Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 4 (2017), pp. 1241–1248

http://www.ripublication.com/gjpam.htm

Some results associated with Hermite-Hadamard type Inequalities in Invariant Harmonic Convex Set

Satya Narayan Mishra and Prasanta Kumar Das[1]

KIIT University,
Department of Mathematics,
Bhubaneshwar-751024, Odisha, India.

Girish Chandra Nayak

Department of Mathematics,
Govt. Junior College, Phulbani,
Kandhamala-762001, Odisha, India.

Abstract

The aim of this paper is to introduce the class of invariant harmonic convex set and to define the invariant harmonic convex function in it. The extension work of Hermite-Hadamard type inequalities are studied in invariant harmonic convex set to develop the result studied by Insan [3]. Some more results are also studied associated with the invariant harmonic convex function and Hermite-Hadamard type inequalities.

AMS subject classification: 26A51, 26B25, 52A41, 26D15.
Keywords: Hermite-Hadamard inequality, Hermite-Hadamard type inequality, invariant harmonic convex set, invariant harmonic convex function.

[1]Corresponding Author.

1. Introduction

Let X be a topological vector space. Let $K \subset X \setminus \{0\}$ be a set satisfying the following conditions. For $x, y \in K$, let $I[y, x]$ be a path joining y and x contained in K and the map $\gamma_{xy} : [0, 1] \to I[y, x]$ be continuous. The set K has the invariant harmonic convex (IHC) combination property in a given direction $v \in X$ if the following are satisfied:

$(P1)$ $y + tv \in K$ for all $t \in [0, 1]$, $v \in X$ and $y \in K$.

$(P2)$ $y + tv = \begin{cases} y, & \text{if } t = 0; \\ x, & \text{if } t = 1. \end{cases}$ if and only if $y + tv = \frac{x+y}{2}$ for $t = \frac{1}{2}$

$(P3)$ for any $z \in I[y, x] \subset K$, we have $z = y + tv = x + (1 - t)v$,

$(P4)$ $\frac{xy}{y + tv} \in I[y, x]$ for all $x, y \in K$.

Let $X = \mathbb{R}$ and for any $v \in X$, the set $K = K_v = [a, b] \subset X \setminus \{0\}$ be a IHC combination properties P_1 to P_4 in the direction $v \in X$. Then the inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

hold if the mapping $f : K \to \mathbb{R}$ is harmonically convex function on K_v defined in equation (3.1).

Remark 1.1. The inequality (1.1) coincides with inequality (2.1) if $v = b - a$.

2. Preliminaries

Hermite and Hadamard have studied an integral inequality associated with a convex function as follows: Let $K \subset \mathbb{R}$ be a convex set and $f : K \to \mathbb{R}$ be a convex function. Then for any interval $[a, b] \subset K$, the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

hold. This integral inequality is known as Hermite and Hadamard inequality. In 2014, I. Insan [3] has introduced the concept of harmonically convex set and harmonically convex functions and has studied the Hermite-Hadamard type inequalities for harmonically convex functions as an extension work of Hermite-Hadamard inequalities. We recall the concepts of harmonic sets and harmonic convex function introduced by Insan [3].

Definition 2.1. Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \to \mathbb{R}$ be any map.

1. K is said to be *harmonic convex set* if

$$\frac{xy}{tx + (1-t)y} \in K$$

for all $x, y \in K$ and $t \in [0, 1]$,

2. f is said to be *harmonically convex function* on the harmonically convex set K if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x)$$

for all $x, y \in K$.

We recall the Hermite-Hadamard type inequalities for harmonically convex functions studied by Insan [3]. Let $L[a, b]$ be the set of integrable functions in the interval $[a, b]$.

Theorem 2.2. Let $f : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in K$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \tag{2.1}$$

The above inequalities are sharp.

Theorem 2.3. Let $f : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on $Int(K)$, the interior of K and $a, b \in K$ with $a < b$. If $f' \in L[a, b]$, then the following inequalities hold

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\ = \; &\frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt. \end{aligned} \tag{2.2}$$

3. Main Results

In this section, we have extended the results (2.1) and (2.2) under the certain conditions. For our need, we make the definition of invariant harmonically convex function in the IHC set K.

Definition 3.1. Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \to \mathbb{R}$ be any map.

1. K is said to be IHC set given in the direction $v \in \mathbb{R}\setminus 0$ if K has the IHC combination properties P_1 to P_4.

2. f is said to be invariant harmonically convex (IHC) function on the IHC set K if

$$f\left(\frac{xy}{y+tv}\right) \le tf(y) + (1-t)f(x) \tag{3.1}$$

for all $x, y \in K$, $t \in [0, 1]$.

For existence of the result (1.1), we show the following result.

Lemma 3.2. For $v \in \mathbb{R}$, let $K = K_v \subset \mathbb{R} \setminus \{0\}$ be a IHC set. Let $f : K \to \mathbb{R}$ be a IHC function on $Int(K)$ with respect to the direction $v \in \mathbb{R} \setminus \{0\}$, then for all $x, y \in K$, the following inequality hold

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}.$$

Proof. By IHC properties of f, we have

$$f\left(\frac{xy}{y+tv}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in K$, $t \in [0, 1]$. At $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{2y+v}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in K$. Since $x = y + v$, we have

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in K$. This completes the proof. ■

Theorem 3.3. For $v \in \mathbb{R}$, let $K = K_v \subset \mathbb{R} \setminus \{0\}$ be a IHC set. For $a, b \in K$ with $a < b$, let there exist vectors $v, w \in \mathbb{R} \setminus \{0\}$ with $v + w = 0$ such that

$$a + tv = \begin{cases} a, & \text{if } t = 0; \\ b, & \text{if } t = 1, \end{cases} \quad \text{and } b + tw = \begin{cases} b, & \text{if } t = 0; \\ a, & \text{if } t = 1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be a IHC function on $Int(K)$ with respect to the direction $v \in \mathbb{R} \setminus \{0\}$. Let $f \in L[a, b]$, then the following inequality hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

Proof. In view of Lemma 3.2, we have

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2} \tag{3.2}$$

for all $x, y \in K$, implying

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f(a)+f(b)}{2}$$

for $a, b \in K$ with $a < b$. Assuming $x = \dfrac{ab}{b+tv}$ and $y = \dfrac{ab}{a+tw}$, we have

$$\frac{2xy}{x+y} = \frac{ab}{a+b+t(v+w)} = \frac{ab}{a+b}.$$

Since $w = -v$, we have

$$\frac{1}{v} - \frac{1}{w} = \frac{2}{v}.$$

If $s = \dfrac{\alpha\beta}{\beta+tz}$ where $t \in [0, 1]$ and $\beta = \alpha + z$. As $t : 0 \to 1$,

$$s : \alpha \to \frac{\alpha\beta}{\beta+z} = \frac{\alpha\beta}{\alpha} = \beta.$$

Again

$$t = \frac{1}{z}\left(\frac{\alpha\beta}{s} - \beta\right) \Rightarrow dt = -\frac{\alpha\beta}{zx^2}ds.$$

Thus

$$\int_0^1 f\left(\frac{\alpha\beta}{\beta+tz}\right)dt = \frac{\alpha\beta}{z}\int_\alpha^\beta \frac{f(s)}{s^2}ds. \tag{3.3}$$

Replacing x by $\dfrac{ab}{b+tv}$ and y by $\dfrac{ab}{a+tw}$ in (3.2), we obtain

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2}\left(f\left[\frac{ab}{b+tv}\right] + f\left(\frac{ab}{a+tw}\right)\right)$$

for all $t \in [0, 1]$, implying

$$\int_0^1 f\left(\frac{2ab}{a+b}\right)dt \le \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{b+tv}\right)dt + \int_0^1 f\left(\frac{ab}{a+tw}\right)dt\right].$$

Using (3.3), we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\le \frac{1}{2}\left[\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx - \frac{ab}{w}\int_a^b \frac{f(x)}{x^2}dx\right] \\ &= \frac{ab}{2}\left(\frac{1}{v} - \frac{1}{w}\right)\int_a^b \frac{f(x)}{x^2}dx \\ &= \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx. \end{aligned} \tag{3.4}$$

For fixed $a, b \in K$, f satisfies

$$f\left(\frac{ab}{b+tv}\right) \le tf(b) + (1-t)f(a),$$

for all $t \in [0, 1]$. Applying (3.3), we have

$$\int_0^1 f\left(\frac{ab}{b+tv}\right)dt \leq \int_0^1 tf(b)dt + \int_0^1 (1-t)f(a)dt = \frac{f(a)+f(b)}{2},$$

i.e.,

$$\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3.5}$$

Hence (3.4) and (3.5), we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

This completes the proof. ∎

Theorem 3.4. For $v \in \mathbb{R}$, let $K = K_v \subset \mathbb{R} \setminus \{0\}$ be a IHC set. For $a, b \in K$ with $a < b$, let there exist vectors $v \in \mathbb{R} \setminus \{0\}$ such that

$$a + tv = \begin{cases} a, & \text{if } t = 0; \\ b, & \text{if } t = 1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be a IHC function on $Int(K)$ with respect to the direction $v \in \mathbb{R} \setminus \{0\}$. Let $f' \in L[a, b]$, then the following inequality hold:

$$\frac{f(a)+f(b)}{2} - \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx = \frac{abv}{2}\int_0^1 \frac{1-2t}{(a+tv)^2} f'\left(\frac{ab}{a+tv}\right)dt.$$

Proof. Assuming $x = \dfrac{ab}{a+tv}$, we get

$$t = \frac{ab-ax}{vx} \Rightarrow 1-2t = \frac{v+2a}{v} - \frac{2ab}{vx} \text{ and } dt = -\frac{(a+tv)^2}{abv}dx.$$

Since $a + v = b$ and $t : 0 \to 1$, we have $x : b \to \dfrac{ab}{a+v} = a$. Thus

$$\begin{aligned}
&\frac{abv}{2}\int_0^1 \frac{1-2t}{(a+tv)^2} f'\left(\frac{ab}{a+tv}\right)dt \\
&= \frac{abv}{2}\int_a^b \frac{1}{(a+tv)^2}\left(\frac{v+2a}{v} - \frac{2ab}{vx}\right) f'(x)\left(-\frac{(a+tv)^2}{abv}\right)dx \\
&= \frac{1}{2}\int_a^b \left(\frac{v+2a}{v} - \frac{2ab}{vx}\right) f'(x)dx \\
&= \frac{b-a}{2v}[f(a)+f(b)] - \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx \\
&= \frac{f(a)+f(b)}{2} - \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx,
\end{aligned} \tag{3.6}$$

since $b - a = v$. This completes the proof. ■

Corollary 3.5. For $w \in \mathbb{R}$, let $K = K_w \subset \mathbb{R} \setminus \{0\}$ be a IHC set. For $a, b \in K$ with $a < b$, let there exists a vector $w \in \mathbb{R} \setminus \{0\}$ such that

$$b + tw = \begin{cases} b, & \text{if } t = 0; \\ a, & \text{if } t = 1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be a IHC function on $Int(K)$ with respect to the direction $w \in \mathbb{R} \setminus \{0\}$. Let $f' \in L[a, b]$, then the following inequality hold:

$$\frac{f(a) + f(b)}{2} - \frac{ab}{w} \int_a^b \frac{f(x)}{x^2} dx = \frac{abw}{2} \int_0^1 \frac{1 - 2t}{(b + tw)^2} f' \left(\frac{ab}{b + tw} \right) dt. \tag{3.7}$$

Proof. Proof of this corollary is similar as the proof of Theorem 3.4. ■

Theorem 3.6. Let for each $v \in \mathbb{R}$, let $K = K_v \subset \mathbb{R} \setminus \{0\}$ be a IHC set. For $a, b \in K$ with $a < b$, let there exist vectors $v, w \in \mathbb{R} \setminus \{0\}$ with $v + w = 0$ such that

$$a + tv = \begin{cases} a, & \text{if } t = 0; \\ b, & \text{if } t = 1, \end{cases} \quad \text{and } b + tw = \begin{cases} b, & \text{if } t = 0; \\ a, & \text{if } t = 1. \end{cases}$$

Let $\theta : K \times K \times \mathbb{R} \times [0, 1] \to X$ be any mapping defined by

$$\theta(a, b; v; t) = \frac{abv}{2\,(a + tv)^2}(1 - 2t).$$

Let $T : K \times K \times \mathbb{R} \times [0, 1] \to X^*$ be a mapping defined by

$$\langle T(a, b; v; t), \theta(a, b; v; t) \rangle = \int_0^1 \theta(a, b; v; t) f' \left(\frac{ab}{a + vt} \right) dt$$

for all $a, b \in K$, $v \in \mathbb{R}$ and $t \in [0, 1]$. Let $f' \in L[a, b]$, then followings equalities hold:

(A) $\langle T(b, a; w; t), \theta(b, a; w; t) \rangle - \langle T(a, b; v; t), \theta(a, b; v; t) \rangle = \dfrac{2ab}{v} \displaystyle\int_a^b \frac{f(x)}{x^2} dx.$

(B) $\langle T(b, a; w; t), \theta(b, a; w; t) \rangle + \langle T(a, b; v; t), \theta(a, b; v; t) \rangle = f(a) + f(b).$

Proof. By (3.6), we have

$$\begin{aligned} \langle T(a, b; v; t), \theta(a, b; v; t) \rangle &= \int_0^1 \theta(a, b; v; t) f' \left(\frac{ab}{a + tv} \right) dt \\ &= \frac{abv}{2} \int_0^1 \frac{1 - 2t}{(a + tv)^2} f' \left(\frac{ab}{a + tv} \right) dt \\ &= \frac{f(a) + f(b)}{2} - \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} dx \end{aligned} \tag{3.8}$$

By (3.7), we have

$$\begin{aligned}\langle T(b,a;w;t),\theta(b,a;w;t)\rangle &= \int_0^1 \theta(b,a;v;t) f'\left(\frac{ab}{b+tw}\right)dt \\ &= \frac{abw}{2}\int_0^1 \frac{1-2t}{(b+tw)^2} f'\left(\frac{ab}{b+tw}\right)dt \\ &= \frac{f(a)+f(b)}{2} - \frac{ab}{w}\int_a^b \frac{f(x)}{x^2}dx. \qquad (3.9)\end{aligned}$$

Subtracting (3.8) from (3.9), we get

$$\begin{aligned}&\langle T(b,a;w;t),\theta(b,a;w;t)\rangle - \langle T(a,b;v;t),\theta(a,b;v;t)\rangle \\ &= ab\left(\frac{1}{v}-\frac{1}{w}\right)\int_a^b \frac{f(x)}{x^2}dx \\ &= \frac{2ab}{v}\int_a^b \frac{f(x)}{x^2}dx,\end{aligned}$$

i.e.,

$$\langle T(b,a;w;t),\theta(b,a;w;t)\rangle - \langle T(a,b;v;t),\theta(a,b;v;t)\rangle = \frac{2av}{v}\int_a^b \frac{f(x)}{x^2}dx.$$

This proves (A). Adding (3.8) and (3.9), we get

$$\begin{aligned}&\langle T(a,b;v;t),\theta(a,b;v;t)\rangle + \langle T(b,a;w;t),\theta(b,a;w;t)\rangle \\ &= f(a)+f(b) - \frac{ab}{2}\left(\frac{v+w}{vw}\right)\int_a^b \frac{f(x)}{x^2}dx \\ &= f(a)+f(b),\end{aligned}$$

since $v+w=0$. Hence

$$\langle T(a,b;v;t),\theta(a,b;v;t)\rangle + \langle T(b,a;w;t),\theta(b,a;w;t)\rangle = f(a)+f(b).$$

This proves (B). This completes the proof. ■

References

[1] Anderson, G. D., Vamanamurthy, M. K. and Vuorinen, M., (2007), “Generalized convexity and inequalities”, Journal of Mathematical Analysis and Applications, 335, pp. 1294–1308.

[2] Hanson, M. A., (1981), “On Sufficiency of the Kuhn-Tucker Conditions”, Journal of Mathematical Analysis and Applications, 80, pp. 545–550.

[3] Insan, I., (2014), “Hermite-Hadamard type inequalities for harmonically convex functions”, Hacettepe Journal of Mathematics and Statistics, 43(6), pp. 935–942.