Characterisation of Duo-Rings in Which Every Weakly Cohopfian Module is Finitely Cogenerated

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Abstract
Let $R$ be a ring. A left $R$-module is said to be weakly cohopfian if every injective endomorphism of $M$ is essential. The ring $R$ is called a weakly left SCI-ring if every weakly left cohopfian module is finitely cogenerated. The purpose of this note is to give a characterization of weakly SCI-duo-rings.

Keywords: Cohopfian module, weakly cohopfian module, duo-rings, weakly SCI-ring

1 INTRODUCTION
Let $R$ an associative ring with $1 \neq 0$ and $M$ an unitary left $R$-module. $M$ is said to be cohopfian (rep. hopfian) if every injective (resp. surjective) endomorphism of $M$ is an automorphism. $M$ is said to be finitely cogenerated if its socle is essential in $M$ and finitely cogenerated. The ring $R$ is called a left $I$-ring (resp. $S$-ring) if every left cohopfian (resp. hopfian) module is Artinian (resp. Noetherian). $R$ is called a $FGI$-ring (rep. $FGS$-ring) if every left cohopfian (resp. hopfian) module is finitely generated. $R$ is called a duo-ring if every one sided ideal is two sided. It have been prouved that Artinian principal ideal duo-ring characterize $I$-duo-rings, $S$-duo-rings, $FGI$-duo-rings and $FGS$-duo-rings (see [2], [3], [5] and [8]). Following [7], $M$ is said to be weakly
cohopfian if every injective endomorphism of $M$ is essential. Obviously, every cohopfian module is weakly cohopfian. $R$ is called a left weakly SCI-ring if every left cohopfian module is finitely cogenerated. The purpose of this note is to prove that the following statements are equivalent: (i) $R$ is an Artinian principal ideal duo-ring; (ii) $R$ is a weakly SCI-duo-ring. Throughout this note all rings are associative with $1 \neq 0$ and all modules are unitary. The reader may refer to [1] for any notion and notation not defined in this paper.

2 CONSTRUCTION OF A NON FINITELY COGENERATED WEAKLY COHOPFIAN MODULE OVER A LOCAL ARTINIAN RING WHOSE MAXIMAL IDEAL IS NOT PRINCIPAL.

Let $R$ be a commutative local Artinian ring which is a non principal ideal ring. We may suppose without loss of generalities that the ring $R$ is local Artinian with Jacobson radical $J = aR + bR$ where $a^2 = b^2 = ab = 0$, $a \neq 0$ and $b \neq 0$. Following [6], lemma 5 we may write $R = C \oplus bC$ where $C$ is an Artinian local subring of $R$ with maximal ideal $J(C) = aC \neq 0$. Let $M$ be the total ring of fraction of the polynomial ring $C[X]$, $\sigma$ the endomorphism ring of the $C$-module defined by $\sigma(m) = aXm$ for $m \in M$, $\varphi: R \to \text{End}_C M$ the homomorphism of rings defined by $\varphi(\alpha + \beta b) = \alpha 1_M + \beta \sigma$ for $\alpha + \beta b \in R$ where $\alpha \in C, \beta \in C$ and $1_M$ is the identity homomorphism of $M$. We consider on $M$ the $R$-module structure defined by $(\alpha + \beta b) m = \varphi(\alpha + \beta b)(m) = \alpha m + \beta aXm$, for $\alpha + \beta b \in R$ ($\alpha$ and $\beta \in C$) and for $m \in M$. If $f$ is a $R$-endomorphism of the $R$-module $M$ we have

$$\sigma. f(m) = bf(m) = f(bm) = f(\sigma(m))$$

Thus, the $R$-endomorphisms of the $R$-module $M$ are the $C$-endomorphisms of $M$ commuting with $\sigma$.

Following [9] proposition 2.3 the $R$-module $M$ is weakly cohopfian.

For $d = \mu a + \gamma b \in J =Ra + Rb$ and for $am \in aC[X]$ we have:

$$d.(am) = (\mu a + \gamma b)am = \varphi(\mu a + \gamma b)(am) = (\mu a 1_M + \gamma \sigma)(am) = \mu a^2 m + \gamma a^2 Xm = 0.$$  

Therefore, the submodule $aC[X] = \bigoplus_{n \geq 0} aCX^n$ of $M$ is annihilated by $J$, then $aC[X]$ is semisimple as $R$-module. Since $aC[X]$ is not finite length as $R$-module, then $M$ is not finitely cogenerated.
3 CHARACTERIZATION OF WEAKLY SCI-DUO-RINGS

Lemma 3.1: Let $P$ and $P'$ two prime ideals of a ring $R$ such that $P \not\subseteq P'$. Then $\text{Hom}(R/P, R/P') = 0$.

Proof: Let $f: R/P \rightarrow R/P'$ be an $R$-homomorphism and set $f(1 + P) = t + P'$ where $t \in R$. Let $x \in P - P'$ and let $r$ be any element of $R$. We have $P' = f(xr + P) = xrt + P'$. Thus, $xrt \in P'$ and since $P'$ is prime we have $t \in P'$ and hence $f(1 + P) = P'$ and $f \equiv 0$. □

Lemma 3.2: Every homomorphic image of a left weakly SCI-ring is a left weakly SCI-ring.

Proof: The proof is straightforward and will be omitted. □

Proposition 3.3: Let $R$ be a weakly SCI-ring. If $R$ is an integral domain, then $R$ is a division ring.

Proof: Let $K$ be the division ring of the integral domain $R$. The $R$-module $K$ is weakly cohopfian. Therefore $K$ is finitely cogenerated. Thus, $\text{Soc}(K) \cap R \neq \{0\}$. Let $S = Ra \ (a \in R - \{0\})$ a simple submodule of $\text{Soc}(K) \cap R$. The map

$$\varphi: R \rightarrow S = Ra$$

$$x \mapsto xa$$

is an isomorphism of $R$-modules. Therefore $R$ is semisimple. For any element $b \in R - \{0\}$ we have $R = Rb = Rb^2$, then $b = cb^2$ for some $c \in R$. It follows that $l = cb$ and $l \in Rb = bR$ which implies that $l = bd$ for some $d \in R$. Thus $b$ is left invertible and right invertible, so $b$ is invertible. □

Proposition 3.4: Let $R$ be a weakly SCI-duo-ring. We have the following results:

1. Every prime ideal of $R$ is maximal;
2. The Jacobson radical of $R$ is nil;
3. The set of the maximal ideals of $R$ is finite;
4. $R$ is semiperfect;
5. $R$ is a finite direct product of local weakly SCI-duo-rings.

Proof: (1) results from proposition 3.3.

(2) By (1) the Jacobson radical is equal to the prime radical and consequently it is nil.
(3) Let $D$ the set of all prime ideals of $R$. By lemma 3.1 the semisimple $R$-module $M = \bigoplus_{P \in D} R/P$ is weakly cohopfian, so $\text{Soc}(M) = M$ is finitely cogenerated. It follows that $M$ is finitely generated and $D$ is a finite set.

(4) By (3) $R/J \cong \prod_{P \in D} R/P$. This implies that $R$ is a semisimple ring and since $J$ is two sided ideal of $R$, then $R$ is semiperfect.

(5) results from (4). □

**Proposition 3.5:** [1] proposition 10.10

For a module $M$ the following statements are equivalent

1. $M$ is Artinian;
2. Every factor module of $M$ is finitely cogenerated.

**Proposition 3.6:** Let $R$ be a weakly SCI-duo-ring. Then $R$ is Artinian.

**Proof:** Since $R$ is semiperfect we may suppose without loss of generalities that $R$ is local. Let $R$ be $f$ an injective endomorphism of the $R$-module $R$ where $f(1) = a$. Then putting $\forall n \in \mathbb{N}, f^n = f \circ f \circ ... \circ f$, we show by induction that $f^n(1) = a^n$. If $a$ is an element of the Jacobson radical of $R$, then by 3.3 there exists $m \in \mathbb{N}$ such that $a^m = 0$, ie $f^m(1) = 0$ which is a contradiction because $f^m$ is injective. Then $a \not\in J$ and consequently $a$ is invertible.

Let $y \in R$, then $y = y1 = y(a^{-1}a) = (ya^{-1})a = f(ya^{-1})$, $f$ is surjective.

Thus $R$ is weakly cohopfian and consequently $R$ is finitely cogenerated. Then a weakly SCI-duo-ring is finitely cogenerated. This implies that $R/I$ is finitely cogenerated for every ideal $I$ of $R$ as an homomorphic image of $R$ which is a weakly SCI-duo-ring and by proposition 3.5 $R$ is Artinian. □

**Proposition 3.7:** Let $R$ be a weakly SCI-duo-ring. Then $R$ is a finite direct product of local Artinian weakly SCI-duo-rings.

**Proof:** It follows from (3.4) and (3.6). □

**Proposition 3.8:** [1] proposition 10.8

For a module $M$ the following statements are equivalent:

1. $R$ is left Artinian;
2. Every finitely generated $R$-module is finitely cogenerated.
**Proposition 3.9:** Let $M$ be a direct sum of an infinite countable of a family $(M_n)_{n \in \mathbb{N}}$ of a nonzero submodule of $M$ such that any two of them are isomorphic. Then $M$ is not weakly cohopfian.

**Proof:** For every integer $n$ let $\varphi_n$ be an isomorphism of $M_n$ onto $M_{n+1}$ and $\varphi$ the endomorphism of $M$ such that $\varphi/M_n = \varphi_n$. Then $\varphi$ is a monomorphism of $M$ such that $\text{Im} \varphi = \bigoplus_{n \geq 1} M_n$ which is not essential in $M$. \hfill \Box

**Proposition 3.10:** A direct summand of a weakly cohopfian module is weakly cohopfian.

**Proof:** Let $M$ be a module and $N$ a direct summand of $M$. We can write $M = N \oplus K$ where $K$ is a submodule of $M$.

If $M$ is a weakly cohopfian module and $g$ an injective endomorphism of $N$, then

$$
\xi: M = N \oplus K \rightarrow M = N \oplus K
$$

$$
n + k \mapsto g(n) + k
$$

is an injective endomorphism of $M$. Then $\text{Im} \xi \cong M$ ie $\text{Im} g \cong N \oplus K$ which implies that $\text{Im} g \cong N$ (We can also see [7] corollary 1.3). \hfill \Box

**Theorem 3.11:** Let $R$ be a duo-ring. Then the following statements are equivalent:

1. $R$ is a weakly SCI-duo-ring;
2. $R$ is an Artinian principal ideal duo-ring.

**Proof:** (1) $\Rightarrow$ (2)

Following (3.7) we may suppose that $R$ is a local Artinian weakly SCI-duo-ring. Then by §1 $R$ is a principal ideal ring.

(2) $\Rightarrow$ (1)

Let $R$ be an Artinian principal ideal duo-ring. Following [8] every $R$-module is a direct sum of cyclic submodules. Let now $M$ be a weakly cohopfian module which is not finitely cogenerated. Then by [1] proposition 10.18 $M$ is not finitely generated. We can write $M = \bigoplus_{i \in I} M_i$ where the $M_i$ are cyclic submodules of $M$. Since there is only a finite number of non isomorphic cyclic $R$-modules, then there is an infinite countable sub-family $(M_n)_{n \in \mathbb{N}}$ of the family $(M_i)_{i \in I}$ such that any two of them are isomorphic. Therefore, we can write

$$
M = K \oplus L \text{ where } L = \bigoplus_{n \in \mathbb{N}} M_n
$$

Following proposition 3.10 $L$ is weakly cohopfian and following proposition 3.9 $L$ is not weakly cohopfian. This is a contradiction. \hfill \Box
Corollary 3.12: Let $R$ be a duo-ring. Then the following conditions are equivalent:

1. $R$ is an Artinian principal ideal duo-ring;
2. $R$ is an I-duo-ring;
3. $R$ is a S-duo-ring;
4. $R$ is a FGI-duo-ring;
5. $R$ is a FGS-duo-ring;

REFERENCES