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Characterisation of Duo-Rings in Which Every Weakly Cohopfian Module is Finitely Cogenerated

Sidy Demba Toure, Khady DIOP, Galaye Traore and Mamadou Sangharé

Département de Mathématiques et Informatique
Faculté des Sciences et Techniques de
L'Université Cheikh Anta Diop de Dakar, Sénégal

Abstract

Let R be a ring. A left R-module is said to be weakly cohopfian if every injective endomorphism of M is essential. The ring R is called a weakly left *SCI*-ring if every weakly left cohopfian module is finitely cogenerated. The purpose of this note is to give a characterization of weakly *SCI*-duo-rings.

Keywords: Cohopfian module, weakly cohopfian module, duo-rings, weakly SCI-ring

1 INTRODUCTION

Let R an associative ring with $1 \neq 0$ and M an unitary left R-module. M is said to be cohopfian (rep. hopfian) if every injective (resp. surjective) endomorphism of M is an automorphism. M is said to be finitely cogenerated if its socle is essential in M and finitely cogenerated**.** The ring R is called a left *I*-ring (resp. *S*-ring) if every left cohpfian (resp. hopfian) module is Artinian (resp. Noetherian). R is called a *FGI*-ring (rep. *FGS*-ring) if every left cohopfian (resp. hopfian) module is finitely generated. R is called a duo-ring if every one sided ideal is two sided. It have been prouved that Artinian principal ideal duo-ring characterize *I*-duo-rings, *S*-duo-rings, *FGI*-duo-rings and *FGS*-duo-rings (see [2], [3], [5] and [8]). Following [7], M is said to be weakly cohopfian if every injective endomorphism of M is essential. Obviously, every cohopfian module is weakly cohopfian. R is called a left weakly *SCI*-ring if every left cohopfian module is finitely cogenerated. The purpose of this note is to prove that the following statements are equivalent: (i) R is an Artinian principal ideal duo-ring; (ii) R is a weakly *SCI*-duo-ring. Throughout this note all rings are associative with $1 \neq 0$ and all modules are unitary. The reader may refer to [1] for any notion and notation not defined in this paper.

2 CONSTRUCTION OF A NON FINITELY COGENERATED WEAKLY COHOPFIAN MODULE OVER A LOCAL ARTINIAN RING WHOSE MAXIMAL IDEAL IS NOT PRINCIPAL.

Let R be a commutative local Atinian ring which is a non principal ideal ring. We may suppose without loss of generalities that the ring R is local Artinian with Jacobson radical $J = aR + bR$ where $a^2 = b^2 = ab = 0$, $a \neq 0$ and $b \neq 0$. Following [6], lemma 5 we may write $R = C \oplus bC$ where C is an Artinian local subring of R with maximal ideal $J(C) = aC \neq 0$. Let M be the total ring of fraction of the polynomial ring $C[X]$, σ the endomorphism ring of the C-module defined by $\sigma(m) = aXm$ for $m \in M$, $\varphi: R \to End\ cM$ the homomorphism of rings defined by $\varphi(\alpha + \beta b) = \alpha 1_M + \beta\sigma$ for $\alpha + \beta b \in R$ where $\alpha \in C, \beta \in C$ and 1_M is the identity homomorphism of M. We consider on M the R-module structure defined by $(\alpha + \beta b)m = \varphi(\alpha + \beta b)(m) = \alpha m + \beta a X m$, for $\alpha + \beta b \in R$ (α and $\beta \in C$) and for $m \in M$. If f is a R-endomorphism of the R-module M we have

$$\sigma . f(m) = bf(m) = f(bm) = f(\sigma(m))$$

Thus, the R-endomorphisms of the R-module M are the C-endomorphisms of M commuting with σ.

Following [9] proposition 2.3 the R-module M is weakly cohopfian.

For $d = \mu a + \gamma b \in J = Ra + Rb$ and for $am \in aC[X]$ we have:

$d.(am) = (\mu a + \gamma b)am = \varphi(\mu a + \gamma b)(am) = (\mu a 1_M + \gamma\sigma)(am) = \mu a^2 m + \gamma a^2 X m = 0.$

Therefore, the submodule $aC[X] = \oplus_{n \geq 0} aCX^n$ of M is annihilated by J, then $aC[X]$ is semisimple as R-module. Since $aC[X]$ is not finite length as R-module, then M is not finitely cogenerated.

3 CHARACTERIZATION OF WEAKLY *SCI*-DUO-RINGS

Lemma 3.1: *Let P and P' two prime* ideals *of a ring R such that* $P \nsubseteq P'$. *Then Hom(R/P, R/P') = 0.*

Proof: Let $f\colon R/P \longrightarrow R/P'$ be an R-homomorphism and set $f(1+P) = t + P'$ where $t \in R$. Let $x \in P - P'$ and let r be any element of R. We have $P' = f(xr + P) = xrt + P'$. Thus, $xrt \in P'$ and since P' is prime we have $t \in P'$ and hence $f(1 + P) = P'$ and $f \equiv 0$. □

Lemma 3.2: *Every homomorphic image of a left weakly SCI-ring is a left weakly SCI-ring.*

Proof: The proof is straightforward and will be omitted. □

Proposition 3.3: *Let R be a weakly SCI-ring. If R is an integral domain, then R is a division ring.*

Proof: Let K be the division ring of the integral domain R. The R-module K is weakly cohopfian. Therefore K is finitely cogenerated. Thus, $Soc(K) \cap R \neq \{0\}$. Let $S = Ra$ $(a \in R - \{0\})$ a simple submodule of $Soc(K) \cap R$. The map

$$\varphi\colon R \longrightarrow S = Ra$$
$$x \longmapsto xa$$

is an isomorphism of R-modules. Therefore R is semisimple. For any element $b \in R - \{0\}$ we have $R = Rb = Rb^2$, then $b = cb^2$ for some $c \in R$. It follows that $1 = cb$ and $1 \in Rb = bR$ which implies that $1 = bd$ for some $d \in R$. Thus b is left invertible and right invertible, so b is invertible. □

Proposition 3.4: *Let R be a weakly SCI-duo-ring. We have the following results:*

(1) Every prime ideal of R is maximal;

(2) The Jacobson radical of R is nil;

(3) The set of the maximal ideals of R is finite;

(4) R is semiperfect;

(5) R is a finite direct product of local weakly SCI-duo-rings.

Proof: (1) results from proposition 3.3.

(2) By (1) the Jacobson radical is equal to the prime radical and consequently it is nil.

(3) Let *D* the set of all prime ideals of *R*. By lemma 3.1 the semisimple *R*-module $M = \bigoplus_{P\in D} R/P$ is weakly cohopfian, so $Soc(M) = M$ is finitely cogenerated. It follows that *M* is finitely generated and *D* is a finite set.

(4) By (3) $R/J \cong \prod_{P\in D} R/P$. This implies that *R* is a semisimple ring and since *J* is two sided ideal of *R*, then *R* is semiperfect.

(5) results from (4). □

Proposition 3.5: *[1] proposition 10.10*

For a module M the following statements are equivalent

(1) M is Artinian;

(2) Every factor module of M is finitely cogenerated.

Proposition 3.6: *Let R be a weakly SCI-duo-ring. Then R is Artinian.*

Proof: Since *R* is semiperfect we may suppose without loss of generalities that *R* is local. Let *R* be *f* an injective endomorphism of the *R*-module *R* where $f(1) = a$. Then putting $\forall n \in \mathbb{N}, f^n = f\circ f \circ \ldots \circ f$, we show by induction that $f^n(1) = a^n$. If a is an element of the Jacobson radical of R, then by 3.3 there exists $m \in \mathbb{N}$ such that $a^m = 0$, *ie* $f^m(1) = 0$ which is a contradiction because f^m is injective. *Then* $a \notin J$ and consequently *a* is invertible.

Let $y \in R$, *then* $y = y1 = y(a^{-1}a) = (ya^{-1})a = f(ya^{-1})$, *f* is surjective.

Thus *R* is weakly cohopfian and consequently *R* is finitely cogenerated. Then a weakly *SCI*-duo-ring is finitely cogenerated. This implies that *R/I* is finitely cogenerated for every ideal *I* of *R* as an homomorphic image of *R* which is a weakly *SCI*-duo-ring and by proposition 3.5 *R* is Artinian. □

Proposition 3.7: *Let R be a weakly SCI-duo-ring. Then R is a finite direct product of local Artinian weakly SCI-duo-rings.*

Proof: It follows from (3.4) and (3.6). □

Proposition 3.8: *[1] proposition 10.8*

For a module M the following statements are equivalent:

(1) R is left Artinian;

(2) Every finitely generated R-module is finitely cogenerated.

Proposition 3.9: *Let M be a direct sum of an infinite countable of a family* $(M_n)_{n\in\mathbb{N}}$ *of a nonzero submodule of M such that any two of them are isomorphic. Then M is not weakly cohopfian.*

Proof: For every integer *n* let φ_n be an isomorphism of M_n onto M_{n+1} and φ the endomorphism of *M* such that $\varphi/M_n = \varphi_n$. Then φ is a monomorphism of *M* such that $Im\varphi = \oplus_{n\geq 1} M_n$ which is not essential in *M*. □

Proposition 3.10: *A direct summand of a weakly cohopfian module is weakly cohopfian.*

Proof: Let *M* be a module and *N* a direct summand of *M*. We can write $M = N \oplus K$ where *K* is a submodule of *M*.

If *M* is a weakly cohopfian module and *g* an injective endomorphism of N, then

$$\xi: M = N \oplus K \longrightarrow M = N \oplus K$$

$$n + k \longmapsto g(n) + k$$

is an injective endomorphism of *M*. Then $Im\xi \trianglelefteq M$ ie $Img \oplus K \trianglelefteq N \oplus K$ which implies that $Img \trianglelefteq N$ (We can also see [7] corollary 1.3). □

Theorem 3.11: *Let R be a duo-ring. Then the following statements are equivalent*:

(1) *R is a weakly SCI-duo-ring;*

(2) *R is an Artinian principal ideal duo-ring.*

Proof: (1) ⟹ (2)

Following (3.7) we may suppose that *R* is a local Artinian weakly *SCI*-duo-ring. Then by §1 *R* is a principal ideal ring.

(2)⇒ (1)

Let *R* be an Artinian principal ideal duo-ring. Following [8] every *R*-module is a direct sum of cyclic submodules. Let now *M* be a weakly cohopfian module which is not finitely cogenerated. Then by [1] proposition 10.18 *M* is not finitely generated. We can write $M = \oplus_{i\in I} M_{i\in I}$ where the M_i are cyclic submodules of *M*. Since there is only a finite number of non isomorphic cyclic *R*-modules, then there is an infinite countable sub-family $(M_n)_{n\in\mathbb{N}}$ of the family $(M_i)_{i\in I}$ such that any two of them are isomorphic. Therefore, we can write

$$M = K \oplus L \text{ where } L = \oplus_{n\in\mathbb{N}} M_n$$

Following proposition 3.10 *L* is weakly cohopfian and following proposition 3.9 *L* is not weakly cohopfian. This is a contradiction. □

Corollary 3.12: *Let R be a duo-ring. Then the following conditions are equivalent:*

(1) R is an Artinian principal ideal duo-ring;

(2) R is an I-duo-ring;

(3) R is a S-duo-ring;

(4) R is a FGI-duo-ring;

(5) R is a FGS-duo-ring;

REFERENCES

[1] **F.W. Anderson and K.R. Fuller,** Rings and categories of modules, New York Springer-Verlag (1973).

[2] **M. Barry, O. Diankha and M. Sanghare,** Characterization of FGI-duo-rings, J. Algebra, Group and Geometry 24 (2007) 105-1118

[3] **M. Barry, M. Sanghare and S.D. Toure;** On FGS-duo-rings; Global Journal of Pure Applied Mathematics, Vol. 3, Num 2 (2007) 125-137.

[4] **C. Faith,** On Kothe rings, Math. Ann. 164 (1966) 207-212.

[5] **A.L. Fall, M. Sanghare,** On I-duo-rings, Théorie des nombres, Besançon (2002) 18-20

[6] **A.M. Kaidi and M. Sanghare,** Une caractérisation des anneaux artiniens à idéaux principaux, Lecture Notes in Maths, Ring Theory, 1328, Springer-Verlag, Berlin, 1998, pp 245-254.

[7] **A. Haghany and M.R. Vedadi,** Modules whose injective endomorphisms are essential, Journal of Algebra 243 (2001) 765-779.

[8] **M. Sanghare,** On S-duo-rings, Communications in Algebra, 20(8)**,** (1992) 2183-2189

[9] **M. Sanghare,** Sur quelques classes d'anneaux lies au lemme de Fiting, Rend. Sem. Mat. Univer. Padova, 87 (1992)**,** 29-37.

[10] **W.V. Vasconcelos,** On finitely generated flat module, Trans. Amer. Math. Soc, 138 (1969) 505-512.

[11] **W.V. Vasconcelos,** On injective endomorphisms of finitely generated modules, Proc. Amer. Math. Soc, 25 (1970) 900-901.