Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 4 (2017), pp. 1285–1302

http://www.ripublication.com/gjpam.htm

Study of Generalized Integral Transforms their Properties and Relations

Bhausaheb R. Sontakke

Department of Mathematics,
Pratishthan College, Paithan,
Dist: Aurangabad (M.S.), India.

Govind P. Kamble

Department of Mathematics,
P.E.S. College of Engineering Nagsenvana,
Aurangabad (M.S.), India.

Abstract

In this article, we study the basic theoretical properties of Mellin-type and Weyl fractional integrals and fractional derivatives. Furthermore, we prove some properties of Weyl fractional transform. Also, we study fractional Mellin transform and we prove relation between fractional Mellin transform and Fourier fractional Mellin transform.

AMS subject classification:
Keywords: Caputo Fractional derivative, Integral transform, Fractional Mellin transform, Weyl transform, Fractional Fourier transform.

1. Introduction

The differential equations have played a central role in every aspects of applied mathematics for every long time and with the advent of the computer their importance has increased further. It is also well known that throughout science,engineering and far beyond, scientific computation is taking efforts to understand and control our natural environment in order to develop new technological processes.Thus investigation and analysis of differential equations arising in applications, led to develop different techniques to solve differential equations.

In order to solve the differential equations,the integral transform were extensively used. Historically, the origin of the integral transform can be traced back to celebrated work of P.S. Laplace (1749–1827). In fact,Laplace classic book on La Theorie Analytique des Probabilities included some basic results of the Laplace transform, which was one of the oldest and most commonly used integral transforms available in the mathematical literature.In the papers [8, 9, 10], a broad study of fractional Mellin analysis was developed. In which the Hadamard-type integrals are considered and they represent the appropriate extensions of the classical Riemann–Liouville and Weyl fractional integral [11].

In the literature there are numerous integral transforms invented and widely used in physics, astronomy and engineering.Therefore large amount of research work was done on the theory and application of integral transform, such as development of the Laplace transform, Mellin transform, and Hankel transform.These integrals are also connected with the theory of moment operators [12, 13].

On the other hand,Fourier treatise provide the modern mathematical theory of heat conduction, Fourier series and Fourier integrals with applications. In an attempt to extend his ideas Fourier discovered an integral transform known as Fourier transform and the inverse Fourier transform. There are many other integral transformation including the Mellin transform, the Hankel transform, the Hilbert transform, the Steiltjes transform and the Radon transform which are widely used to solve initial and boundary value ordinary differential equations and partial differential equations.Recently, fractional integral transform is one of the flourishing fields of active research due to its wide range of applications. It is used in signal processing, image reconstruction,pattern recognition and acoustic signal processing etc. We organize this paper as follows: In second section, we study some definitions of integral transforms and fractional integral and fractional derivatives. Third section is devoted for Weyl fractional derivative and fractional integrals and their integral transforms, also we prove some properties of Weyl fractional derivative and integrals. In section 4, we study fractional Mellin transform and Fourier fractional Mellin transform and their relation.

2. Technical Background

Definition 2.1. (Integral Transform) The integral transform of the function f(x) in $a \leq x \leq b$ is defined by

$$F\{f(x)\} = F(k)$$

and is defined by

$$F\{f(x)\} = F(k) = \int_a^b K(x,k) f(x) dx \tag{2.1}$$

where $K(x,k)$ given function of two variables x and k, is called kernel of the transform. The operator F is usually called an integral transform operator or simply an integral transformation. The transform function $F(k)$ is often referred to as the image of the given object function $f(x)$ and k is called the transform variable.

Similarly, the integral transform of a function of several variables is defined by

$$F\{f(X)\} = F(K) = \int_S K(X, K) f(X) dx \tag{2.2}$$

where $x = (x_1, x_2, \ldots, x_n)$, $K = (k_1, k_2, \ldots, k_n)$.

Definition 2.2. (Fourier Transform) The Fourier transform of $f(x)$ is denoted by $F\{f(x)\} = F(k)$, $k \in R$, and is defined by the integral

$$F\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-ikx) f(x) dx \tag{2.3}$$

where F is called the Fourier transform operator or the Fourier transformation and k is the Fourier transform variable which is complex number.

The inverse Fourier transform, denoted by $F^{-1}\{F(k)\} = f(x)$, is defined by

$$F^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(ikx) F(k) dk \tag{2.4}$$

Definition 2.3. (Mellin Transform) The Mellin transform of $f(x)$ is defined by $M\{f(x)\} = F(p)$ and is defined by

$$M\{f(x)\} = F(p) = \int_0^{\infty} x^{p-1} f(x) dx \tag{2.5}$$

Where M is the the Mellin transform operator and p is the Mellin transform variable which is complex number.

The inverse Mellin transform is denoted by $M^{-1}\{F(p)\} = f(x)$ and is defined by

$$M^{-1}\{F(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} F(p) dp \tag{2.6}$$

Where M^{-1} is the inverse Mellin transform operator.

Definition 2.4. (Riemann–Liouville Fractional Integral) If $f(t) \in C[a, b]$ and $a < t < b$, then

$${}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{2.7}$$

where $\alpha \in (-\infty, \infty)$ is called the Riemann–Liouville fractional integral of order α.

Definition 2.5. (Riemann–Liouville Fractional Derivative) [5] If $f(t) \in C[a, b]$ and $a < t < b$, then

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{2.8}$$

where $\alpha \in (0, 1)$ is called the Riemann–Liouville fractional derivative of order α.

Definition 2.6. (Caputo Fractional Derivative) If $f(t) \in C[a, b]$ and $a < t < b$, then the Caputo fractional derivative of order α is defined as follows

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.9}$$

where $n - 1 < \alpha < n$.

Definition 2.7. (Grunwald–Letnikov Fractional Derivatives) The Grunwald-Letnikov definition of fractional derivative of a function generalize the notion of backward difference quotient of integer order. In this case $\alpha = 1$ if the limit exists the Grunwald-Letnikov fractional Derivative is the left derivative of the function. The Grunwald–Letnikov fractional Derivative of order α of the function $f(t)$ is defined as

$$ {}_aD_t^\alpha f(t) = lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^{j\alpha} C_j f(t - jk) \tag{2.10}$$

where $\dfrac{t-a}{h}$ is integer and $\alpha \in C$. If $\alpha = -1$, we have a Riemann sum which is the first integral.

Definition 2.8. (Mittag–Leffler function) The Mittag–Leffler function of one parameter is denoted by $E_\alpha(z)$ and is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha \in C, Re(\alpha) > 0. \tag{2.11}$$

Definition 2.9. (Mittag–Leffler function) The Mittag–Leffler function of two parameter is denoted by $E_{\alpha,\beta}(z)$ and is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0. \tag{2.12}$$

The function $E_\alpha(z)$ was defined and studied by Mittag–Leffler in the year 1903. It is a direct generalization of the exponential series. For $\alpha = 1$ we have the exponential series. The function defined by (2.12) gives generalization of (2.11). This generalization was studied by Wiman in 1905, Agrawal in 1953, Humbert and Agarwal in 1953, and others.

Definition 2.10. (Fourier transform of Riemann–Liouville Fractional integral) The Fourier transform of Riemann–Liouville Fractional integral of order α is given as

$$F\{{}_{-\infty}I_t^\alpha f(t); k\} = F\{{}_{-\infty}D_t^{-\alpha} f(t); k\} = (ik)^{-\alpha} F\{f(t); k\} \tag{2.13}$$

where k is real.

Definition 2.11. (Fourier transform of fractional derivative) The Fourier transform of fractional derivative of order α is given as

$$F\{D_t^{\alpha} f(t); k\} = (ik)^{\alpha} F\{f(t); k\} \tag{2.14}$$

where the operator D_t^{α} represents any of the mentioned fractional derivative i.e. Riemann–Liouville fractional derivative ${}_0D_t^{\alpha} f(t)$ defined by (2.8), Caputo fractional derivative ${}_a^cD_t^{\alpha} f(t)$ defined by (2.9), Grunwald–Letnikov fractional derivative ${}_aD_t^{\alpha} f(t)$ defined by (2.10).

Definition 2.12. (Mellin transform of Riemann–Liouville Fractional integral) The Mellin transform of Riemann–Liouville Fractional integral of order α is given as

$$M\{{}_0I_t^{\alpha} f(t); p\} = M\{{}_0D_t^{-\alpha} f(t); s\} = \frac{\Gamma(1-p-\alpha)}{\Gamma(1-p)} F(p+\alpha) \tag{2.15}$$

where p is complex.

Definition 2.13. (Mellin transform of Fractional derivative) The Mellin transform of Fractional derivative of order α is given as

$$M\{D_t^{\alpha} f(t); p\} = \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} F(p-\alpha) \tag{2.16}$$

where the operator D_t^{α} represents any of the mentioned fractional derivative i.e. Riemann–Liouville fractional derivative ${}_0D_t^{\alpha} f(t)$ defined by (2.8), Caputo fractional derivative ${}_a^cD_t^{\alpha} f(t)$ defined by (2.9), Grunwald–Letnikov fractional derivative ${}_aD_t^{\alpha} f(t)$ defined by (2.10).

Example 2.14. The Mellin transform and Fourier transform of fractional derivative of following functions

(i) $f(t) = sint$.
we have

$$M\{f(t)\} = \Gamma(p) sin\left(\frac{p\pi}{2}\right)$$

$$\begin{aligned} M\{D_t^{\alpha} f(t); p\} &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} [M\{f(t)\}]_{p\to(p-\alpha)} \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} \left[\Gamma(p) sin\left(\frac{p\pi}{2}\right)\right]_{p\to(p-\alpha)} \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} \left[\Gamma(p-\alpha) sin\left(\frac{(p-\alpha)\pi}{2}\right)\right] \end{aligned}$$

(ii) $f(t) = exp(-at^2)$
we have

$$M\{exp(-at^2)\} = \frac{1}{2}a^{-(\frac{p}{2})}\Gamma\left(\frac{p}{2}\right)$$

$$\begin{aligned} M\{D_t^\alpha exp(-at^2); p\} &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)}\left[M\{exp(-at^2)\}\right]_{p\to(p-\alpha)} \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)}\left[\frac{1}{2}a^{-(\frac{p}{2})}\Gamma\left(\frac{p}{2}\right)\right]_{p\to(p-\alpha)} \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)}\left[\frac{1}{2}a^{-(\frac{p-\alpha}{2})}\Gamma\left(\frac{p-\alpha}{2}\right)\right] \end{aligned}$$

Example 2.15. The Fourier transform of fractional derivative of the following function:

$$f(t) = exp(-at^2)$$

we have

$$F\{exp(-at^2); k\} = \frac{1}{\sqrt{2a exp\left(-\frac{k^2}{4a}\right)}}$$

$$\begin{aligned} F\{D_t^\alpha exp(-at^2); k\} &= (ik)^\alpha F\{exp(-at^2); k\} \\ &= (ik)^\alpha\left[\frac{1}{\sqrt{2a exp\left(-\frac{k^2}{4a}\right)}}\right] \\ &= \frac{(ik)^\alpha}{\sqrt{2a exp\left(-\frac{k^2}{4a}\right)}} \end{aligned}$$

3. Main Result

3.1. Weyl Fractional Derivative and Integrals and their Integral Transforms

Definition 3.1. (Weyl Fractional Integral) The Weyl fractional integral of f(x) is denoted by ${}_xW_\infty^{-\alpha}$ and is defined as

$${}_xW_\infty^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt, \tag{3.1}$$

where $0 < Re\alpha < 1$ and $x > 0$. Above result can be interpreted as Weyl transform.

Example 3.2. The Weyl fractional integral of following functions:

(i) $f(t) = exp(-at)$

$$ {}_xW_\infty^{-\alpha}[exp(-at)] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} exp(-at)dt $$

by the change of variable $t - x = y$

$$ {}_xW_\infty^{-\alpha}[exp(-at)] = \frac{exp(-ax)}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} exp(-ay)dy $$

by substituting $ay = u$

$$ \begin{aligned} {}_xW_\infty^{-\alpha}[exp(-at)] &= \frac{exp(-ax)}{a^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} exp(-u)du \\ &= \frac{exp(-ax)}{a^\alpha} \end{aligned} $$

(ii) $f(t) = t^{-\mu}$

$$ {}_xW_\infty^{-\alpha}[t^{-\mu}] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\mu} dt $$

by the change of variable $t - x = y$

$$ \begin{aligned} {}_xW_\infty^{-\alpha}[t^{-\mu}] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1}(y+x)^{-\mu} dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1}}{(y+x)^\mu} dy \\ &= \frac{x^{\alpha-\mu}}{\Gamma(\alpha)} \beta(\alpha, \mu-\alpha) \\ &= \frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)} x^{\alpha-\mu} \end{aligned} $$

(iii) $f(t) = sinat$

$$ {}_xW_\infty^{-\alpha}[sinat] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} sinatdt $$

by the change of variable $t - x = y$

$$ \begin{aligned} {}_xW_\infty^{-\alpha}[sinat] &= \frac{1}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} sin(ax+ay)dy \\ &= \frac{1}{a^\alpha} cos\left[ax + (\alpha-1)\frac{\pi}{2}\right] \\ &= \frac{1}{a^\alpha} sin\left(ax + \frac{\alpha\pi}{2}\right) \end{aligned} $$

where $0 < Re\alpha < 1$ and $\alpha > 0$.

(iv) $f(t) = cosat$
similar to above, we get

$$ {}_xW_\infty^{-\alpha}[cosat] = \frac{1}{a^\alpha} cos\left(ax + \frac{\alpha\pi}{2}\right) $$

where $0 < Re\alpha < 1$ and $\alpha > 0$.

Note: The modified form of the Dirichlet formula is

$$ \int_x^a (t-x)^\alpha dt \int_t^a (u-t)^{\beta-1} f(u)du = \beta(\alpha,\beta) \int_t^a (u-t)^{\alpha+\beta-1} f(u)du \tag{3.2} $$

Property 3.3. Weyl fractional integral obeys the semigroup property that is

$$ {}_xW_\infty^{-\alpha}\,{}_xW_\infty^{-\beta} = {}_xW_\infty^{-(\alpha+\beta)} = {}_xW_\infty^{-\beta}\,{}_xW_\infty^{-\alpha} \tag{3.3} $$

Proof. We have

$$ \begin{aligned} {}_xW_\infty^{-\alpha}\,{}_xW_\infty^{-\beta} &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt \ \frac{1}{\Gamma(\beta)} \int_x^\infty (u-t)^{\beta-1} f(u)du \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_t^a (u-t)^{\alpha+\beta-1} f(u)du \\ &= {}_xW_\infty^{-(\alpha+\beta)} \end{aligned} $$

by considering $a \to \infty$ in formula (3.2). ■

Property 3.4. Relation between n^{th} derivative and Weyl fractional integral

$$ D^n[{}_xW_\infty^{-\alpha} f(x)] = {}_xW_\infty^{-\alpha}[D^n f(x)] \tag{3.4} $$

Proof. We prove this property by mathematical induction
For $n = 1$

$$ D[{}_xW_\infty^{-\alpha} f(x)] = D\left[\frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt\right] $$

by the changing of variable $t - x = y$

$$ \begin{aligned} D[{}_xW_\infty^{-\alpha} f(x)] &= D\left[\frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} f(y+t)dy\right] \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1}[Df(y+t)]dy \\ &= {}_xW_\infty^{-\alpha}[Df(x)] \end{aligned} $$

We assume that this property is true for $n = k$ that is

$$D^k[{}_xW_\infty^{-\alpha} f(x)] = {}_xW_\infty^{-\alpha}[D^k f(x)]$$

Let us prove this property for $n = k + 1$

$$D^{k+1}[{}_xW_\infty^{-\alpha} f(x)] = D^{k+1}\left[\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt\right]$$

by the changing of variable $t - x = y$

$$\begin{aligned} D^{k+1}[{}_xW_\infty^{-\alpha} f(x)] &= D^{k+1}\left[\frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1} f(y+t)dy\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{k+1}[D^{k+1} f(y+t)]dy \\ &= {}_xW_\infty^{-\alpha}[D^{k+1} f(x)] \end{aligned}$$

Hence by induction, we prove

$$D^n[{}_xW_\infty^{-\alpha} f(x)] = {}_xW_\infty^{-\alpha}[D^n f(x)]$$

■

Definition 3.5. (Weyl Fractional Derivative) If β is a positive number and n is the smallest integer greater than β such that $n - \beta = \alpha > 0$, the Weyl fractional derivative of a function f(x) is defined by

$$ {}_xW_\infty^{\beta}[f(x)] = (-D)^n\ {}_xW_\infty^{-(n-\beta)}[f(x)] = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_x^\infty (t-x)^{(n-\beta-1)} f(t)dt \tag{3.5}$$

Property 3.6. Weyl fractional derivatives obeys the law of exponents.

$$ {}_xW_\infty^{\beta}\,{}_xW_\infty^{\gamma} = {}_xW_\infty^{(\beta+\gamma)} = {}_xW_\infty^{\gamma}\,{}_xW_\infty^{\beta} \tag{3.6}$$

Example 3.7. Weyl fractional derivative of following functions.

(i) ${}_xW_\infty^{\beta}[exp(-ax)] = a^\beta exp(-ax)$

(ii) ${}_xW_\infty^{\beta}[x^{-\mu}] = \dfrac{\Gamma(\beta+\mu)}{\Gamma(\mu)}x^{-(\beta+\mu)}$

(iii) ${}_xW_\infty^{\beta}[cosat] = a^\beta cos\left(ax - \dfrac{\beta\pi}{2}\right)$

(iv) ${}_xW_\infty^{\beta}[sinat] = a^\beta sin\left(ax - \dfrac{\beta\pi}{2}\right)$.

Definition 3.8. Mellin transform of Weyl fractional Integral

$$M\{{}_xW_{\infty}^{-\alpha}[f(x)];\,p\} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}M\{f(x)\}_{p\to(p+\alpha)} \tag{3.7}$$

Definition 3.9. Mellin transform of Weyl fractional derivative:

$$M\{{}_xW_{\infty}^{\beta}[f(x)];\,p\} = \frac{\Gamma(p)}{\Gamma(p-\beta)}M\{f(x)\}_{p\to(p-\beta)} \tag{3.8}$$

Example 3.10. We obtain the Mellin transform of Weyl fractional integrals and derivatives of some standard functions as follows.

(i) Mellin transform of Weyl fractional integral
We have

$$M\{exp(-ax)\} = n^{-p}\ \Gamma(p),\, whereRe(p) > 0$$

$$\begin{aligned}
M\{{}_xW_{\infty}^{-\alpha}[exp(-ax)];\,p\} &= \frac{\Gamma(p)}{\Gamma(p+\alpha)}M\{exp(-ax)\}_{p\to(p+\alpha)} \\
&= \frac{\Gamma(p)}{\Gamma(p+\alpha)}\ \left[n^{-p}\,\Gamma(p)\right]_{p\to(p+\alpha)} \\
&= \frac{\Gamma(p)}{\Gamma(p+\alpha)}\ n^{-(p+\alpha)}\ \Gamma(p+\alpha))
\end{aligned}$$

Mellin transform of Weyl fractional derivative

$$\begin{aligned}
M\{{}_xW_{\infty}^{\beta}[exp(-ax)];\,p\} &= \frac{\Gamma(p)}{\Gamma(p-\beta)}\ M\{exp(-ax)\}_{p\to(p-\beta)} \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)}\ \left[n^{-p}\,\Gamma(p)\right]_{p\to(p-\beta)} \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)}\,n^{-(p-\beta)}\,\Gamma(p-\beta)
\end{aligned}$$

(ii) Mellin transform of Weyl fractional integral
We have

$$M\{sinax\} = a^{-p}\,\Gamma(p)\,sin\frac{\pi p}{2}$$

where $0 < Re(p) < 1$.

$$\begin{aligned}
M\{{}_xW_{\infty}^{-\alpha}[sinax];\,p\} &= \frac{\Gamma(p)}{\Gamma(p+\alpha)}M\ \{sinax\}_{p\to(p+\alpha)} \\
&= \frac{\Gamma(p)}{\Gamma(p+\alpha)}\left(a^{-p}\Gamma(p)sin\frac{\pi p}{2}\right)_{p\to(p+\alpha)} \\
&= \frac{\Gamma(p)}{\Gamma(p+\alpha)}a^{-(p+\alpha)}\Gamma(p+\alpha)sin\frac{\pi(p+\alpha)}{2}
\end{aligned}$$

Mellin transform of Weyl fractional derivative

$$\begin{aligned}M\{{}_xW_\infty^{\beta}[sinax];\,p\} &= \frac{\Gamma(p)}{\Gamma(p-\beta)}M\,\{sinax\}_{p\to(p-\beta)}\\ &= \frac{\Gamma(p)}{\Gamma(p-\beta)}\left(a^{-p}\Gamma(p)sin\frac{\pi p}{2}\right)_{p\to(p-\beta)}\\ &= \frac{\Gamma(p)}{\Gamma(p-\beta)}a^{-(p-\beta)}\Gamma(p-\beta)sin\frac{\pi(p-\beta)}{2}\end{aligned}$$

(iii) Mellin transform of Weyl fractional integral
We have

$$M\{cosax\} = a^{-p}\Gamma(p)cos\frac{\pi p}{2}$$

where $0 < Re(p) < 1$.

$$\begin{aligned}M\{{}_xW_\infty^{-\alpha}[cosax];\,p\} &= \frac{\Gamma(p)}{\Gamma(p+\alpha)}M\,\{cosax\}_{p\to(p+\alpha)}\\ &= \frac{\Gamma(p)}{\Gamma(p+\alpha)}\left(a^{-p}\Gamma(p)cos\frac{\pi p}{2}\right)_{p\to(p+\alpha)}\\ &= \frac{\Gamma(p)}{\Gamma(p+\alpha)}a^{-(p+\alpha)}\Gamma(p+\alpha)cos\frac{\pi(p+\alpha)}{2}\end{aligned}$$

Mellin transform of Weyl fractional derivative

$$\begin{aligned}M\{{}_xW_\infty^{\beta}[cosax];\,p\} &= \frac{\Gamma(p)}{\Gamma(p-\beta)}M\,\{cosax\}_{p\to(p-\beta)}\\ &= \frac{\Gamma(p)}{\Gamma(p-\beta)}\left(a^{-p}\Gamma(p)cos\frac{\pi p}{2}\right)_{p\to(p-\beta)}\\ &= \frac{\Gamma(p)}{\Gamma(p-\beta)}a^{-(p-\beta)}\Gamma(p-\beta)cos\frac{\pi(p-\beta)}{2}\end{aligned}$$

Definition 3.11. The Fourier transform of Weyl fractional integral

$$F\{{}_xW_\infty^{-\alpha}f(x)\} = exp\left(-\frac{\pi i\alpha}{2}\right)k^{-\alpha}F\{f(x)\} \tag{3.9}$$

Theorem 3.12. The Riemann–Liouville fractional integral operator ${}_0I_x^{\alpha}f(x)$ defined in (2.8) and Weyl fractional integral operator ${}_xW_\infty^{-\alpha}f(x)$ are adjoint of each other. i.e.

$$\left\langle {}_0I_x^{\alpha}f(x),\, g(x)\right\rangle = \left\langle f(x),\ {}_xW_\infty^{-\alpha}g(x)\right\rangle \tag{3.10}$$

where $< f(x), g(x) >$ is inner product notation.

Proof.

$$
\begin{aligned}
\langle {}_0I_x^{\alpha} f(x), g(x)\rangle &= \int_0^{\infty} \{{}_0I_x^{\alpha} f(x) g(x)\} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} g(x)dx \int_0^{x} (x-t)^{\alpha-1} f(t)dt \\
&= \int_0^{\infty} f(t)dt \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (x-t)^{\alpha-1} g(x)dx \\
&= \int_0^{\infty} f(t)dt \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} g(t)dt \\
&= \int_0^{\infty} f(t)\, {}_xW_{\infty}^{-\alpha} g(x) \\
&= \langle f(x),\, {}_xW_{\infty}^{-\alpha} g(x)\rangle
\end{aligned}
$$

Hence proved. ■

4. Fractional Mellin transform and Fourier fractional Mellin transform

Definition 4.1. (Fractional Mellin Transform) Fractional Mellin transform in the range $(0, \infty)$. The one dimensional fractional Mellin transform with parameter θ of f(x) denoted by $M^{fr}\{[f(x), \theta, 0, \infty]\}$ performs a linear operation, given by integral transform

$$M^{fr}\left\{[f(x), \theta, 0, \infty]\right\} = \int_0^{\infty} f(x)\, x^{\frac{2\pi i u}{sin\theta}-1}\, exp\left[\frac{\pi i}{tan\theta}(u^2 + log^2 x)\right] dx \tag{4.1}$$

where $0 < \theta \leq \dfrac{\pi}{2}$.

Property 4.2. Relation between Mellin transform and fractional Mellin transform. The fractional Mellin transform is given as

$$M^{fr}\left\{[f(x), \theta, 0, \infty]\right\} = exp\left[\frac{\pi i}{tan\theta} u^2\right] M\left\{\left[f(x) exp\left(\frac{\pi i}{tan\theta} log^2 x\right), \theta, 0, \infty\right]\right\}$$

Proof.

$$
\begin{aligned}
M^{fr}\{f(x)\} &= \int_0^\infty f(x)\, x^{\frac{2\pi i u}{sin\theta}-1}\, exp\left[\frac{\pi i}{tan\theta}(u^2 + log^2 x)\right] dx \\
&= exp\left[\frac{\pi i}{tan\theta}u^2\right] \int_0^\infty f(x)\, x^{\frac{2\pi i u}{sin\theta}-1}\, exp\left(\frac{\pi i}{tan\theta} log^2 x\right) dx \\
&= exp\left[\frac{\pi i}{tan\theta}u^2\right] \int_0^\infty g(x)\, x^{\frac{2\pi i u}{sin\theta}-1} dx \\
&= exp\left[\frac{\pi i}{tan\theta}u^2\right] \int_0^\infty g(x)\, x^{p-1} dx \\
&= exp\left[\frac{\pi i}{tan\theta}u^2\right] M\{[g(x), \theta, 0, \infty]\}
\end{aligned}
$$

where $g(x) = f(x) exp\left[\frac{\pi i}{tan\theta} log^2 x\right]$ and $p = \frac{2\pi i u}{sin\theta}$. ■

Definition 4.3. Fractional Mellin transform in the range $\left(0, \frac{1}{a}\right)$ Fractional Mellin transform with parameter of f(t) in the range $\left(0, \frac{1}{a}\right)$ is denoted as $M^{fr}\left\{\left[f(t), p, 0, \frac{1}{a}\right]\right\}$ and is defined as

$$
M^{fr}\left\{\left[f(t), p, 0, \frac{1}{a}\right]\right\} = \int_0^{\frac{1}{a}} a^p\, t^{p-1} f(t) dt \tag{4.2}
$$

where $p > 0, a > 0$ is a parameter and $\frac{1}{a}$ is fractional.

4.1. Linearity Property

$$
\begin{aligned}
&M^{fr}\left\{\left[\alpha f(t) + \beta g(t), p, 0, \frac{1}{a}\right]\right\} \\
&\quad = \alpha M^{fr}\left\{\left[f(t), p, 0, \frac{1}{a}\right]\right\} + \beta M^{fr}\left\{\left[g(t), p, 0, \frac{1}{a}\right]\right\}
\end{aligned} \tag{4.3}
$$

Proof.

$$
\begin{aligned}
&M^{fr}\left\{\left[\alpha f(t)+\beta g(t), p, 0, \frac{1}{a}\right]\right\}\\
&=\int_0^{\frac{1}{a}} a^p t^{p-1}[\alpha f(t)+\beta g(t)]dt\\
&=\alpha\int_0^{\frac{1}{a}} a^p t^{p-1} f(t)dt+\beta\int_0^{\frac{1}{a}} a^p t^{p-1} g(t)dt\\
&=\alpha M^{fr}\left\{\left[f(t), p, 0, \frac{1}{a}\right]\right\}+\beta M^{fr}\left\{\left[g(t), p, 0, \frac{1}{a}\right]\right\}
\end{aligned}
$$

Hence proved. ■

4.2. Power Property

$$
M^{fr}\left\{\left[f(t^b), p, 0, \frac{1}{a}\right]\right\}=\frac{1}{b}M^{fr}\left\{\left[f(t^b), \frac{p}{b}, 0, \frac{1}{a^b}\right]\right\} \tag{4.4}
$$

Proof.

$$
\begin{aligned}
M^{fr}\left\{\left[f(t^b), p, 0, \frac{1}{a}\right]\right\}&=\int_0^{\frac{1}{a}} a^p\, t^{p-1} f(t^b)dt\\
&=\int_0^{\frac{1}{a^b}} a^p\, \tau^{\frac{p-1}{b}}\frac{1}{b}\, f(\tau)\, \tau^{\frac{1}{b}-1}d\tau\\
&=\frac{1}{b}\int_0^{\frac{1}{a^b}} a^{b\frac{p}{b}} f(\tau)\, \tau^{\frac{p}{b}-1}d\tau\\
&=\frac{1}{b}M^{fr}\left\{\left[f(t^b), \frac{p}{b}, 0, \frac{1}{a^b}\right]\right\}
\end{aligned}
$$

where $\tau=t^b$ hence proved. ■

Example 4.4. Find fractional Mellin transform in the range $\left(0, \frac{1}{a}\right)$ of following functions
(i) $exp(t)$ and (ii) $sinat$

Solution: (i)

$$\begin{aligned}
M^{fr}\left\{\left[exp(t),\, p, 0, \frac{1}{a}\right]\right\} &= \int_0^{\frac{1}{a}} a^p\, t^{p-1} exp(t)dt \\
&= \int_0^{\frac{1}{a}} a^p\, t^{p-1} \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} dt \\
&= \sum_{k=0}^{\infty} \frac{a^p}{\Gamma(k+1)} \int_0^{\frac{1}{a}} t^{k+p-1} dt \\
&= \sum_{k=0}^{\infty} \frac{a^p}{\Gamma(k+1)} \left[\frac{t^{k+p}}{(k+p)}\right]_{k=0}^{\frac{1}{a}} \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{a})^k}{(k+p)\Gamma(k+1)}
\end{aligned}$$

where k is an integer.
(ii)

$$\begin{aligned}
M^{fr}\left\{\left[sinat,\, p, 0, \frac{1}{a}\right]\right\} &= \int_0^{\frac{1}{a}} a^p\, t^{p-1} sinat\, dt \\
&= \int_0^{\frac{1}{a}} a^p\, t^{p-1} \sum_{k=0}^{\infty} (-1)^k \frac{(at)^{2k+1}}{\Gamma(2k+2)} dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{a^p}{\Gamma(2k+2)} \int_0^{\frac{1}{a}} (at)^{2k+p} dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{a^p}{\Gamma(2k+2)} \left[\frac{(at)^{2k+p+1}}{(2k+p+1)}\right]_{k=0}^{k=\frac{1}{a}} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{a})^{2k+1}}{\Gamma(2k+2)(2k+p+1)}
\end{aligned}$$

where k is an integer.

Definition 4.5. Fourier Fractional Mellin transform in the range $\left(0, \frac{1}{a}\right)$ is denoted as $FM^{fr}\left\{\left[f(x,t), s, 0, \infty, p, 0, \frac{1}{a}\right]\right\}$ and defined as

$$FM^{fr}\left\{\left[f(x,t), s, 0, \infty, p, 0, \frac{1}{a}\right]\right\} = \int_0^{\infty} \int_0^{\frac{1}{a}} exp(-isx)\, a^p\, t^{p-1} f(x,t)dt \quad (4.5)$$

where $a > 0$, p and s are parameters and $\frac{1}{a}$ is fractional.

4.3. Linearity Property

$$FM^{fr}\{\alpha f(x,t) + \beta g(x,t)\} = \alpha FM^{fr}\{f(x,t)\} + \beta FM^{fr}\{g(x,t)\} \tag{4.6}$$

Proof.

$$\begin{aligned}
&FM^{fr}\{\alpha f(x,t) + \beta g(x,t)\}\\
&= \int_0^\infty \int_0^{\frac{1}{a}} exp(-isx)\, a^p\, t^{p-1}[\alpha f(x,t) + \beta g(x,t)]dxdt\\
&= \alpha \int_0^\infty \int_0^{\frac{1}{a}} exp(-isx)\, a^p\, t^{p-1} f(x,t)dxdt\\
&+ \beta \int_0^\infty \int_0^{\frac{1}{a}} exp(-isx)\, a^p\, t^{p-1} g(x,t)dxdt\\
&= \alpha FM^{fr}\{f(x,t)\} + \beta FM^{fr}\{g(x,t)\}
\end{aligned}$$

Hence proved. ■

4.4. Power Property

$$\begin{aligned}
&FM^{fr}\left\{\left[f(x,t^n), s, 0, \infty, p, 0, \frac{1}{a}\right]\right\}\\
&= \frac{1}{n} FM^{fr}\left\{\left[f(x,t), s, 0, \infty, \frac{p}{n}, 0, \left(\frac{1}{a}\right)^n\right]\right\}
\end{aligned} \tag{4.7}$$

Proof.

$$\begin{aligned}
&FM^{fr}\left\{\left[f(x,t^n), s, 0, \infty, p, 0, \frac{1}{a}\right]\right\}\\
&= \int_0^\infty \int_0^{\frac{1}{a}} exp(-isx)\, a^p\, t^{p-1} f(x,t^n)dxdt\\
&= \int_0^\infty \int_0^{(\frac{1}{a})^n} exp(-isx)\, a^p\, z^{\frac{p-1}{n}} f(x,z) \frac{1}{n} z^{\frac{1}{n}-1} dxdz\\
&= \frac{1}{n} \int_0^\infty \int_0^{(\frac{1}{a})^n} exp(-isx)(a^n)^{\frac{p}{n}} z^{\frac{p}{n}-1} f(x,z)dxdz\\
&= \frac{1}{n} FM^{fr}\left\{\left[f(x,t), s, 0, \infty, \frac{p}{n}, 0, \left(\frac{1}{a}\right)^n\right]\right\}
\end{aligned}$$

where $t^n = z$. Hence proved. ■

5. Conclusions

(i) We study various integral transforms and fractional integral transforms.

(ii) We study Weyl fractional integral and fractional derivative. Also, we study Mellin transform of Weyl fractional integral and fractional derivative. Furthermore, we obtain Mellin transform of Weyl fractional derivative and integrals of some standard functions.

(iii) We study fractional Mellin transform and studied their properties and we prove some relations between fractional Mellin transform and Fourier fractional Mellin transform.

References

[1] A.P. Bhadane, K.C. Takale, Basic Developments of fractional calculus and its applications, *Bulletin of Marathwada Mathematical Society*, Vol. 12, No. 2 (2011), pp. 1–7.

[2] L.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, series and Products, *Academic Press,* 7^{th} *edition (p. 438).*

[3] Lokenath Debnath, Dambare Bhatta, Integral Transforms and their applications, *CRC press,* 3^{rd} *edition.*

[4] Mathai A.M., Haubold H.J., Special functions for applied Scientists, (2008) *(XXVI), ISBN-978-0-387-75893-0*, pp. 79–111.

[5] I. Podlubny, Fractional Differential equations, *Academic Press, San Diago* (1999).

[6] B. Ross (Ed.), Fractional Calculus and its Applications, *Lecture Notes in Mathematics*, Springer-Verlag, New York, Vol. 457 (1975).

[7] K.C. Takale, V.R. Nikam, A.S. Shind, Mittag–Leffler functions, Its computations and applications to differential equation of fractional order, *International conference on mathematical modlling and applied soft computing*, Vol. 1 (2012), pp. 561–575.

[8] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Composition of Hadamard-type fractional integration operators and the semigroup property, *J. Mat. Anal. Appl.*, 269, (2002), pp. 387–400.

[9] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Mat. Anal. Appl.*, 270, (2002), pp. 1–15.

[10] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Stirling functions of the second kind in the setting of difference and fractional calculus, *Numer. Funct. Anal. Optimiz.*, 4(7–8), 270, (2003), pp. 673–711.

[11] A.A. Kilbas, H. M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, *Elsevier, Amsterdam*, (2006).

[12] C. Bardaro, I. Mantellini, On the moments of the bivarite Mellin–Picard type kernels and applications, *Integral Transform Spec. Funct.*, 23(2), (2012), pp. 135–148.

[13] C. Bardaro, I. Mantellini, On Mellin convolution operators: a direct approach to the asymptotic formulae, *Integral Transform Spec. Funct.*, 25(3), (2014), pp. 182–195.