(c + A^*) = \wedge\nu_C(c + A^*) \geq \wedge\nu_B(c + A^*) = \nu_{B/A}(c + A^*)$ or $B/A \leq C/A$.

(b) Since A is an L-if/v-ideal of both B and C and, L is a strongly regular complete infinite distributive lattice, by 2.7, both B/A and C/A are L-if/v-subrings of B^*/A^* and C^*/A^*, respectively.

(c) First, since B^* is a ideal of C^*, $\dfrac{B^*}{A^*}$ is a ideal of $\dfrac{C^*}{A^*}$.

Next, for all $a_1, a_2 \in A^*$, $b \in B^*$ and $c \in C^*$, since B is an L-if/v-ideal of C and A^* is an ideal of B^*, we get that for each $a_1, a_2 \in A^*$,

$\mu_B(b + a_1) \ \wedge \ \mu_C(c + a_2) \ \leq \mu_B(c + a_2 \ b + a_1 \ (c + a_2)) = \mu_B(bc + a_1a_2) \ \leq \vee\mu_B(bc + A^*)$.

Since L is a complete infinite meet distributive lattice, from the above,

$$\mu_{B/A}(b + A^*) \wedge \mu_{C/A}(c + A^*) = \vee\mu_B(b + A^*) \wedge \vee\mu_C(c + A^*)$$
$$\leq \vee\mu_B((b + A^*).(c + A^*)) = \mu_{B/A}((b + A^*).(c + A^*)).$$

Similarly, B is an L-if/v-ideal C, A^* is an ideal of B^* and L is complete infinite join distributive lattice, imply

$\nu_{B/A}(b + A^*) \vee \nu_{C/A}(c + A^*) \geq \nu_{B/A}((b + A^*) \cdot (c + A^*))$.

Therefore, $\dfrac{B}{A}$ is an L-if/v-ideal of $\dfrac{C}{A}$.

Now by 2.7 and (2) above, $\dfrac{C/A}{B/A}:\dfrac{C^*/A^*}{B^*/A^*}\to L$ is an L-if/v-subring of $\dfrac{C^*/A^*}{B^*/A^*}$.

(3) By 2.7, 2.8 and (2) above, $\dfrac{C/A}{B/A}:\dfrac{C^*/A^*}{B^*/A^*}\to L$ is defined by:

for each $x \in C^*$, $\mu_{\frac{C/A}{B/A}}((x + A^*).(B^*/A^*)) = \vee\mu_{C/A}((x + A^*).(B^*/A^*))$ and $\nu_{\frac{C/A}{B/A}}((x + A^*) \cdot (B^*/A^*)) = \wedge\nu_{C/A}((x + A^*) \cdot (B^*/A^*))$.

On the other hand $\eta : \dfrac{C^*/A^*}{B^*/A^*} \to \dfrac{C^*}{B^*}$ is defined by:

for each $x \in C^*$, $\eta\left((x + A^*) \cdot \dfrac{B^*}{A^*}\right) = x + B^*$.

Now we show that this η defines the necessary isomorphism between $\dfrac{C/A}{B/A}$ and C/B. Since (a) η is 1-1, $\eta^{-1}(x+B^*) = (x+A^*)\cdot\dfrac{B^*}{A^*}$, (b) $(x+A^*)\cdot\dfrac{B^*}{A^*} = \{(x+A^*)\cdot b+A^*/b \in B^*\} = \{xb + A^*/b \in B^*\}$, we shall get that

(i) $\mu_{\eta(\frac{C/A}{B/A})}(x + B^*) = \vee\mu_{\frac{C/A}{B/A}}\eta^{-1}(x + B^*) = \mu_{\frac{C/A}{B/A}}\left((x + A^*) \cdot \dfrac{B^*}{A^*}\right)$
$= \vee\mu_{C/A}\left((x + A^*) \cdot \dfrac{B^*}{A^*}\right) = \vee_{b\in B^*}\mu_{C/A}(xb + A^*) = \vee_{b\in B^*}(\vee\mu_C(xb + A^*))$
$= \vee_{b\in B^*} \vee_{z\in xb+A^*} \mu_C z = \vee_{z(\in\cup_{b\in B^*}xb+A^*)}\mu_C z$.
On the other hand, $\mu_{C/B}(x + B^*) = \vee\mu_C(x + B^*) = \vee_{z\in x+B^*}\mu_C z$.

(ii) $\nu_{\eta(\frac{C/A}{B/A})}(x + B^*) = \wedge\nu_{\frac{C/A}{B/A}}\eta^{-1}(x + B^*) = \nu_{\frac{C/A}{B/A}}\left((x + A^*) \cdot \dfrac{B^*}{A^*}\right)$
$= \wedge\nu_{C/A}\left((x + A^*) \cdot \dfrac{B^*}{A^*}\right) = \wedge_{b\in B^*}\nu_{C/A}(xb + A^*) = \wedge_{b\in B^*}(\wedge\nu_C(xb + A^*))$
$= \wedge_{b\in B^*} \wedge_{z\in xb+A^*} \nu_C z = \wedge_{z\in(\cup_{b\in B^*}xb+A^*)}\nu_C z$.

On the other hand, $\nu_{C/B}(x + B^*) = \wedge\nu_C(x + B^*) = \wedge_{z\in x+B^*}\nu_C z$.

Now $z \in x + B^*$ implies $z \in xb + A^*$ because $0 \in A^*$. Therefore from (i) and (ii) above, $\mu_{C/B}(x + B^*) \leq \mu_{\eta(\frac{C/A}{B/A})}(x + B^*)$ and $\nu_{C/B}(x + B^*) \geq \nu_{\eta(\frac{C/A}{B/A})}(x + B^*)$.

On the other hand, $z \in xb+A^*$ for some $b \in B^*$ implies $z = xb+a, a \in A^* \subseteq B^*$ or $z = x+b', b' \in B^*$, implying $z \in x+B^*$. Therefore from (i) and (ii) above, $\mu_{C/B}(x+B^*) \geq \mu_{\eta(\frac{C/A}{B/A})}(x + B^*)$ and $\nu_{C/B}(x + B^*) \leq \nu_{\eta(\frac{C/A}{B/A})}(x + B^*)$.

Therefore, $\mu_{\eta(\frac{C/A}{B/A})}(x + B^*) = \mu_{C/B}(x + B^*)$ and $\nu_{\eta(\frac{C/A}{B/A})}(x + B^*) = \nu_{C/B}(x + B^*)$ or $\eta\left(\dfrac{C/A}{B/A}\right) = \dfrac{C}{B}$. Therefore $\eta\left(\dfrac{C/A}{B/A}\right) = \dfrac{C}{B}$. Hence $\dfrac{C/A}{B/A}$ is L-if/v-isomorphic onto C/B. ■

Acknowledgements

The author would like to express her gratitude to Professor Nistala V.E.S. Murthy, Andhra university for his continuous help throughout the preparation of this document. She also wants to express her thankfulness to the management of Aditya Institute of Technology and Management, Tekkali.

References

[1] Zadeh, L., Fuzzy Sets, Information and Control, Vol. 8, P. 338–353, 1965.

[2] Atanassov, K., Intuitionistic Fuzzy Sets, V. Sgurev, Ed., VII ITKR's Session, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, (1984).

[3] Gau, W.L. and Buehrer, D.J., Vague Sets, IEEE Transactions on Systems,Man and Cybernetics, Vol. 23, P. 610–614, 1993.

[4] Liu, W.J., Fuzzy Invariant Subgroups and Fuzzy Ideals, Fuzzy Sets and Systems, Vol. 8, P. 133–139, 1982.

[5] Ren, Y.C., Fuzzy Ideals and Quotient Rings, Fuzzy Math., Vol. 4, P. 19–26, 1985.

[6] A. Rosenfeld, Fuzzy Groups, J. Math. Anal. Appl., Vol. 35, P. 512–517, 1971.

[7] Banerjee, B. and Basnet, D., Intuitionistic Fuzzy Sub Rings and Ideals, Journal of Fuzzy Mathematics, Vol. 11, No. 1, P. 139–155, 2003.

[8] Hur, K., Kang, H.W. and Song, H.K., Intuitionistic Fuzzy Subgroups and Sub Rings, Honam Mathematical Journal, Vol. 25, No. 1, P. 19–41, 2003.

[9] Hur, K., Kim, K.J. and Song, H.K., Intuitionistic Fuzzy Ideals and bi-Ideals,Honam Mathematical Journal, Vol. 26, No. 3, P. 309–330, 2004.

[10] Jun, Y.B., ztrk, M.A. and Park, C.H., Intuitionistic Nil Radicals of Intuitionistic Fuzzy Ideals and Euclidean Intuitionistic Fuzzy Ideals in Rings, Information Sciences, Vol. 177, No. 21, P. 4662–4677, 2007.

[11] Wang, J. and Lin, X., Intuitionistic Fuzzy Ideals with Thresholds of Rings, International Mathematical Forum, Vol. 4, No. 23, P. 1119–1127, 2009.

[12] Palaniappan, N., Arjunan, K. and Palanivelrajan, M., A Study on Intuitionistic L-Fuzzy Sub Rings, NIFS 14, Vol 3, P. 5–10, 2008.

[13] Meena, K. and Thomas, K.V., Intuitionistic LFuzzy Sub Rings, International Mathematical Forum, Vol. 6, No. 52, P. 2561–2572, 2011.

[14] Murthy V.E.S. Nistala and Vasanti Gopal, Properties of (Inverse) Images of Intutionistic L-Fuzzy Subsets And Correspondence and IsomorphismTheorems for Intuitionistic L-fuzzy subgroups, Internationa Journal of Computational Cognition (IJCC), Vol. 10, No. 1, P. 18–37, 2011, U.S.A.

[15] Vasanti Gopal, More On L-Intuitionistic Zuzzy Or L-Vague Quotient Rings, Advanced Fuzzy Sets And Systems(AFSS), Vol. 16, No. 2, 2013.