Causal and Topological Aspects in Special and General Theory of Relativity

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Abstract

In this article we present a review of a geometric and algebraic approach to causal cones and describe cone preserving transformations and their relationship with the causal structure related to special and general relativity. We describe Lie groups, especially matrix Lie groups, homogeneous and symmetric spaces and causal cones and certain implications of these concepts in special and general relativity, related to causal structure and topology of space-time. We compare and contrast the results on causal relations with those in the literature for general space-times and compare these relations with K-causal maps. We also describe causal orientations and their implications for space-time topology and discuss some more topologies on space-time which arise as an application of domain theory. For the sake of completeness, we reproduce proofs of certain theorems which we proved in our earlier work.

1.   Introduction

The notion of causal order is a basic concept in physics and in the theory of relativity in particular. A space-time metric determines causal order and causal cone structure. Alexandrov [1,2,3] proved that a causal order can determine a topology of space-time called Alexandrov topology which, as is now well known, coincides with manifold topology if the space time is strongly causal. The books by Hawking-Ellis, Wald and Joshi [4,5,6] give a detailed treatment of causal structure of space-time. However, while general relativity employs a Lorentzian metric, all genuine approaches to quantum gravity are free of space-time metric. Hence the question arises whether there exists a structure which gets some features of causal cones (light cones) in a purely topological or order-theoretic manner. Motivated by the requirement on suitable structures for a theory of quantum gravity, new notions of causal structures and cone structures were deployed on a space-time.

The order theoretic structures, namely causal sets have been extensively used by Sorkin and his co-workers in developing a new approach to quantum gravity [7]. As a part of this program, Sorkin and Woolgar [8] introduced a relation called K - causality and proved interesting results by making use of Vietoris topology. Based on this work and other recent work, S. Janardhan and R.V. Saraykar [9, 10] and E.Minguzzi [11, 12] proved many interesting results. Especially after good deal of effort, Minguzzi [12] proved that K - causality condition is equivalent to *stably causal* condition.

More recently, K.Martin and Panangaden [13] making use of domain theory, a branch of theoretical computer science, proved fascinating results in the causal structure theory of space-time. The remarkable fact about their work is that only *order* is needed to develop the theory and topology is an outcome of the *order*. In addition to this consequence, there are abstract approaches, algebraic as well as geometric to the theory of cones and cone preserving mappings. Use of quasi-order (a relation which is reflexive and transitive) and partial order is made in defining the cone structure. Such structures and partial orderings are used in the optimization problems [14], game theory and decision making etc [15]. The interplay between ideas from theoretical computer science and causal structure of space-time is becoming more evident in the recent works [16, 17].

Keeping these developments in view, in this article, we present a review of geometric and algebraic approach to causal cones and describe cone preserving transformations and their relationship with causal structure. We also describe certain implications of these concepts in special and general theory of relativity related to causal structure and topology of space-time.

Thus in section 2, we begin with describing Lie groups, especially matrix Lie groups, homogeneous spaces and then causal cones. We give an algebraic description of cones by using quasi-order. Furthermore, we describe cone preserving transformations. These maps are generalizations of causal maps related to causal structure of space-time which we shall describe in section 3. We then describe explicitly Minkowski space as an illustration of these concepts and note that some of the space-time models in general theory of relativity can be described as homogeneous spaces.

In section 3, we describe causal structure of space time, causality conditions, K-

causality and hierarchy among these conditions in the light of recent work of S. Janardhan and R.V. Saraykar and E. Minguzzi and M. Sanchez [9, 10, 12, 18]. We also describe geometric structure of causal group, a group of transformations preserving causal structures or a group of causal maps on a space-time.

In section 4, we describe causal orientations and their implications for space-time topology. We find a parallel between these concepts and concepts developed by Martin and Panangaden [13] to describe topology of space-time, especially a globally hyperbolic one. Finally we discuss some more topologies on space-time which arise as an application of domain theory. Some material from Sections 2 and 4 is borrowed from the book by Hilgert and Olafsson [19].

We end the article with concluding remarks where we discuss more topologies which are different from, but physically more significant than manifold topology.

2. Lie Groups, Homogeneous Spaces, Causal Cones and cone preserving transformations

2.1. Lie Groups, Matrix groups and Homogeneous Spaces

To begin with, we describe Lie groups, matrix Lie groups, homogeneous and symmetric spaces and state some results about them. These will be used in the discussion on causal cones. We refer to the books [20, 21] for more details.

Definition 2.1. Lie groups and matrix Lie groups:
Lie group: A finite dimensional manifold G is called a Lie group if G is a group such that the group operations, composition and inverse are compatible with the differential structure on G. This means that the mappings
$G \times G \to G : (x, y) \mapsto x.y$ and
$G \to G : x \mapsto x^{-1}$
are C^∞ as mappings from one manifold to other.

The n-dimensional real Euclidean space R^n, n-dimensional complex Euclidean space C^n, unit sphere S^1 in R^2, the set of all $n \times n$ real matrices $M(n, R)$ and the set of all $n \times n$ complex matrices $M(n, C)$ are the simplest examples of Lie groups. $M(n, R)$ (and $M(n, C)$) have subsets which are Lie groups in their own right. These Lie groups are called *matrix Lie groups*. They are important because most of the Lie groups appearing in physical sciences such as classical and quantum mechanics, theory of relativity - special and general, particle physics etc are matrix Lie groups. We describe some of them here, which will be used later in this article.

Gl(n,R): General linear group of $n \times n$ real invertible matrices. It is a Lie group and topologically an open subset of $M(n, R)$. Its dimension is n^2.
Sl(n,R): Special linear group of $n \times n$ real invertible matrices with determinant +1. It is a closed subgroup of $Gl(n, R)$ and a Lie group in its own right, with dimension $n^2 - 1$.
O(n): Group of all $n \times n$ real orthogonal matrices. It is called an orthogonal group. It

is a Lie group of dimension $\dfrac{n(n-1)}{2}$.

SO(n): Special orthogonal group- It is a connected component of $O(n)$ containing the identity I and also a closed (compact) subgroup of $O(n)$ consisting of real orthogonal matrices with determinant +1. In particular $SO(2)$ is isomorphic to S^1.

The corresponding Lie groups which are subsets of $M(n, C)$ are $GL(n, C)$, $SL(n, C)$, $U(n)$ and $SU(n)$ respectively, where *orthogonal* is replaced by *unitary*. $SU(n)$ is a compact subgroup of $GL(n, C)$. For $n = 2$, it can be proved that $SU(2)$ is isomorphic to S^3, the unit sphere in R^4. Thus S^3 is a Lie group. [However for topological reasons, S^2 is not a Lie group, though it is C^∞-differentiable manifold]

O(p,q) and SO(p,q): Let p and q be positive integers such that $p + q = n$. Consider the quadratic form $Q(x_1, x_2, \ldots, x_n)$ given by
$Q = x_1^2 + x_2^2 + \cdots x_p^2 - x_{p+1}^2 - x_{p+2}^2 \cdots - x_n^2$.

The set of all $n \times n$ real matrices which preserve this quadratic form Q is denoted by $O(p, q)$ and a subset of $O(p, q)$ consisting of those matrices of $O(p, q)$ whose determinant is +1, is denoted by $SO(p, q)$. Both $O(p, q)$ and $SO(p, q)$ are Lie groups. Here preserving quadratic form Q means the following:

Consider standard inner product η on $R^{p+q} = R^n$ given by the diagonal matrix: $\eta = diag(1, 1 \ldots 1, -1, -1 \ldots -1)$, (1 appearing p times). Then η gives the above quadratic form $Q(x_1, x_2, \ldots, x_n)$, i.e. $X\eta X^T = Q(x_1, x_2, \ldots, x_n)$ where $X = [x_1, x_2, \ldots, x_n]$. $n \times n$ matrix A is said to preserve the quadratic form Q if $A^T \eta A = \eta$.

$O(p, q)$ is called *indefinite orthogonal group* and $SO(p, q)$ is called *indefinite special orthogonal group*. Dimension of $O(p, q)$ is $\dfrac{n(n-1)}{2}$.

Assuming both p and q are nonzero, neither of the groups $O(p, q)$ or $SO(p, q)$ are connected. They have respectively four and two connected components. The identity component of $O(p, q)$ is denoted by $SO_o(p, q)$ and can be identified with the set of elements in $SO(p, q)$ which preserves both orientations.

In particular $O(1, 3)$ is the Lorentz group, the group of all Lorentz transformations, which is of central importance for electromagnetism and special theory of relativity. $U(p, q)$ and $SU(p, q)$ are defined similarly. For more details, we refer the reader to [20, 22]

We now define Homogeneous spaces and discuss some of their properties:

Definition 2.2. We say that a Lie group G is *represented as a Lie group of transformations of a C^∞ manifold M (or has a left (Lie)-action on M)* if to each $g \in G$, there is associated a diffeomorphism from M to itself: $x \mapsto \psi_g(x), x \in M$ such that $\psi_{gh} = \psi_g \psi_h$ for all $g, h \in G$ and $\psi_e = Id.$, Identity map of M, and if further-more $\psi_g(x)$ depends smoothly on the arguments g, x. i.e. the map $(g, x) \mapsto \psi_g(x)$ is a smooth map from $G \times M \to M$.

The Lie group G is said to have a *right action* on M if the above definition is valid with the property $\psi_g \psi_h = \psi_{gh}$ replaced by $\psi_g \psi_h = \psi_{hg}$.

If G is any of the matrix Lie groups $GL(n, R), O(n, R), O(p, q)$ or $GL(n, C), U(n), U(p, q)$ (where $p + q = n$), then G acts in the obvious way on the manifold R^n or $R^{2n} = C^n$. In these cases, the elements of G act as linear transformations.

The action of a group G is said to be *transitive* if for every two points x, y of M, there exists an element $g \in G$ such that $\psi_g(x) = y$.

Definition 2.3. A manifold on which a Lie group acts transitively is called a *homogeneous space* of the Lie group.

In particular, any Lie group G is a homogeneous space for itself under the action of left multiplication. Here G is called the *Principal left homogenous space* (of itself). Similarly the action $\psi_g(h) = hg^{-1}$ makes G into its own *Principal right homogeneous space*.

Let x be any point of a homogeneous space of a Lie group G. The *isotropy group* (or stationary group) H_x of the point x is the stabilizer of x under the action of G: $H_x = \{g \in G / \psi_g(x) = x\}$.

We have the following lemma.

Lemma 2.4. All isotropy groups H_x of points x of a homogeneous space are isomorphic.

Proof. Let x, y be any two points of the homogeneous space. Let $g \in G$ be such that $\psi_g(x) = y$. Then the map $H_x \rightarrow H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism. (Here we have assumed the left action). ∎

We thus denote simply by H, the isotropy group of some (and hence of every element modulo isomorphism) element of M on which G acts on the left.

We now have the following theorem.

Theorem 2.5. There is a one- one correspondence between the points of a homogeneous space M of the Lie group G, and the left cosets gH of H in G, where H is the isotropy group and G is assumed to act on the left.

Proof. Let x_0 be any point of the manifold M. Then with each left coset gH_{x_0} we associate the point $\psi_g(x_0)$ of M. Then this correspondence is well-defined, i.e. independent of the choice of representative of the coset, one - one and onto.

It can be shown under certain general conditions that the isotropy group H is a closed sub group of G, and the set G/H with the natural quotient topology can be given a unique (real) analytic manifold structure such that G is a Lie transformation group of G/H. Thus $M \approx G/H$. ∎

Examples of homogeneous spaces are:

1. Stiefel manifolds: For each $n, k(k \leq n)$, the Stiefel manifold $V_{n,k}$ has as its points all orthonormal frames $x = (e_1, e_2, \ldots, e_k)$ of k vectors in Euclidean n-space i.e. ordered sequences of k orthonormal vectors in R^n. Then $V_{n,k}$ is embeddable as a non-singular surface of dimension $nk - k(k+1)/2$ in R^{nk} and can be visualized as $SO(n)/SO(n-k)$. In particular we have $V_{n,n} \cong O(n)$, $V_{n,n-1} \cong SO(n)$, $V_{n,1} \cong S^{n-1}$.

2. Grassmannian manifolds: The points of the Grassmannian manifold $G_{n,k}$, are by definition, the k-dimensional planes passing through the origin of n-dimensional Euclidean space. This is a smooth manifold and it is given by $G_{n,k} \cong O(n)/O(k) \times O(n-k)$. We now define symmetric spaces.

Definition 2.6. A simply connected manifold M with a metric g_{ab} defined on it, is called a *symmetric space* (*symmetric manifold*) if for every point x of M, there exists an isometry (motion) $s_x : M \to M$ with the properties that x is an isolated fixed point of it, and that the induced map on the tangent space at x reflects (reverses) every tangent vector at x i.e. $\xi \mapsto -\xi$. Such an isometry is called *a symmetry of M of the point x*. For every symmetric space, covariant derivative of Riemann curvature tensor vanishes. For a homogeneous symmetric manifold M, let G be the Lie group of all isometries of M and let H be the isotropy group of M with respect to left action of G on M. Then, as we have seen above, M can be identified with G/H, the set of left cosets of H in G. As examples of such spaces in general relatively, we have the following space-times:

Space of constant curvature with isotropy group $H = SO(1, 3)$:

1. Minkowski space R^4.

2. The de Sitter space
   $S_+ = SO(1, 4)/SO(1, 3)$. Here S_+ is homeomorphic to $R \times S^3$ and the curvature tensor R is the identity operator on the space of bivectors $\Lambda^2(R^4)$, $R = Id$.

3. The anti-de Sitter space
   $S = SO(2, 3)/SO(1, 3)$. This space is homeomorphic to $S^1 \times R^3$ and its universal covering space is homeomorphic to R^4. Here curvature tensor $R = -Id$.

Another example of symmetric space-time is the symmetric space M_t of plane waves. For these spaces the isotropy group is abelian, and the isometry group is soluble (*solvable*). (A group G is called *solvable* if it has a finite chain of normal subgroups $\{e\} < G_1 < \ldots < G_r = G$, beginning with the identity subgroup and ending with G, all of whose factors G_{i+1}/G_i are abelian). In terms of suitable coordinates, the metric has the form $ds^2 = 2dx_1dx_4 + [(cost)x_2^2 + (sint)x_3^2] dx_4^2 + dx_2^2 + dx_3^2, cost \geq sint$. The curvature tensor is constant (refer [21]).

Gödel universe [4] is also an example of a homogeneous space but it is not a physically reasonable model since it contains closed time like curve through every point. We now turn our attention to Causal cones and cone preserving transformations.

2.2. Causal cones and cone preserving transformations

We note that all genuine approaches to quantum gravity are free of space-time metric while general relativity employs a Lorentzian space-time metric. Hence, the question arises whether there exists a structure which gets some features of light cones in a purely topological manner. Motivated by the requirements on suitable structures for a theory of quantum gravity, new notions of causal structure and cone structures were developed on a space-time M. Here we describe these notions.

The definition of *causal cone* is given as follows:
Let M be a finite dimensional real Euclidean vector (linear) space with inner product $<, >$. Let R^+ be the set of positive real numbers and $R_0^+ = R^+ \cup \{0\}$. A subset C of M is a *cone* if $R^+ C \subset C$ and is a *convex cone* if C, in addition, is a convex subset of M. This means, if $x, y \in C$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in C$. In other words, C is a convex cone if and only if for all $x, y \in C$ and $\lambda, \mu \in R^+, \lambda x + \mu y \in C$. We call cone C as *non-trivial* if $C \neq -C$. If C is non-trivial, then $C \neq \{0\}$ and $C \neq M$.
We use the following notations:

i. $M^c = C \cap -C$

ii. $< C >= C - C = \{x - y / x, y \in C\}$

iii. $C^* = \{x \in M / \forall y \in C, (x, y) \geq 0\}$

Then M^c and $< C >$ are vector spaces. They are called the *edge* and the *span* of C. The set C^* is a closed convex cone called the *dual cone* of C. This definition coincides with the usual definition of the dual space M^* of M by using inner product $(\ ,\ )$. If C is a closed convex cone, we have $C^{**} = C$, and $(C^* \cap -C^*) =< C >^\perp$, where for $U \subset M, U^\perp = \{y \in M / \forall u \in U, (u, y) = 0\}$.

Definition 2.7. Let C be a convex cone in M. Then C is called *generating* if $< C >= M$. C is called *pointed* if there exists a $y \in M$ such that for all $x \in C - \{0\}$, we have $(x, y) > 0$. If C is closed , it is called *proper* if $M^c = \{0\}$. C is called *regular* if it is generating and proper. Finally, C is called *self-dual*, if $C^* = C$.

If M is an ordered linear space, the Clifford's theorem [23] states that M is directed if and only if C is generating. The set of interior points of C is denoted by C^o or $int(C)$. The interior of C in its linear span $< C >$ is called the *algebraic interior* of C and is denoted by alg int(C).

Let $S \subset M$. Then the closed convex cone generated by S is denoted by *Cone(S)*:
Cone$(S) =$ closure of $\{ \sum_{finite} r_s s / s \in S, r_s \geq 0\}$.

If C is a closed convex cone, then its interior C^o is an open convex cone. If Ω is an open convex cone, then its closure $\overline{\Omega} = cl(\Omega)$ is a closed convex cone. For an open convex cone, we define the dual cone by
$\Omega^* = \{x \in v / \forall y \in \overline{\Omega} - \{0\}(x, y) > 0\} = int(\overline{\Omega^*})$.
If $\overline{\Omega}$ is proper, we have $\Omega^{**} = \Omega$
We now have the following results: (cf [19, 24]).

Proposition 2.8. Let C be a closed convex cone in M. Then the following statements are equivalent:

   i. C^o is nonempty

   ii. C contains a basis of M.

   iii. $< C >= M$.

Proposition 2.9. Let C be a nonempty closed convex cone in M. Then the following properties are equivalent:

   i. C is pointed

   ii. C is proper

   iii. $\text{int}\,(C^*) \neq \phi$.

As a consequence, we have

Corollary 2.10. Let C be a closed convex cone. Then C is proper if and only if C^* is generating.

Corollary 2.11. Let C be a convex cone in M. Then $C \in Cone(M)$ if and only if $C^* \in Cone(M)$. Here $Cone(M)$ is the set of all closed regular convex cones in M.

To proceed further along these lines, we need to make ourselves familiar with more terminology and notations. The linear automorphism group of a convex cone is defined as follows:
Aut $(C) = \{a \in GL(M)/\alpha(C) = C\}$. GL (M) is the group of invertible linear transformations of M. If C is open or closed, Aut (C) is closed in GL (M). In particular Aut(C) is a linear Lie group.

Definition 2.12. Let G be a group acting linearly on M. Then a cone $C \in M$ is called *G-invariant* if $G.C = C$. We denote the set of invariant regular cones in M by $Cone_G(M)$. A convex cone C is called *homogeneous* if Aut (C) acts transitively on C.

For $C \in Cone_G(M)$, we have Aut $(C) =$ Aut (C^o) and $C = \partial C \cup C^o = (C - C^o) \cup C^o$ is a decomposition of C into Aut (C) - invariant subsets. In particular a non-trivial closed *regular* cone can never be homogeneous. We have the following theorem:

Theorem 2.13. Let G be a Lie group acting linearly on the Euclidean vector space M and $C \in Cone_G(M)$. Then the stabilizer in G of a point in C^o is compact.

Proof. Let $\Omega = C^o$, interior of a convex cone C. Here, $C \in Cone_G(M)$, the set of G-invariant regular cones in M. We first note that for every $v \in \Omega$, the set $U = \Omega \bigcap (v - \Omega)$ is open (being intersection of two open sets), non-empty $\left(\dfrac{v}{2} \in U\right)$ and bounded. Hence

we can find closed balls $\overline{B_r\left(\frac{v}{2}\right)} \subset U \subset \overline{B_R\left(\frac{v}{2}\right)}$ (by property of open sets in a metric space). Let $a \in Aut(\Omega)^v = \{b \in Aut(\Omega)/b.v = v\}$. Then $a.\Omega \subset \Omega$ and $a.v = v$. Thus we obtain $a(U) \subset U$. Hence, $a\left(\overline{B_r\left(\frac{v}{2}\right)}\right) \subset a(U) \subset U \subset \overline{B_R\left(\frac{v}{2}\right)}$. Therefore, $a\left(\frac{v}{2}\right) = \frac{v}{2}$ implies $\parallel a \parallel \leq \frac{R}{r}$. Thus $Aut(\Omega)^v$ is closed and bounded, that is, compact.

In the abstract mathematical setting, cones are described using quasi-order relation [25] as follows:

1. Let $M \neq 0$ be a set and $*$ be a mapping of $M \times M$ into $P^*(M)$ (the set of all non-empty subsets of M). The pair $(M, *)$ is called a hypergroupoid. For $A, B \in P^*(M)$, we define $A * B = \bigcup\{a * b : a \in A, b \in B\}$.

2. A hypergroupoid $(M, *)$ is called a hypergroup, if $(a * b) * c = a * (b * c)$ for all $a, b, c \in M$, and the reproduction axiom, $a * M = M = M * a$, for any $a \in M$, is satisfied.

3. For a binary relation R on A and $a \in A$ denote $U_R(a) = \{b \in A/ < a, b >\in R\}$. A binary relation Q on a set A is called quasiorder if it is reflexive and transitive. The set $U_Q(a)$ is called a cone of a. In the case when a quasiorder Q is an equivalence, $U_Q(A) = \{x \in M/\exists y \in A, < x, y >\in Q\}$ for any $A \subseteq M$. Analogously, for $B \subseteq A$ we set $U_Q(B) = \bigcap\{U_Q(a)/a \in B\}$.

   item In the light of this definition, we shall observe in section 3 that causal cones and K-causal cones fall in this category since causal relation $<$ and K-causal relation $\prec$ are reflexive and transitive. ■

In the literature, (see for example [26, 27, 28]), cone preserving mappings are defined as follows:

1. Let $\boldsymbol{A} = (A, R)$ and $\boldsymbol{B} = (B, S)$ be quasi-ordered sets. A mapping $h : A \to B$ is called *cone preserving* if $h(U_R(a)) = U_S(h(a))$ for each $a \in A$.

2. To illustrate the concepts described above, we consider the example of the Minkowski space.

2.3. Example of a Forward Light cone in Minkowski space

Note: In the paper by Gheorghe and Mihul [29], forward light cone is called *'positive cone'* and is defined as follows:

Let M be a n-dimensional real linear space. A causal relation of M is a partial ordering relation $\geq$ of M with regard to which M is *directed*, i.e. for any $x, y \in M$ there is $z \in M$ so that $z \geq x, z \geq y$. Then the positive cone is defined as $C = \{x/x \in M; x \geq 0\}$.

Let p and q be two positive integers and $n = p + q$. Let $M = R^n$. We write elements of M as $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x \in R^p$ and $y \in R^q$. For $p = 1$, x is a real number. We write projections p_{r_1} and p_{r_2} as $p_{r_1}(v) = x$ and $p_{r_2}(v) = y$.

As discussed earlier, connected component of identity in $O(p, q)$ denoted by $O(p, q)_o = SO_0(p, q) = SO(p, q)_0$. Also Let $Q_{+r} = \{x \in R^{n+1} / Q_{p+1,q}(x, x) = r^2\}, r \in R^+, p, q \in N, n = p + q \geq 1$. Clearly, $O(p + 1, q)$ acts on Q_{+r}. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for R^n. Then we have the following result.

Proposition 2.14. For $p, q > 0$, the group $SO_0(p + 1, q)$ acts transitively on Q_{+r}. The isotropy sub group at re_1 is isomorphic to $SO_0(p, q)$. As a manifold,

$$Q_{+r} \simeq SO_0(p + 1, q)/SO_0(p, q).$$

In particular for $n \geq 2$, $q = n - 1$ and $p = 1$, we define the semi algebraic cone C in R^n by $C = \{v \in R^n / Q_{1,q}(v, v) \geq 0, x \geq 0\}$ and set $C^* = \Omega = \{v \in R^n / Q_{1,q}(v, v) > 0, x > 0\}$. C is called the *forward light cone* in R^n. We have $M = \begin{pmatrix} x \\ y \end{pmatrix} \in C$ if and only if $x \geq \| y \|$.

(Gheorghe and Mihul [29] state in Lemma 2.4 that *There is a norm $\| \|$ in $\overline{M}$ (a n-1 dimensional linear real space) so that: $Q = \{x/x \in M; \varepsilon x^0 = \| \overline{x} \|\}, int C = \{x/x \in M; \varepsilon x^0 > \| \overline{x} \|\}$, where $\varepsilon = 1$ if $(-1, \overline{0})$ is not in C and $\varepsilon = -1$ if $(1, \overline{0})$ is not in C).*

Boundary of C and C^o are described as follows: $\partial C = \{v \in R^n / \epsilon x = \| y \|\}, C^o = \{v \in R^n / \epsilon x > \| y \|\}$ where $\epsilon = 1$ if $(-1, 0)$ is not in C and $\epsilon \neq 1$ if $(1, 0) \in C$. If $v \in C \cap -C$, then $0 \leq x \leq 0$ and hence $x = 0$. Then $\| y \| = 0$ and thus $y = 0$. Thus $v = 0$ and C is proper. For $v, v' \in C$, we calculate

$$(v, v') = (v', v) = x'x + (y', y) \geq \| y' \| \| y \| + (y', y) \geq 0.$$

Thus $C \subset C^*$.

Conversely, let $v = \begin{pmatrix} x \\ y \end{pmatrix} \in C^*$. Then testing against e_1, we get $x \geq 0$. We may assume $y \neq 0$. Define ω by $p_{r_1}(\omega) = \| y \|$ and $p_{r_2}(\omega) = -y$. Then $\omega \in C$ and $0 \leq (w, v) = x \| y \| - \| y \|^2 = (x - \| y \|) \| y \|$. Hence $x \geq \| y \|$. Therefore $y \in C$ and thus $C^* \subset C$. So $C = C^*$ and C is self-dual. Similarly, we can show that Ω is self dual.

Moreover, the forward light cone C is invariant under the usual operation of $SO_o(1, q)$ and under all dilations, $\lambda I_n, \lambda > 0$. (I_n is the $n \times n$ identity matrix). We now prove that the group $SO_o(1, q)R^+ I_{q+1}$ acts transitively on $\Omega = C^o$ if $q \geq 2$ ($q = 3$ for Minkowski space). Thus Ω will be homogeneous. For this we prove that $\Omega = SO_o(1, q)R^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using

$$a_t = \begin{pmatrix} cosh(t) & sinh(t) & 0 \\ sinh(t) & cosh(t) & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in SO_o(1, q),$$

we get

$$a_t \begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \lambda^t (cosh(t), sinh(t), 0, \cdots, 0)$$

for all $t \in R$. Let S^{q-1} denote a unit sphere in R^q. Now $SO(q)$ acts transitively on S^{q-1} and $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in SO_o(1, q)$ for all $A \in SO(q)$. Hence the result follows by noting the fact that coth(t) runs through $(1, \infty)$ as t varies in $(0, \infty)$.

3. Causal Structure of Space-times, Causality Conditions and Causal group

3.1. Causal Structure and K-Causality

In this section, we begin with basic definitions and properties of causal structure of space-time. Then we define different causality conditions and their hierarchy. Furthermore we discuss causal group and causal topology on space-time in general, and treat Minkowski space as a special case. We take a space-time (M, g) as a connected C^2 - Hausdorff four dimensional differentiable manifold which is paracompact and admits a Lorentzian metric g of signature $(-, +, +, +)$. Moreover, we assume that the space-time is space and time oriented.

We say that an event x *chronologically* precedes another event y, denoted by $x \ll y$ if there is a smooth future directed timelike curve from x to y. If such a curve is non-spacelike, i.e., timelike or null, we say that x causally precedes y or $x < y$. The chronological future $I^+(x)$ of x is the set of all points y such that $x \ll y$. The chronological past $I^-(x)$ of x is defined dually. Thus

$$I^+(x) = \{y \in M / x \ll y\} \quad \text{and}$$
$$I^-(x) = \{y \in M / y \ll x\}.$$

The causal future and causal past for x are defined similarly:

$$J^+(x) = \{y \in M / x < y\} \quad \text{and}$$
$$J^-(x) = \{y \in M / y < x\}$$

As Penrose [30] has proved, the relations $\ll$ and $<$ are transitive. Moreover, $x \ll y$ and $y < z$ or $x < y$ and $y \ll z$ implies $x \ll z$. Thus $\overline{I^+(x)} = \overline{J^+(x)}$ and also $\partial I^+(x) = \partial J^+(x)$, where for a set $X \subset M$, $\overline{X}$ denotes closure of X and ∂X denotes topological boundary of X. The chronological future and causal future of any set $X \subset M$ is defined as

$$I^+(X) = \bigcup_{x \in X} I^+(x) \quad \text{and}$$
$$J^+(X) = \bigcup_{x \in X} J^+(x)$$

The chronological and causal pasts for subsets of M are defined similarly.

An ordering which is reflexive and transitive is called quasi - ordering. This ordering was developed in a generalized sense by Sorkin and Woolgar [8] and these concepts were further developed by Garcia Parrado and Senovilla [31, 32] and S. Janardhan and Saraykar [9] to prove many interesting results in causal structure theory in General Relativity.

In the recent paper, Zapata and Kreinovich [28] call chronological order as open order and causal order as closed order and prove that under reasonable assumptions, one can uniquely reconstruct an open order if one knows the corresponding closed order. For special theory of relativity, this part is true and hence every one-one transformation preserving a closed order preserves open order and topology. This fact in turn implies that every order preserving transformation is linear. The conserve part is well known namely, the open relation uniquely determines both the topology and the closed order.

We now introduce the concept of K-causality and give causal properties of space-times in the light of this concept. For more details we refer the reader to [9], [11, 12] and [31, 32].

Definition 3.1. K^+ is the smallest relation containing I^+ that is topologically closed and transitive. If q is in $K^+(p)$ then we write $p \prec q$.

That is, we define the relation K^+, regarded as a subset of $M \times M$, to be the intersection of all closed subsets $R \supseteq I^+$ with the property that $(p, q) \in R$ and $(q, r) \in R$ implies $(p, r) \in R$. (Such sets R exist because $M \times M$ is one of them.) One can also describe K^+ as the closed-transitive relation generated by I^+.

Definition 3.2. An open set O is *K-causal* iff the relation '$\prec$' induces a reflexive partial ordering on O. i.e. $p \prec q$ and $q \prec p$ together imply $p = q$.

If we regard C^o as the interior of future light cone in a Minkowski space-time ($p = 1, q = 3$), then under standard chronological structure I^+, $M(a, b)$ becomes $I^-(b) \cap I^+(a)$. As it is well known, such sets form a base for Alexandrov topology and since Minkowski space-time is globally hyperbolic and hence strongly causal, Alexandrov topology coincides with the manifold topology (Euclidean topology). Thus, lemma 2 of [9] is a familiar result in the language of Causal structure theory.

Analogous to usual causal structure, we defined in [9] strongly causal and future distinguishing space-times with respect to K^+ relation.

Definition 3.3. A C^0-space-time M is said to be *strongly causal at p with respect to* K^+, if p has arbitrarily small K-convex open neighbourhoods.

Analogous definition would follow for K^-. M is said to be *strongly causal with respect to* K^+, if it is strongly causal with respect to K^+ at each and every point of it. Thus, lemma 16 of [8] implies that K-causality implies strong causality with respect to K^+.

Definition 3.4. A C^0-space-time M is said to be *K-future distinguishing* if for every $p \neq q$, $K^+(p) \neq K^+(q)$. *K-past distinguishing* spaces can be defined analogously.

Definition 3.5. A C^0-space-time M is said to be *K-distinguishing* if it is both K-future and K-past distinguishing.

Analogous result would follow for K^-. Hence, in a C^0-space-time M, strong causality with respect to K implies K-distinguishing.

Remark 3.6. K-conformal maps preserve K-distinguishing, strongly causal with respect to K^+ and globally hyperbolic properties.

Definition 3.7. A C^0-space-time M is said to be *K-reflecting* if $K^+(p) \supseteq K^+(q) \Leftrightarrow K^-(q) \supseteq K^-(p)$.
However, since the condition $K^+(p) \supseteq K^+(q)$ always implies $K^-(q) \supseteq K^-(p)$ because of transitivity and $x \in K^+(x)$, and vice versa, a C^0-space-time with K-causal condition is always K-reflecting. Moreover, in general, K-reflecting need not imply reflecting. Since, any K-causal space-time is K-reflecting, any non-reflecting open subset of the space-time will be K-causal but non-reflecting.

We now give the interesting hierarchy of K-causality conditions as follows: We have proved that strong causality with respect to K^+ implies K-future distinguishing. Thus, K-causality $\Rightarrow$ strongly causality with respect to K $\Rightarrow$ K-distinguishing.

Since a K-causal space-time is always K-reflecting, it follows that the K-causal space-time is K-reflecting as well as K-distinguishing. In the classical causal theory, such a space-time is called causally continuous [33]. (Such space-times have been useful in the study of topology change in quantum gravity [34]). Thus if we define K-causally continuous space-time analogously then we get the result that a K-causal C^0-space-time is K-causally continuous. Moreover, since $K^{\pm}(x)$ are topologically closed by definition, analogue of causal simplicity is redundant and causal continuity (which is implied by causal simplicity) follows from K-causality.

In [9], we proved the following theorems. Here we recall their proofs for the sake of completeness.

Theorem 3.8. Let V be a globally hyperbolic C^0-space-time. If $S \subseteq V$ is compact then $K^+(S)$ is closed.

Proof. Let $S \subseteq V$ be compact. Let $q \in cl(K^+(S))$. Then there exists a sequence q_n in $K^+(S)$ such that q_n converges to q. Hence there exists a sequence p_n in S corresponding to q_n and future directed K-causal curves Γ_n from p_n to q_n. Then p_n has a subsequence p_{n_k} converging to $p \in S$ since S is compact, which gives a subsequence Γ_{n_k} of future directed K-causal curves from p_{n_k} to q_{n_k} where p_{n_k} converges to p and q_{n_k} converges to q. Define P $= \{p_{n_k}, p\}$ and $Q = \{q_{n_k}, q\}$. Then P and Q are compact subsets of V. Hence the set C of all future directed K-causal curves from P to Q is compact. Now, $\{\Gamma_{n_k}\}$ is a subset of C. Thus, $\{\Gamma_{n_k}\}$ is a sequence in a compact set and hence has a convergent subsequence say $\Gamma_{n_{k_l}}$ of future directed K-causal curves from $p_{n_{k_l}}$ to $q_{n_{k_l}}$ where $p_{n_{k_l}}$

converges to p and $q_{n_{k_l}}$ converges to q. Let Γ be the Vietoris limit of $\Gamma_{n_{k_l}}$. Then Γ is a future directed K-causal curve from p to q. Since $p \in S$, we have, $q \in K^+(S)$. Hence $cl(K^+(S)) \subseteq K^+(S)$. Thus $K^+(S)$ is closed. ∎

The next two theorems show that in a globally hyperbolic C^0-space-time V, it is possible to express $K^+(x)$ in terms of $I^+(x)$.

Theorem 3.9. If V is a globally hyperbolic C^0-space-time, then $K^+(p) = cl(int(K^+(p)))$, $p \in V$.

Proof. Let V be globally hyperbolic. It is enough to prove that $K^+(p) \subseteq cl(int(K^+(p)))$, $p \in V$. For this we show that $cl(int(K^+(p)))$ is closed with respect to transitivity. So, let $x, y, z \in cl(int(K^+(p)))$ such that $x \prec y$ and $y \prec z$. We show that $x \prec z$. Since x, y, z are limit points of $int(K^+(p))$, there are sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $int(K^+(p))$ such that $x_n \to x$, $y_n \to y$, $z_n \to z$. Using first countability axiom, we may assume, without loss of generality, that these sequences are linearly ordered in the past directed sense [13]. Thus, for sufficiently large n, we can assume that $x_n \prec y_n$ and $y_n \prec z_n$. Since $x_n, y_n, z_n \in K^+(p)$, by transitivity, $x_n \prec z_n$ for sufficiently large n. We claim that $x \prec z$. Let x be not in $K^-(z)$. Then as $K^-(z)$ is closed, using local compactness, there exists a compact neighbourhood N of x such that $N \cap K^-(z) = \emptyset$, and so, z is not in $K^+(N)$. Now as V is globally hyperbolic and N is compact, $K^+(N)$ is closed. Hence, there exists a K-convex neighbourhood N' of z such that $N' \cap K^+(N) = \emptyset$, which is a contradiction as $x_n \prec z_n$ for large n. Hence, $x \prec z$. Thus, $cl(int(K^+(p)))$ is closed with respect to transitivity. Since, by definition, $K^+(p)$ is the smallest closed set which is transitive, we get, $K^+(p) \subseteq cl(int(K^+(p)))$. Hence $K^+(p) = cl(int(K^+(p)))$. Similarly, $K^-(p) = cl(int(K^-(p)))$. ∎

Theorem 3.10. If V is a globally hyperbolic C^0-space-time then $int(K^\pm(x)) = I^\pm(x)$, $x \in V$.

Proof. Let V be globally hyperbolic and $x \in V$. That $I^+(x) \subseteq int(K^+(x))$ is obvious by definition of $K^+(x)$. To prove the reverse inclusion, we prove that, if $x \prec y$ then there exists a K-causal curve from x to y and if $y \in int(K^+(x))$, then this curve must be a future-directed time-like curve.

Let $x \prec y$ and there is no K-causal curve from x to y. Then image of [0,1] will not be connected, compact or linearly ordered. This is possible, only when a point or a set of points has been removed from the compact set $K^+(x) \cap K^-(y)$, that is, when some of the limit points have been removed from this set, which will imply that this set is not closed.

But, since V is globally hyperbolic, $K^+(x) \cap K^-(y)$ is compact and hence closed. Hence, there must exist a K-causal curve from x to y.

Suppose, $y \in int(K^+(x))$. Then, there exists a neighbourhood $I^+(p) \cap I^-(q)$ of y such that $y \in I^+(p) \cap I^-(q) \subseteq K^+(x)$. To show that a K-causal curve from x to y is time-like, it is enough to prove that x and y are not null-related, that is, there exists a

non-empty open set in $K^+(x) \cap K^-(y)$.

Consider, $I^+(p) \cap I^-(q) \cap I^+(x) \cap I^-(y)$, which is open. Take any point say z, on the future-directed time-like curve from p to y. Then, $z \in I^+(p) \cap I^-(q) \cap I^+(x) \cap I^-(y) \subseteq K^+(x) \cap K^-(y)$. (Here, $z \in I^+(x)$ because, if x and z are null-related then $K^+(x) \cap K^-(z)$ will not contain an open set. But $I^+(p) \cap I^-(z) \subseteq K^+(x) \cap K^-(z)$). That is, $K^+(x) \cap K^-(y)$ has a non-empty open subset. Hence, x and y are not null-related, and so, the K-causal curve from x to y is time-like. That is, $y \in I^+(x)$. Thus, $int(K^+(x)) \subseteq I^+(x)$ which proves that $int(K^+(x)) = I^+(x)$. Similarly, we can prove that $int(K^-(x)) = I^-(x)$.

We now discuss important contribution by Minguzzi [12]. We recall that, (M, g) is stably causal if there is $g' > g$ with (M, g') causal. Here $g' > g$ if the light cones of g' are everywhere strictly larger than those of g. Equivalence of K-causality and stable causality uses the concept of *compact stable causality* introduced in [11]. A space-time is compactly stably causal if for every compact set, the light cones can be widened on the compact set while preserving causality. In [12], Minguzzi proved that K-causality implies compact stable causality, and he also gave examples which showed that the two properties differ. It will not be out of place here to mention relationship between stable causality, Seifert future $J_s^+(x)$, almost future $A^+(x)$ and smooth and temporal time functions.

For detailed discussion of these concepts, we refer the reader to [6] and a more recent review by M. Sanchez [35]. Seifert future is defined as

$$J_s^+(x) = \bigcap_{g' > g} J^+(x, g').$$

Then, J_s^+ is closed, transitive and contains J^+. The space-time is stably causal if and only if J_s^+ is anti-symmetric and hence a partial ordering on M. (for proof, we refer to Seifert [36]). ∎

Another causality condition related to Seifert future is Almost future [37], which is defined as follows:
An event x almost causally precedes another event y, denoted by xAy, if for all $z \in I^-(x)$, $I^+(z) \subseteq I^+(y)$. We now define $A^+(x) = \{y \in M / xAy\}$. $A^-(x)$ is defined similarly. It is clear that $y \in A^+(x)$ if and only if $x \in A^-(y)$. A space-time is called W-causal if $x \in A^+(y)$ and $y \in A^+(x)$ implies $x = y$ for all $x, y \in M$.

It is proved in [6] [Prop. 4.12] that the almost future $A^+(x)$ is closed in the manifold topology for all $x \in M$. Moreover [Prop.4.15], for all $x \in M$, $A^+(x) \subseteq J_S^+$. In general, stable causality implies W-causality, though converse is not always true. Also, there is an interesting relationship between stable causality and existence of time functions.

We give the following definition: Let (M, g) be a space-time. A (non-necessarily continuous) function $t : M \to R$ is:

(i) A generalized time function if t is strictly increasing on any future-directed causal curve γ.

(ii) A time function if t is a continuous generalized time function.

(iii) A temporal function if t is a smooth function with past-directed time-like gradient ∇t.

Then, we have the following theorem:

Theorem 3.11. For a space-time (M, g) the following properties are equivalent:

(i) To be stably causal.

(ii) To admit a time function t

(iii) To admit a temporal function T

See [35] for the proof and more detailed discussion on Causal hierarchy. See also Joshi [6], section 4.6, for a general discussion on causal functions and relationship with stably causal space-times. Coming back to relation K^+, we recall that K^+ is the smallest closed and transitive relation which contains J^+. A space-time is K-causal if K^+ is anti-symmetric. By definition, K^+ is contained in J_S^+, but they do not coincide. However, K-causality is equivalent to stable causality and in this case $K^+ = J_S^+$. In [12], Minguzzi proves the equivalence of K-causality and stable causality. For this, he develops a good deal of new terminology and proves a series of lemmas, and uses results proved in earlier papers [11, 38, 39]. Once this equivalence is proved, it also follows that in a K-causal space-time, K^+ relation coincides with the Seifert relation, as mentioned above.

This equivalence, which follows after a laborious work extended over a series of four papers, considerably simplifies the hierarchy of Causality conditions, which now reads as:

Global hyperbolicity $\Rightarrow$ Stably causal $\Leftrightarrow$ K-causality $\Rightarrow$ Strong causality $\Rightarrow$ K-Distinguishing.

3.2.   Causal Groups and Causal Topology

We now discuss causal groups and causal topology and then compare these notions with those in section 2. If R^n is a directed set with respect to a certain partial ordering relation '$\geq$' of R^n, then such a relation is called a *Causal relation*. Thus in a globally hyperbolic space-time (or in a Minkowski space time) J^+ and K^+ are causal relations (In a C^2 globally hyperbolic space-time, $J^+ = K^+$, whereas in a C^0 - globally hyperbolic space-time, only K^+ is valid). The *Causal group G* relative to causal relation is then defined as the group of permutations $f : R^n \to R^n$ which leaves the relation '$\geq$' invariant. i.e. $f(x) \geq f(y)$ if and only if $x \geq y$. Such maps are called causal maps. They preserve causal order. These maps are special cases of cone preserving maps defined in section 2.

We define the K-causal map and discuss their properties briefly. A K-causal map is a causal relation which is a homeomorphism between the two topological spaces and at the same time preserves the order with respect to K^+. To define it, we first define an order preserving map with respect to K^+:

Definition 3.12. Let V and W be C^0-space-times. A mapping $f : V \to W$ is said to be *order preserving with respect to K^+* or simply *order preserving* if whenever $p, q \in V$ with $q \in K^+(p)$, we have $f(q) \in K^+(f(p))$. i.e., $p \prec q$ implies $f(p) \prec f(q)$.

Definition 3.13. Let V and W be C^0-space-times. A homeomorphism $f : V \to W$ is said to be *K-causal* if f is order preserving.

Remark 3.14. In general K-causal maps and causal maps defined by A. Garcia-Parrado and J.M. Senovilla [31, 32] are not comparable as $r \in K^+(p)$ need not imply that $r \in I^+(p)$ (refer figure 1 of [9]).

Using the definition of K-causal map, we now prove a series of properties which follow directly from the definition. We give their proofs for the sake of completeness:

Proposition 3.15. A homeomorphism $f : V \to W$ is order preserving iff

$$f((K^+(x)) \subseteq K^+(f(x)), \forall x \in V.$$

Proof. Let $f : V \to W$ be an order preserving homeomorphism and let $x \in V$. Let $y \in f(K^+(x))$. Then $y = f(p), x \prec p$ which implies $f(x) \prec f(p)$ as f is order preserving. i.e., $f(x) \prec y$ or $y \in K^+(f(x))$.
Hence $f(K^+(x)) \subseteq K^+(f(x)), \forall x \in V.$
Conversely let $f : V \to W$ be a homeomorphism such that $f((K^+(x)) \subseteq K^+(f(x)), x \in V.$
Let $p \prec q$. Then $f(q) \in f(K^+(p))$. By hypothesis, this gives $f(q) \in K^+(f(p))$. Hence $f(p) \prec f(q)$. Thus if f is a K-causal map then $f(K^+(x)) \subseteq K^+(f(x)), \forall x \in V.$ ■

Similarly we have the property:

Proposition 3.16. If $f : V \to W$ be a homeomorphism then f^{-1} is order preserving iff $K^+(f(x)) \subseteq f((K^+(x)), x \in V$.

Proof. We now define, for $S \subseteq V$, $K^+(S) = \bigcup_{x \in S} K^+(x).$ ■

In general, $K^+(S)$ is neither open nor closed. We shall show that in a globally hyperbolic C^0 space-time, if S is compact, then $K^+(S)$ is closed. However at present, we can prove the following property:

Proposition 3.17. If $f : V \to W$ is an order preserving homeomorphism then $f((K^+(S)) \subseteq K^+(f(S)), S \subseteq V.$

Proof. If $f : V \to W$ be an order preserving homeomorphism and $S \subseteq V$ then by definition, $K^+(S) = \bigcup_{x \in S} K^+(x)$. Let $y \in f(K^+(S))$. Then there exists x in S such that

$y \in f(K^+(x))$. This gives $y \in K^+(f(x))$.
i.e., $y \in K^+(f(S))$. Hence $f(K^+(S)) \subseteq K^+(f(S))$. ∎

Analogously we have,

Proposition 3.18. If $f : V \to W$ be a homeomorphism, and f^{-1} is order preserving then $K^+(f(S)) \subseteq f(K^+(S))$, $S \subseteq V$.

We know that causal structure of space-times is given by its conformal structure. Thus, two space-times have identical causality properties if they are related by a conformal diffeomorphism. Analogously, we expect that a K-conformal map should preserve K-causal properties. Thus we define a K-conformal map as follows.

Definition 3.19. A homeomorphism $f : V \longrightarrow W$ is said be K-conformal if both f and f^{-1} are K-causal maps.

Remark 3.20. A K-conformal map is a causal automorphism in the sense of E.C. Zeeman [40].

This definition is similar to *chronal / causal isomorphism* of Zeeman [40], Joshi [6] and Garcia–Parrado and Senovilla [31, 32].
Combining the above properties, we have the following:

Proposition 3.21. If $f : V \to W$ is K-conformal then $f(K^+(x)) = K^+(f(x))$, $\forall x \in V$.

By definition, K-conformal map will preserve different K-causality conditions. If a map is only K-causal and not K-conformal, then we have the following properties:

Proposition 3.22. If $f : V \to W$ is a K-causal mapping and W is K-causal, so is V.

Proof. Let $f : V \to W$ be a K-causal map and W be K-causal. Let $p \prec q$ and $q \prec p$, $p, q \in V$. Then $f(p), f(q) \in W$ such that $f(p) \prec f(q)$ and $f(q) \prec f(p)$ as f is order preserving. Therefore $f(p) = f(q)$ since W is K-causal. Hence $p = q$. ∎

Analogous result would follow for f^{-1}.
In addition, a K-causal mapping takes K-causal curves to K-causal curves. This is given by the property:

Proposition 3.23. If V be a K-causal space-time and $f : V \to W$ be a K-causal mapping, then f maps every K-causal curve in V to a K-causal curve in W.

Proof. Let $f : V \to W$ be a K-causal map. Therefore f is an order preserving homeomorphism. Let Γ be a K-causal curve in V. Then Γ is connected, compact and linearly ordered. Since f is continuous, it maps a connected set to a connected set and a compact set to a compact set. Since f is order preserving and Γ is linearly ordered, $f(\Gamma)$ is a K-causal curve in W. Analogous result would follow for f^{-1}. ∎

From the above result we can deduce the following:

Proposition 3.24. If f be a K-causal map from V to W, then for every future directed K-causal curve Γ in V, any two points $x, y \in f(\Gamma)$ satisfy $x \prec y$ or $y \prec x$.

Definition 3.25. Let V and W be two C^0-space-times. Then W is said to be *K-causally related* to V if there exists a K-causal mapping f from V to W. i.e., $V \prec_f W$.

The following property follows easily from this definition, which shows that the relation ' $\prec'_f$ is transitive also.

Proposition 3.26. If $V \prec_f W$ and $W \prec_g U$ then $V \prec_{g \circ f} U$.

Proposition 3.27. If $f : V \longrightarrow W$ is a K-causal map then $C \subseteq V$ is K-convex if $f(C)$ is a K-convex subset of W.

Proof. Let $f : V \longrightarrow W$ be K-causal and $f(C)$ be a K-convex subset of W. Let $p, q \in C$ and $r \in V$ such that $p \prec r \prec q$. Since f is order preserving we get $f(p) \prec f(r) \prec f(q)$ where $f(p), f(q) \in f(C)$ and $f(r) \in W$. Since $f(C)$ is K-convex, $f(r) \in f(C)$. i.e., $r \in C$. Hence C is a K-convex subset of V. ∎

Remark 3.28. Concept of a convex set is needed to define *strong causality*, as we shall see below.

We now discuss briefly the algebraic structure of the set of all K-causal maps from V to V. We define the following:

Definition 3.29. If V is a C^0–space–time then *Hom(V)* is defined as the group consisting of all homeomorphisms acting on V.

Definition 3.30. If V is a C^0-space-time then *K(V)* is defined as the set of all K-causal maps from V to V.

Proposition 3.31. K(V) is a submonoid of Hom(V).

Proof. If $f_1, f_2, f_3 \in K(V)$ then $f_1 \circ f_2 \in K(V)$. Also, $f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3$ and identity homeomorphism exists. Hence K(V) is a submonoid of Hom(V).

It is obvious that K(V) is a bigger class than the class of K-conformal maps.

Thus in a C^0 globally hyperbolic space-time, every K-causal map f where f^{-1} is also order preserving is a causal relation and causal group is the group of all such mapping which we called K-conformal groups.

In the light of the definition of quasiorder given in section 2, we observe that causal cones and K-causal cones fall in this category, since causal relation '$<$' and K-causal relation '$\prec$' are reflexive and transitive. If we replace quasi-order by a causal relation (or K-causal relation), then we see that an order preserving map is nothing but a causal map. Thus an order preserving map is a generalization of a causal map (or K-causal map). These concepts also appear in a branch of theoretical computer science called

domain theory. Martin and Panangaden [12] and S. Janardhan and Saraykar [10] have used these concepts in an abstract setting and proved some interesting results in causal structure of space times. They proved that order gives rise to a topological structure.

As far as the *causal topology* on R^n is concerned, it is defined as the topology generated by the fundamental system of neighbourhoods containing open ordered sets $M(a, b)$ defined for any $a, b \in R^n$ with $b - a \in int C$ as: $M(a, b) = \{y \in R^n / b - y, y - a \in int C\}$. Gheorghe and Mihul [29] describe *'causal topology'* on R^n and prove that the causal topology of R^n is equivalent to the Euclidean topology. Causal group G is thus comparable to conformal group of space-time under consideration. Further any $f \in G$ is a homeomorphism in causal topology and hence it is a homeomorphism in Euclidean topology.

If C is a Minkowski cone as discussed in the above example, then Zeeman [40] has proved that G is generated by translations, dilations and orthochronous Lorentz transformations of Minkowski space $R^n (n = 4)$. We can say more for the causal group G of Minkowski space. Let $G_0 = \{f \in G / f(0) = 0\}$. Then G_0 contains the identity homeomorphism. Gheorghe and Mihul [29] proved that G is generated by the translations of R^n and by linear transformation belonging to G_0. Hence G is a subgroup of the affine group of R^n. This is the main result of [29]. ∎

Let $G_0' = G_0 \cap SL(n, R)$. Then G_0' is the orthochronus Lorentz group under the norm

$$\parallel y \parallel = \left[\sum_{i=1}^{q} \mid y^i \mid^2 \right]^{\frac{1}{2}} \quad \text{for } y \in R^q, y = (y^1, y^2, \cdots, y^q).$$

For

$$\parallel y \parallel = \left[\sum_{i=1}^{q} \mid y^i \mid^\alpha \right]^{\frac{1}{\alpha}}, \alpha > 2, G_0'$$

is the discrete group of permutations and the symmetries relative to the origin of the basis vectors of R^q. The factor group G_0/G_0' is the dilation group of R^n. Also, G is the semi-direct product of the translation group with the subgroup G_0' of SL(n,R). Moreover G_0' is a topological subgroup of SL(n,R). Similar results have been proved by Borchers and Hegerfeldt [41]. Thus we have,

Theorem 3.32. Let M denote n-dimensional Minkowski space, $n \geq 3$ and let T be a 1 - 1 map of M onto M. Then T and T^{-1} preserve the relation $(x - y)^2 > 0$ if and only if they preserve the relation $(x - y)^2 = 0$. The group of all such maps is generated by

(i) The full Lorentz group (including time reversal)

(ii) Translations of M

(iii) Dilations (multiplication by a scalar)

In our terminology, T is a causal map. In the same paper [41], the following theorem is also proved.

Theorem 3.33. Let $dim M \geq 3$, and let T be a 1 - 1 map of M onto M, which maps light like lines onto (arbitrary) straight lines. Then T is linear.

This implies that constancy of light velocity c alone implies the Poincare group upto dilations.

Thus, for Minkowski space, things are much simpler. For a space-time of general relativity (a Lorentz manifold) these notions take a more complicated form where partial orders are J^+ or K^+.

4. Causal Orientations and order theoretic approach to Global Hyperbolicity

4.1. Causal Orientations

In this section, we discuss briefly the concepts of *Causal orientations, causal structures and causal intervals* which lead to the definition of a *'Globally hyperbolic homogeneous space'*. These notions cover Minkowski Space and homogeneous cosmological models in general relativity. We also discuss domain theoretic approach to causal structure of space-time and comment on the parallel concepts appearing in these approaches.

Let M be a C^1 (respectively smooth) space-time. For $m \in M$, $T_m(M)$ denotes the tangent space of M at m and $T(M)$ denotes the tangent bundle of M. The derivative of a differentiable map $f : M \rightarrow N$ at m will be denoted by $d_m f : T_m M \rightarrow T_{f(m)} N$. A C^1 (respectively smooth) causal structure on M is a map which assigns to each point $m \in M$ a nontrivial closed convex cone C(m) in $T_m M$ and it is C^1 (smooth) in the following sense:

We can find an open covering $\{U_i\}_{i \in I}$ of M, smooth maps $\phi_i : U_i \times R^n \rightarrow T(M)$ with $\phi_i(m, M) \in T_m(M)$ and a cone C in R^n such that $C(m) = \phi_i(m, C)$.

The causal structure is called *generating* (respectively proper, regular) if C(m) is generating (proper, regular) for all m. A map $f : M \rightarrow M$ is called *causal* if $d_m f(C(m)) \subset C(f(m))$ for all $m \in M$. These definitions are obeyed by causal structure J^+ in a causally simple space-time and causal maps of García-Parrado and Senovilla [32]. If we consider C^0-Lorentzian manifold with a C^1-metric so that we can define null cones, then these definitions are also satisfied by causal structure K^+ and K-causal maps. Thus the notions defined above are more general than those occurring in general relativity at least in a special class of space-times. Rainer [42] called such a causal structure an *ultra weak cone structure* on M where $m \in int M$.

We now define G-invariant causal structures where G is a Lie group and discuss some properties of such structures. If a Lie group G acts smoothly on M via $(g, m) \mapsto g.m.$, we denote the diffeomorphism $m \mapsto g.m$ by l_g.

Definition 4.1. Let M be a manifold with a causal structure and G a Lie group acting on M. Then the causal structure is called *G-invariant* if all l_g, $g \in G$, are causal maps. If H

is a Lie subgroup of G and $M = G/H$ is homogeneous then a G-invariant causal structure is determined completely by the cone $C = C(0) \subset T_oM$, where $o = H \in G/H$. Moreover C is proper, generating etc if and only if this holds for the causal structure. We also note that C is invariant under the action of H on $T_o(M)$ given by $h \mapsto d_0 l_h$. On the other hand, if $C \in Cone_H(T_o(M))$, then we can define a field of cones by $M \to T_{\alpha.0}(M) : aH \mapsto C(\alpha H) = d_0 l_a(C)$.

This cone field is G-invariant, regular and satisfies $C(0) = C$. Moreover the mapping $m \mapsto C(m)$ is also smooth in the sense described above. If this mapping is only continuous in the topological sense, for all m in M, then Rainer [42] calls such cone structure, a *weak local cone structure* on M.
We have the following theorem.

Theorem 4.2. Let $M = G/H$ be homogeneous. Then $C \mapsto (\alpha H \mapsto d_0 l_a(C))$ defines a bijection between $Cone_H(T_o(M))$ and the set of G-invariant, regular causal structures on M.

We call a mapping $\nu : [a, b] \to M$ as *absolutely continuous* if for any coordinate chart $\phi : U \to R^n$, the curve $\eta = \phi \circ \nu : \nu^{-1}(U) \to R^n$ has absolutely continuous coordinate functions and the derivatives of these functions are locally bounded.

Further, we define a *C-causal curve*: Let $M = G/H$ and $C \in Cone_G(T_oM)$. An absolutely continuous curve $\nu : [a, b] \to M$ is called C - causal (*Cone causal or conal*) if $\nu'(t) \in C(\nu(t))$ whenever the derivative exists.

Next, we define a relation '$\leq_s$' (s for strict) of M by $m \leq_s n$ if there exists a C-causal curve ν connecting m with n. This relation is obviously reflexive and transitive. Such relations are called *causal orientations* or *quasi-orders*. They give rise to causal cones as we saw in section 2.

Note: A reader who is familiar with the books by Penrose [30], Hawking and Ellis [4] or Joshi [6] will immediately note that the above relation is our familiar causal order $J^{\pm}$ in the case when M is a space-time in general relativity.

We ask the question: Which of the space-times M can be written as G/H? Gödel universe, Taub universe and Bianchi universe are some examples of such space-times. They are all spatially homogeneous cosmological models. Isometry group of a spatially homogeneous cosmological model may or may not be abelian. If it is abelian, then these are of Bianchi type I, under Bianchi classification of homogeneous cosmological models. Thus above discussion applies to such models.

As an example to illustrate above ideas, we again consider a finite dimensional vector space M and let C be a closed convex cone in M. Then we define a causal Aut(C) - invariant orientation on M by $u \leq v$ iff $v - u \in C$. Then '$\leq$' is antisymmetric iff C is proper. In particular $H^+(n, R)$ defines a $GL(n, R)$-invariant global ordering in H(n,R). Here H(n,R) are $n \times n$ real orthogonal matrices (Hermitian if R is replaced by C) and $H^+(n, R) = \{X \in H(m, R)/X \text{ is positive definite }\}$ is an open convex cone in H(n,R). (the closure of $H^+(n, R)$ is the closed convex cone of all positive semi definite matrices in

H(n,R)). Also, the light cone $C \subset R^{n+1}$ defines a $SO_O(n, 1)$-invariant ordering in R^{n+1}. The space R^{n+1} together with this global ordering is the (n+1)-dimensional Minkowski space.

Going back to the general situation we note that in general, the graph $M_{\leq_s} = \{(m, n) \in M \times M / m \leq_s n\}$ of '$\leq_s$' is not closed in $M \times M$. However, if we define $m \leq n \Leftrightarrow (m, n) \in \overline{M_{\leq_s}}$, then it turns out that '$\leq$' is a causal orientation. This can be seen as follows:

'$\leq$' is obviously reflexive. We show that it is transitive: Suppose $m \leq n \leq p$ and let m_k, n_k, n_k', p_k be sequences such that $m_k \leq_s n_k$, $n_k' \leq_s p_k, m_k \to m, n_k \to n, n_k' \to n$ and $p_k \to p$. Now we can find a sequence g_k in G converging to the identity such that $n_k' = g_k n_k$. Thus $g_k m_k \to m$ and $g_k n_k \leq_s p_k$ implies $m \leq p$.

The above result resembles the way in which K^+ was constructed from I^+. The following definitions are analogous to $I^\pm$, $J^\pm$ or $K^\pm$ and so is the definition of interval as $I^+(p) \cap I^-(q)(J^+(p) \cap J^-(q)$ or $K^+(p) \cap K^-(q))$:

Given any causal orientation '$\leq$' on M, we define for $A \subset M$,
$\uparrow A = \{y \in M / \exists a \in A$ with $a \leq y\}$ and
$\downarrow A = \{y \in M / \exists a \in A$ with $y \leq a\}$.
Also, we write $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$.
The intervals with respect to this causal orientation are defined as
$[m, n]_\leq = \{z \in M / m \leq z \leq n\} = \uparrow m \cap \downarrow n$.
Finally we introduce some more definitions.

Definition 4.3. Let M be a space-time.

(1) a causal orientation '$\leq$' on M is called *topological* if its graph $M_\leq$ in $M \times M$ is closed.

(2) a space $(M, \leq)$ with a topological causal orientation is called a *causal space*. If '$\leq$' is, in addition, antisymmetric, that is a partial order, then $(M, \leq)$ is called *globally ordered* or *ordered*.

(3) Let $(M, \leq)$ and $(N, \leq)$ be two causal spaces and let $f : M \to N$ be continuous. Then f is called *order preserving* or *monotone* if $m_1 \leq m_2 \Rightarrow f(m_1) \leq f(m_2)$.

(4) Let G be a group acting on M. Then a causal orientation $\leq$ is called *G-invariant* if $m \leq n \Rightarrow a.m \leq a.n, \forall a \in G$.

(5) A triple $(M, \leq, G)$ is called a *Causal G-Manifold* or *causal* if '$\leq$' is a topological G-invariant causal orientation.

Thus referring to partial order K^+, we see, in the light of above definitions (1) and (2), that $\leq_K$ is topological and $(M, \leq_K)$ is a causal space. A K-causal map satisfies definition (3).

For a homogeneous space $M = G/H$ carrying a causal orientation such that $(M, \leq, G)$ is causal, the intervals are always closed subsets of M. If the intervals are compact, we say that $M = G/H$ is *globally hyperbolic*. We use the same definition for a space-time where intervals are $J^+(p) \cap J^-(q)$. Thus globally hyperbolic space-times can be defined by using causal orientations for homogeneous spaces. In this setting, intervals are always closed, as in causally continuous space-times.

4.2. Domain Theory and Causal Structure

As the last part of our article, we discuss the central concepts and definitions of domain theory, as we observe that these concepts are related to causal structure of space-time and also to space-time topologies.

The relations $< \ and \ \ll$ discussed in section 3 have been generalised to abstract orderings using the concepts in Domain Theory and also many interesting results have been proved related to causal structures of space-time in general relativity. For definitions and preliminary results in domain theory, we follow Abramsky and Jung [43] and Martin and Panangaden [13]. We have expanded some of the proofs which follow in this section, as it gives a better understanding of these concepts and their applications.

Definition 4.4. A *poset* is defined as a partially ordered set, i.e. a set together with a reflexive, anti-symmetric and transitive relation.

Domain theory deals with partially ordered sets to model a domain of computation and the elements of such an order are interpreted as pieces of information or results of a computation where elements of higher order extend the information of the elements below them in a consistent way.

Definition 4.5. Let $(P, \sqsubseteq)$ be a partially ordered set. An *upper bound* of a subset S of a poset P is an element b of P, such that $x \sqsubseteq b, \ \forall x \in S$. The dual notion is called *lower bound*.

A concept that plays an important role in domain theory is the one of a directed subset of a domain, i.e. of a non-empty subset in which each two elements have an upper bound.

Definition 4.6. A nonempty subset $S \subseteq P$ is *directed* if for every x, y in S, $\exists z \in S \ni: x, y \sqsubseteq z$. The supremum of $S \subseteq P$ is the least of all its upper bounds provided it exists and is denoted by $\bigsqcup S$.

This means that every two pieces of information with in the directed subset are consistently extended by some other element in the subset.

A nonempty subset $S \subseteq P$ is *filtered* if for every x, y in S ,$\exists z \in S \ni: z \sqsubseteq x, y$. The infimum of $S \subseteq P$ is the greatest of all its lower bounds provided it exists and is denoted by $\bigwedge S$.

In the partially ordered set $(R, \leq)$ where R is the set of real numbers and $\leq$ denotes

the relation *less than or equal to*, the subset [0 , 1] is directed with supremum 1 and is filtered with infimum 0.

Remark 4.7.

(i) $\forall x \in P$, $\{x\}$ is a directed set.

(ii) In the theory of metric spaces, sequences play a role that is analogous to the role of directed sets in domain theory in many aspects.

(iii) In the formalization of order theory, *limit* of a directed set is just the *least upper bound* of the directed set. As in the case of limits of sequences, least upper bounds of directed sets do not always exist.

The domain in which all consistent specifications converge is of special interest and is defined as follows:

Definition 4.8. A *dcpo*(directed complete partial order) P is a poset in which every directed subset has a supremum.

The poset $(R, \leq)$ is not a dcpo, as the directed subset $(0, \infty)$ does not have a supremum.

Using partial order, some topologies can be derived. For example,

Definition 4.9. A subset U of a poset P is *Scott open* if

(i) U is an upper set: i.e. $x \in U$ and $x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: i.e. for every directed $S \subseteq P$ with a supremum, $\bigsqcup S \in U \Rightarrow S \cap U \neq \phi$.

The collection of all Scott open sets on P is called the *Scott topology*. For the poset $(R, \leq)$, $(1, \infty)$ is Scott open. A more elaborate approach leads to the definition of order of *approximation* denoted by ' $\ll$' which is also called the *way - below relation*.

Definition 4.10. For elements x, y of a poset, $x \ll y$ iff for all directed sets S with a supremum, $y \sqsubseteq \bigsqcup S \Rightarrow \exists s \in S \ni: x \sqsubseteq s$. Define, $\Downarrow x = \{a \in P / a \ll x\}$ and $\Uparrow x = \{a \in P / x \ll a\}$.

In an ordering of sets, an infinite set is way above any of its finite subsets. On the other hand, consider the directed set of finite sets $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$.... The supremum of this set is the set N of all natural numbers. i.e., no infinite set is way below N.

Definition 4.11. An element x in a poset P is said to be *compact* if $x \ll x$.

Proposition 4.12. $x \ll y \Rightarrow x \sqsubseteq y$.

Proof. Let $x \ll y$. Consider the directed set $\{y\}$. Since $\bigsqcup \{y\} = y$, by definition $x \sqsubseteq y$.

∎

Proposition 4.13. The relation ' $\ll$ ' is not necessarily reflexive.

Proof. Let S be a directed set with $x = \bigsqcup S$ and x is not in S. Then $x \sqsubseteq s$, $s \in S$ is false. ∎

Proposition 4.14. $x \sqsubseteq y \ll z \Rightarrow x \ll z$.

Proof. Let S be a directed set with $z \sqsubseteq \bigsqcup S$. Now $y \ll z \Rightarrow \exists \, s \in S$ such that $y \sqsubseteq s$. Since $x \sqsubseteq y$, we have $x \sqsubseteq s$. This holds for each directed set with $\bigsqcup S \sqsupseteq z$. Hence $x \ll z$.

Let $x \ll y \sqsubseteq z$. If S is a directed set with $z \sqsubseteq \bigsqcup S$ then $y \sqsubseteq \bigsqcup S$. Hence $x \ll y \Rightarrow \exists s \in S$ such that $x \sqsubseteq s$. Thus $x \ll z$. ∎

Definition 4.15. For a subset X of a poset P, define $\uparrow X := \{y \in P / \exists x \in X, x \sqsubseteq y\}$ and $\downarrow X := \{y \in P / \exists x \in X, y \sqsubseteq x\}$ Then, $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$ for $x \in X$.

In $(R, \leq)$, $\uparrow \{x\} = [x, \infty)$ and $\downarrow \{x\} = (-\infty, x]$ A subset of elements which is sufficient for getting all other elements as least upper bounds can be defined as follows:

Definition 4.16. A *basis* for a poset P is a subset B such that $B \cap \Downarrow x$ contains a directed set with supremum x for all x in P.

A poset is *continuous* if it has a basis. A poset is *ω-continuous* if it has a countable basis. Continuous posets have an important property that they are interpolative.

Proposition 4.17. $\Downarrow x$ is a directed set in a continuous poset P.

Proof. Let B be a basis in P. Then $B \cap \Downarrow x$ is a directed set with $x = \bigsqcup S$. Let $y, z \in \Downarrow x$. Then $y \ll x$ and $z \ll x$. Now $y \ll x$ implies $\exists s_1 \in S$ such that $y \sqsubseteq s_1$. $z \ll x$ implies $\exists s_2 \in S$ such that $z \sqsubseteq s_2$. Now both $s_1, s_2 \in S$ and S is directed. Therefore, $\exists s \in S$ such that $s_1, s_2 \sqsubseteq s$. Hence $s \in \Downarrow x$ and $y \sqsubseteq s$, $z \sqsubseteq s$. Thus $\Downarrow x$ is a directed set. ∎

Proposition 4.18. $\bigsqcup \Downarrow x = x$, in a continuous poset P.

Proof. For every $y \in \Downarrow x$, $y \ll x$. Therefore, $y \sqsubseteq x$. i.e. x is an upper bound of $\Downarrow x$. Let a be any other upper bound of $\Downarrow x$. Since P is a continuous poset, $B \cap \Downarrow x$ contains a directed set S with $\bigsqcup S = x$. Obviously, $S \subseteq \Downarrow x$. Hence $\bigsqcup S \sqsubseteq \bigsqcup \Downarrow x \sqsubseteq$ any upper bound of $\Downarrow x$. Therefore, $x \sqsubseteq a$. Thus $x = \bigsqcup \Downarrow x$ where P is a continuous poset. ∎

Proposition 4.19. If $x \ll y$ in a continuous poset P, then there is $z \in P$ with $x \ll z \ll y$ (that is, continuous posets are interpolative). Actually a more general result is true namely, if G is a finite subset of P with $G \ll y$, i.e. $\forall x \in G, x \ll y$, then $\exists z \in P$ such that $G \ll z \ll y$.

Proof. Let $A = \{a \in P/\exists a' \in P$ with $a \ll a' \ll y\}$. We claim that A is non-empty. Consider $x \in M, x \ll y$. Now $B \cap \Downarrow x$ contains a directed set S with $\bigsqcup S = x$. Let $a \in S$. Then $a \in \Downarrow x$. Therefore, $a \ll x$ and $x \ll y$. Hence $a \in A$.

Now we claim that A is a directed set. Let $a, b \in A$. Then $\exists a', b' \in P$ such that $a \ll a' \ll y$ and $b \ll b' \ll y$. Since $a', b' \in \Downarrow y$ and $\Downarrow y$ is a directed set, $\exists c' \in \Downarrow y$ such that $a', b' \sqsubseteq c', c' \ll y$. Using directedness of $\Downarrow c'$, we have, $a \ll a' \sqsubseteq c', b \ll b' \sqsubseteq c'$. Therefore, $a, b \in \Downarrow c'$ and hence, $\exists c \in \Downarrow c' \ni: a, b \sqsubseteq c$.

As $c \ll c'$ and $c' \ll y$, we have $c \in \Downarrow y$. Thus, given $a, b \in A, \exists c \in A \ni: a, b \sqsubseteq c$. Hence A is a directed set.

We now show that $y = \bigsqcup \Downarrow y = \bigsqcup A$. Let $y' \ll y$. Then for each $r \in \Downarrow y'$, $r \ll y' \ll y$. Therefore, $r \in A$ which implies $\Downarrow y' \subseteq A$. Hence $\bigsqcup \Downarrow y' \sqsubseteq \bigsqcup A$. i.e., $y' = \bigsqcup \Downarrow y' \sqsubseteq \bigsqcup A$. This holds holds for each $y' \ll y$. Since $B \cap \Downarrow y$ contains a directed set S with $\bigsqcup S = y$, for each $y' \in S, y' \ll y$. Therefore, $y' \sqsubseteq \bigsqcup A$. Hence $\bigsqcup S \sqsubseteq \bigsqcup A$. But $\bigsqcup S = y$. Therefore, $y \sqsubseteq \bigsqcup A$. But by definition, each element of A is below y. Therefore, $\bigsqcup A \sqsubseteq y$. Hence $y = \bigsqcup A$.

For each $x \in G, x \ll y = \bigsqcup A$, and A is a directed set. So, by definition of '$\ll$', $\exists z_x \in A, \ni: x \sqsubseteq z_x$. Since G is finite, z_x in A are finite in number. So, by directedness of A, $\exists z' \in A \ni: x \sqsubseteq z' \forall x \in G$. Now $z' \in A \Rightarrow \exists z \ni: z' \ll z \ll y$. Therefore, $x \ll z \ll y, \forall x \in G$. i.e. $G \ll z \ll y$. ∎

Then we have,

Theorem 4.20. The collection $\{\Uparrow x/x \in P\}$ is a basis for the Scott topology on a continuous poset.

Proof. We first show that $\Uparrow x$ is Scott open for each x in P. Let $y \in \Uparrow x, y \sqsubseteq z$. Then $x \ll y \sqsubseteq z$. So, we have $x \ll z$ and hence $z \in \Uparrow x$. Thus $\Uparrow x$ is an Upper set. Let S be any directed set with a supremum such that $\bigsqcup S \in \Uparrow x$. Let $y = \bigsqcup S$. Thus $y \gg x$. By interpolativeness of '$\ll$', $\exists z \in P \ni: y \gg z \gg x, z \ll y$ and $y = \bigsqcup S$.

Therefore $\exists s \in S \ni: z \sqsubseteq s$. Then $x \ll z \sqsubseteq s$ and hence $x \ll s$. Further $s \in S$. So, $s \in \Uparrow x$ and $s \in S$. Therefore, $S \cap \Uparrow x \neq \phi$. Thus $\Uparrow x$ is Scott open for each $x \in P$. Let $x \in P$, and U be a Scott open set with $x \in U$. Consider $B \cap \Uparrow x$. It contains a directed set say S with $\bigsqcup S = x$. Since $x \in U$, it follows that $S \cap U \neq \phi$. Let $y \in S \cap U$. Obviously $x \in \Uparrow y$. Let $z \in \Uparrow y$. Since $y \in U$ and $y \sqsubseteq z$ we must have $z \in U$. Thus for each $x \in P$ and Scott open set U, $x \in U, \exists y \ni: x \in \Uparrow y$ and $\Uparrow x \subseteq U$. Hence, $\{\Uparrow x/x \in P\}$ forms a basis for the Scott topology. ∎

Lawson topology can be defined as,

Definition 4.21. The *Lawson topology* on a continuous poset P has as a basis all sets of

the form $\Uparrow x \sim \uparrow F$, for $F \subseteq P$ finite.

Definition 4.22. A continuous poset P is *bicontinuous* if for all x, y in P
$x \ll y$ iff for all filtered $S \subseteq P$ with an infimum, $\bigwedge S \sqsubseteq x \Rightarrow \exists s \in S \ni: s \sqsubseteq y$ and for
each $x \in P$, the set $\Uparrow x$ is filtered with infimum x.

Definition 4.23. A *domain* is a continuous poset which is also a *dcpo*.

Proposition 4.24. On a bicontinuous poset P, sets of the form $(a, b) := \{x \in P / a \ll x \ll b\}$ form a basis for a topology. This topology is called the *interval topology*.

Proof. For any $x \in P$, $\Uparrow x$ is filtered with infimum x and $\Downarrow x$ is directed with supremum x. Due to bicontinuity, $\Uparrow x$, $\Downarrow x$ are non-empty. Let $a \in \Downarrow x$, $b \in \Uparrow x$. Then $a \ll x \ll b$. Let $x \in P$ be such that $a \ll x \ll b$ and $a_1 \ll x \ll b_1$. Then $a, a_1 \in \Downarrow x$. Since $\Downarrow x$ is a directed set, $\exists a_2 \in \Downarrow x \ni: a, a_1 \sqsubseteq a_2$. Similarly, $b, b_1 \in \Uparrow x$ which is filtered. Therefore, $\exists b_2 \in \Uparrow x$ such that $b_2 \sqsubseteq b, b_1$. Obviously, $a_2 \ll x \ll b_2$. Further, if y is such that $a_2 \ll y \ll b_2$, then $a \sqsubseteq a_2 \ll y$ and $y \ll b_2 \sqsubseteq b \Rightarrow a \ll y \ll b$. i.e., $y \in a \ll \cdots \ll b$. Similarly, $y \in a_1 \ll \cdots \ll b_1$. Hence (a, b) forms a topology on P. We recall some more definitions regarding causal structure of space-time and elaborate and modify proofs of certain theorems regarding causality conditions. ∎

Definition 4.25. The relation J^+ is defined as $p \sqsubseteq q \equiv q \in J^+(p)$.

Proposition 4.26. Let $p, q, r \in M$. Then

 (i) The sets $I^+(p)$ and $I^-(p)$ are open.

 (ii) $p \sqsubseteq q$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$

 (iii) $q \in I^+(p)$ and $q \sqsubseteq r \Rightarrow r \in I^+(p)$

 (iv) $\overline{I^+(p)} = \overline{J^+(p)}$ and $\overline{I^-(p)} = \overline{J^-(p)}$.

We assume strong causality which can be characterized as follows:

Theorem 4.27. A space-time M is strongly causal iff its Alexandrov topology is Hausdorff iff its Alexandrov topology is the manifold topology.

Definition 4.28. A space-time M is *globally hyperbolic* if it is strongly causal and if $\uparrow a \cap \downarrow b$ is compact in the manifold topology, for all a, b in M.

Lemma 4.29. If (x_n) is a sequence in a globally hyperbolic space-time M with $x_n \sqsubseteq x$ for all n, then

$$\lim_{n \to \infty} x_n = x \Rightarrow \bigsqcup_{n \geq 1} x_n = x.$$

Lemma 4.30. For any $x \in M$, $I^-(x)$ contains an increasing sequence with supremum x.

Proposition 4.31. In a globally hyperbolic space-time M, $x \ll y \Leftrightarrow y \in I^+(x)$ for all x, y in M. Here M is a bicontinuous poset.

Proof. Let $x \ll y$. Then, there is an increasing sequence (y_n) in $I^-(y)$ with $y = \bigsqcup y_n$. Since $x \ll y$, there exists n such that $x \sqsubseteq y_n$. Hence, $x \sqsubseteq y_n$ and $y_n \in I^-(y) \Rightarrow x \in I^-(y)$. That is, $y \in I^+(x)$.

Let $y \in I^+(x)$. To prove $x \ll y$, we have to prove that if S is any directed set with $y \sqsubseteq \bigsqcup S$, then $\exists s \in S$ such that $x \sqsubseteq s$. Since $y \sqsubseteq \bigsqcup S, \bigsqcup S \in J^+(y)$, we have $y \in I^+(x)$ and hence $\bigsqcup S \in I^+(x)$.

Case 1: If $\bigsqcup S \in S$, then we take $s = \bigsqcup S$, and hence the proof.

Case 2: Let $\bigsqcup S$ is not in S. Then S must be infinite. Let $\bigsqcup S = z$. Consider $s_1, s_2 \in S$. Then we can find s_3 such that $s_1 \sqsubseteq s_3, s_2 \sqsubseteq s_3$. (If s_3 coincides with $s_1 or s_2$ in that case, we have $s_1 \sqsubseteq s_2 or s_2 \sqsubseteq s_1$). Consider then another element of S different from s_1, s_2, s_3, $\exists s_4 \in S \ni: s_3 \sqsubseteq s_4$.... We can proceed in this way to get a strictly increasing sequence in S. If we denote this set by S', then $\bigsqcup S' = \bigsqcup S = z$. (For, $\bigsqcup S' = \bigsqcup S, as S' \subseteq S$. If $\bigsqcup S' \sqsubseteq \bigsqcup S$ and $\bigsqcup S' \neq \bigsqcup S$ then either there exists s in S such that $\bigsqcup S'$ and s are not related or $\bigsqcup S' \sqsubseteq s$ and $\bigsqcup S' \neq s$. Both are ruled out as S is a directed set. Thus, S' is a strictly increasing chain in S with $\bigsqcup S' = \bigsqcup S$.)

For this $S' = \{s_1, s_2...\}$, we consider compact sets $J^+(s_i) \cap J^-(z)$. Then, $\{J^+(s_i) \cap J^-(z)\}$ will be a decreasing sequence of compact sets whose intersection is z which is in the open set $I^+(x)$. Hence, for some s_i, $J^+(s_i) \cap J^-(z) \subseteq I^+(x)$. Otherwise, from each of the above compact sets we can find x_i such that x_i is not in $I^+(x)$, where x_i is an increasing sequence with $z = \sup\{x_i\}$ and the open set $I^+(x)$ not intersecting the sequence. This is not possible. Therefore $J^+(s_i) \cap J^-(z) \subseteq I^+(x)$ which implies $s_i \in I^+(x)$ as $s_i \sqsubseteq z$. i.e. $x \sqsubseteq s_i$ and hence $x \ll y$.

The above proof is a modified version of that given in [13]. ∎

Theorem 4.32. If M is globally hyperbolic then $(M, \sqsubseteq)$ is a bicontinuous poset with $\ll = I^+$ whose interval topology is the manifold topology.

Causal simplicity also has a characterization in order-theoretic terms.

Theorem 4.33. Let $(M, \sqsubseteq)$ be a continuous poset with $\ll = I^+$. Then the following are equivalent:

(i) M is causally simple.

(ii) The Lawson topology on M is a subset of the interval topology on M.

We now give definitions and results from a recent article by K. Martin and P. Panangaden [13].

Definition 4.34. Let $(X, \leq)$ be a globally hyperbolic poset. A subset $\pi \subseteq X$ is a causal curve if it is compact, connected and linearly ordered. Let $\pi(0) = \bot$ and $\pi(1) = \top$ where $\bot$ and $\top$ are the least and greatest elements of π. For $P, Q \subseteq X$,
$C(P, Q) = \{\pi / \pi \, causal \, curve, \pi(0) \in P, \pi(1) \in Q\}$
called the space of causal curves between P and Q.

It is clear that a subset of a globally hyperbolic space-time M is the image of a causal curve iff it is the image of a continuous monotone increasing $\pi : [0, 1] \to M$ iff it is a compact connected linearly ordered subset of $(M, \leq)$.

Theorem 4.35. If $(X, \leq)$ is a separable globally hyperbolic poset, then the space of causal curves $C(P, Q)$ is compact in the Vietoris topology and hence in the upper topology.

This result plays an important role in the proofs of certain singularity theorems in [5], in establishing the existence of maximum length geodesics in [4] and in the proof of certain positive mass theorems in [45].

Also, Globally hyperbolic posets are very much like the real line. A well-known domain theoretic construction pertaining to the real line extends in perfect form to the globally hyperbolic posets:

Theorem 4.36. The closed intervals of a globally hyperbolic poset X, $\mathbf{IX} = \{[a, b]/a \leq b \, and \, a, b \in X\}$ ordered by reverse inclusion $[a, b] \subseteq [c, d] \equiv [c, d] \subseteq [a, b]$ form a continuous domain with $[a, b] \ll [c, d] \equiv a \ll c \, and \, d \ll b$. The poset X has a countable basis iff $\mathbf{IX}$ is ω-continuous. Finally, $max(\mathbf{IX}) \simeq X$ where the set of maximal elements has the relative Scott topology from $\mathbf{IX}$.

The observation that the space-time has a canonical domain theoretic model, teaches that from only a countable set of events and the causality relation, space-time can be reconstructed in a purely order theoretic manner using domain theory.

In [13], K. Martin and P. Panangaden construct the space-time from a discrete causal set as follows:

An abstract basis is a set $(C, \ll)$ with a transitive relation that is interpolative from the $-$ direction: $F \ll x \Rightarrow \exists y \in C \ni: F \ll y \ll x$ for all finite subsets $F \subseteq C$ and all $x \in F$. Suppose, it is also interpolative from the $+$ direction: $x \ll F \Rightarrow \exists y \in C \ni: x \ll y \ll F$. Then a new abstract basis of intervals can be defined as, $int(C) = \{(a, b)/a \ll b\} = \ll \subseteq C^2$ whose relation is $(a, b) \ll (c, d) \equiv a \ll c \, and \, d \ll b$.

Let $\mathbf{IC}$ denote the ideal completion of the abstract basis $int(C)$.

Theorem 4.37. Let C be a countable dense subset of a globally hyperbolic space-time M and $\ll = I^+$ be timelike causality. Then $max(\mathbf{IC}) \simeq M$ where the set of maximal elements have the Scott topology.

This theorem is very different because, a process by which a countable set with a causality relation determines a space, is identified here in abstract terms. The process is

entirely order theoretic in nature and space-time is not required to understand or execute it. In this sense, the understanding of the relation between causality and the topology of space-time is explainable independently of geometry.

In a C^0- globally hyperbolic space-time, we can now extend some of the order theoretic concepts to K-causality. To generalize some of these concepts in the context of K-causality, we first prove the following.

Proposition 4.38. In a C^0-globally hyperbolic space-times, $x \ll y \Rightarrow y \in K^+(x)$ where the partial order is $\prec = K^+$.

Proof. Let $x \ll y$. Consider $int\,K^-(y)$ which is an open set not containing y. Since $y \in K^-(y)$, y is a limit point of $int\,K^-(y)$. Hence there exists a sequence y_n in $int\,K^-(y)$ such that $\lim y_n = y$. We can choose y_n as increasing sequence. (using second countability as in Lemma 4.3 of [13]). Thus $\sup y_n = y$. Now $\{y_n\}$ is a directed set with supremum y. Hence $\exists y_n$ in $int\,K^-(y)$ such that $x \prec y_n \prec y$, as $x \ll y$. Thus $y \in K^+(x)$.

It must be noted that above analysis does not require any kind of differentiability conditions on a space-time manifold, and results are purely topological and order-theoretic. ∎

We also have, analogous to above,

Definition 4.39. $\Downarrow x = \{a \in M/a \ll x\}$ and $\Uparrow x = \{a \in M/x \ll a\}$.

Since $a \ll x \Rightarrow a \in K^-(x)$, we have,
$\Downarrow x \in K^-(x)$ and $\Uparrow x \in K^+(x)$.

We illustrate, for Lawson topology, as to how the concepts above can be generalized to K-causality.

Proposition 4.40. Lawson topology, in K-sense, is contained in the manifold topology.

Proof. Let us take a basis for Lawson topology as the sets of the form $\{\Uparrow x \sim \uparrow F/F$ is finite $\}$. Since F is finite, F is compact in the manifold topology and hence $\uparrow F$ is closed. Since the sets $\Downarrow x$ and $\Uparrow x$ are open in the manifold topology (in a C^0-globally hyperbolic space-time), $\Uparrow x \sim \uparrow F$ are also open in the manifold topology.

Thus Lawson open sets are open in the manifold topology also and hence the result follows.

Similar analysis can be given for Scott topology and interval topology also. The intervals defined above, with appropriate cone structure coincide with causal intervals and hence so does the definition of global hyperbolicity. When the partial order is J^+, interval topology coincides with Alexandrov topology and as is well-known, in a strongly causal space-time, Alexandrov topology coincides with the manifold topology. ∎

5. Concluding Remarks

We note that there are a large number of space-times (solutions of Einstein field equations) which are inhomogeneous (see Krasinski [46]) and hence do not fall in the above

class: $M = G/H$. M.Rainer [42] defines yet another partial order using cones as subsets of a topological manifold and a differential manifold (space-time) which is a causal relation in the sense defined above and which is more general than J^+. Rainer, furthermore defines analogous causal hierarchy like in the classical causal structure theory. Of course, for Minkowski space, the old and new definitions coincide. For a C^2-globally hyperbolic space-time J^+, K^+ and Rainer's relation all coincide, whereas for a C^0-globally hyperbolic space-time, K^+ and Rainer's relation on topological manifold coincide.

Moreover if the cones are characteristic surfaces of the Lorentzian metric, then all his definitions of causal hierarchy coincide with the classical definitions. (cf theorem 2 of Rainer [42]). For more details on this partial order, we refer the reader to this paper.

B. Carter [47] discusses causal relations from a different perspective and discusses in detail many features of this relationship. Topological considerations in the light of time-ordering have been discussed by E.H.Kronheimer [48].

Using cone structure, a Causal Topology on Minkowski space was first discussed by Zeeman [49] way back in 1967. This topology has many interesting features. At the same time, it is difficult to handle mathematically because it is not a normal topological space. Gheorghe and Mihul [29] introduced another topology on Minkowski space by using causal relation and where it was assumed that a positive cone is closed in the Euclidean topology. They further proved that this topology coincides with Euclidean topology. R. Gobel [50] worked out in details many features of Zeeman like topologies in the context of space-time of general relativity. Around the same time, Hawking, King and McCarty [51] and Malament [52] worked out interesting features of a topology on space-time in general relativity using time-like curves. Though this work is mathematically interesting, it did not receive much response from people working in General Relativity. In 1992, D. Fullwood [53] constructed another causal topology F from a basis of sets obtained by taking the union of two Alexandrov intervals $< x, y, z > \equiv < x, y > \cup < y, z > \cup y$. These sets are not open in the manifold topology since they include the intermediate point y. F contains information about the space-time dimension and F is Hausdorff iff the space-time is future and past distinguishing and is moreover, strictly finer than the manifold topology.

Fullwood showed that F can also be obtained via a causal convergence criterion on time-like sequences of events. Recently, Onkar Parrikar and Sumati Surya [54] generalized this definition to include all monotonic causal sequences. This gives rise to yet another causal topology which is denoted by P. They showed that P is strictly coarser than F and also strictly finer than the manifold topology. The paper by Parrikar and Surya gives a non-trivial generalization of the MHKM (Malament-Hawking-King-McCarty) theorem and suggests that there is a causal topology for FPD (Future and Past Distinguishing) space-times which encodes manifold dimension and which is strictly finer than the Alexandrov topology. The construction uses a convergence criterion based on sequences of *chain-intervals* which are the causal analogs of null geodesic segments. They also show that when the region of strong causality violation satisfies a local achronality condition, this topology is equivalent to the manifold topology in an FPD space-time. This work is motivated by Sorkin's Causal sets approach to Quantum Gravity. A somewhat

different and more topological approach has been adopted by Martin Kovar [55]. For more details on Zeeman-like topologies and their relationship with manifold topology of space-time, we refer the reader to [56].

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